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TOPOLOGICAL METHODS FOR C^*-ALGEBRAS I: SPECTRAL SEQUENCES

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Let A be a C^*-algebra filtered by an increasing sequence of closed ideals $\{A_n\}$ with $\overline{\cup_n A_n}=A$. Then there is a spectral sequence which converges to $K_*(A)$ and has $E^1_{p,*}=K_*(A_p/A_{p-1})$. More generally, such spectral sequences obtain for the Brown-Douglas-Fillmore functors $\mathscr{E}xt_*(A)$, the Pimsner-Popa-Voiculescu functors $\mathscr{E}xt_*(Y; A)$, the Kasparov functors $\mathscr{E}xt_*(A, B)$, or indeed for any sequence of covariant or contravariant functors on C^*-algebras which satisfies an exactness axiom.

In recent years various functors on C^*-algebras have been created by analysts interested in diverse problems in single operator theory, index theory, and classification of certain types of extensions of C^*-algebras. Notable among these is the functor $\mathscr{E}xt(A)$ of Brown-Douglas-Fillmore [1], the Pimsner-Popa-Voiculescu functor $\mathscr{E}xt(Y; A)$ [13] and the Kasparov functor $\mathscr{E}xt(A, B)$ [9]. In addition there is the variant of K-theory for Banach categories developed by Karoubi and others in great generality which yields functors K_0, K_1 on C^*-algebras (cf. [20]).

In certain cases these functors have been identified with well-known objects. For example, if X is a compact space then

$$K_q(C(X)) \cong K^q(X)$$

where $C(X)$ is the C^*-algebra of complex-valued continuous functions on X and K^q is topological K-theory which, for X compact, corresponds to formal differences of vector bundles over X. More recently, the groups $\mathscr{E}xt(C(X))$ have been computed in terms of topological K-theory [1], [7] for X compact metric of finite dimension, the groups $\mathscr{E}xt(Y; C(X))$ are known [18], [19], and the groups $\mathscr{E}xt(A, B)$ are partially understood in some special cases [9], [16].

The K_0-group of an AF-algebra has the structure of an ordered "dimension group". These groups have been characterized by Effros, Handelman, and Shen, following earlier work of Bratteli and G. Elliott. (See Effros [6] for a survey.) The groups $K_*(A \times_\alpha G)$ are known in terms of K_*A for $G = \boldsymbol{Z}$ and $\boldsymbol{R}$ by the deep work of Pimsner-Voiculescu [14] and Connes [2] respectively. These results are strong enough to determine the K-groups for many C^*-algebras of interest in applications (c.f. the striking results of Cuntz [3]).

Seen in perspective, however, the field would seem to be in its infancy. There are precious few *general* techniques for calculating functors on (noncommutative) C^*-algebras. The situation seems very analogous to algebraic topology before World War II (or perhaps we should say, before Serre and Eilenberg-Steenrod).

We propose to search for general techniques which will apply to wide classes of C^*-algebras and all of the functors known to date as well as those waiting to be discovered. As our model we look to algebraic topology. In this paper we introduce the analogue of the Atiyah-Hirzebruch spectral sequences. In subsequent papers we hope to discuss geometric realization, Künneth theorems, and other techniques.

It seems very unlikely that topological techniques will enable one to do all possible computations desired. Our hope is far more modest. It is to clarify for those in the field which calculations must be done by concrete analytic methods and which calculations may be avoided by judicious use of topological techniques.

1. Statement of results.

Let A be a C^*-algebra and let

$$A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_n \subset A_{n+1} \subset \cdots \subset A \tag{1.1}$$

be a filtration of A by closed ideals with

$$\mathrm{A} = \overline{\bigcup_n A_n} \,.$$

Let $A_n = A_0$ for $n < 0$. This paper is devoted to the construction of a spectral sequence which relates the C^*-algebra K-theory groups K_*A to the groups $K_*(A_p/A_{p-1})$.

A *spectral sequence* is a sequence $\{E^r, d^r\}_{r\geq 1}$ where each E^r is an abelian group (bigraded, in general), $d^r\colon E^r \to E^r$ is a homomorphism with $d^r \circ d^r = 0$ (called a differential), and

$$E^{r+1} \cong \frac{\mathrm{Ker}\, d^r}{\mathrm{Im}\, d^r} \,.$$

As a general reference we recommend MacLane [11]. Spectral sequences are a natural generalization of exact sequences which arise when filtered objects are considered. If $A_0 \subset A_1 = A$ then

$$0 \longrightarrow A_0 \longrightarrow A \longrightarrow A/A_0 \longrightarrow 0$$

is exact and the long exact sequence in K_* may be used to determine K_*A given knowledge of K_*A_0 and $K_*(A/A_0)$. However if we go to a slightly more complicated situation, such as $A_0 \subset A_1 \subset A_2 = A$, then we must consider the various long exact K_* sequences associated to

$$0 \longrightarrow A_0 \longrightarrow A_1 \longrightarrow A_1/A_0 \longrightarrow 0$$
$$0 \longrightarrow A_0 \longrightarrow A \longrightarrow A/A_0 \longrightarrow 0$$
$$0 \longrightarrow A_1 \longrightarrow A \longrightarrow A/A_1 \longrightarrow 0 ,$$

and the intertwining of these sequences. Spectral sequences organize, store, and process this data in a systematic way.

There are two situations where filtered C^*-algebras occur naturally. The first of these is in the general structure theory of Type I C^*-algebras. If A is a separable Type I C^*-algebra then it has a countable composition series $(J_\rho)_{0 \leqq \rho \leqq \alpha}$ where each $(J_{\rho+1})/J_\rho$ is continuous trace [4]. For such C^*-algebras, then, K_*A may be approached by determining the K-groups of continuous trace C^*-algebras, feeding this information into the spectral sequence, computing differentials, and passing limit ordinals via the isomorphism $K_*(\varinjlim A_n) = \varinjlim K_*(A_n)$.

The second instance of natural filtrations may be seen in the work of A. Dynin [5]. Dynin defines a C^*-algebra A to be *solvable* if it has a filtration $\{A_j\}$ with $A_N = A$ for some N (though this restriction we delete) and with

$$A_p/A_{p-1} \cong C_o(X_p, \mathscr{K}(\mathscr{H}_p))$$

where $\mathscr{H}_p$ is a separable Hilbert space (of finite or infinite dimension), $\mathscr{K}(\mathscr{H}_p)$ denotes the compact operators on $\mathscr{H}_p$, X_p is a locally compact topological space, and $C_o(X_p, \mathscr{K}(\mathscr{H}_p))$ is the C^*-algebra of continuous functions from X_p to $\mathscr{K}(\mathscr{H}_p)$ which vanish at infinity. Dynin considers several C^*-algebras of singular integral operators and shows they are solvable in the process of studying the inversion problem for such operators. For solvable C^*-algebras one has

$$K_*(A_p/A_{p-1}) \cong K^*(X_p)$$

where $K^*(X_p)$ denotes the topological K-theory of X_p (by which we mean

$$K^*(X) = \mathrm{Ker}\,(K^*(X^+) \longrightarrow K^*(+))$$

where X^+ is the one-point compactification of X). Thus the spectral sequence relates $\{K^*(X_p)\}$ to K_*A.

A sequence of covariant functions $\{h_n\}$ *satisfies the exactness axiom* if for each short exact sequence

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0$$

there exists natural transformations $\partial\colon h_n(A/J) \to h_{n-1}(J)$ (called boundary maps) such that the sequence

$$\longrightarrow h_n(J) \longrightarrow h_n(A) \longrightarrow h_n(A/J) \xrightarrow{\partial} h_{n-1}(J) \longrightarrow$$

is a long exact sequence. Similarly, a sequence of contravariant functors $\{h^n\}$ *satisfies the exactness axiom* if there exist natural transformations $\delta: h^n(J) \to h^{n+1}(A/J)$ such that the sequence

$$\longrightarrow h^n(A/J) \longrightarrow h^n(A) \longrightarrow h^n(J) \xrightarrow{\delta} h^{n+1}(A/J) \longrightarrow$$

is a long exact sequence.

The following is a brief statement of our main results. They are in a more abstract format than is necessary for K_*; the reader may replace h_q by K_q throughout. The added generality is not entirely superfluous and it helps clarify just which properties of K_* are being used, in line with our goal of finding general techniques.

THEOREM 1.2. *Suppose given a sequence $\{h_n: n \in Z\}$ of covariant functors from some category of C^*-algebras to abelian groups which satisfies the exactness axiom. Let A be a filtered C^*-algebra in the category. Then there is a spectral sequence $\{E^r, d^r\}$ with $d^r: E^r_{p,q} \to E^r_{p-r,q+r-1}$, which converges to the graded object associated to the natural filtration on $\varinjlim_n h_*(A_n)$ and with*

$$E^1_{p,q} = h_{p+q}(A_p/A_{p-1}) .$$

For K-theory it is convenient to regard all indices involving q as taking values in $Z/2$. Using the fact that

$$K_*(A) \cong \varinjlim_n K_*(A_n)$$

we deduce from Theorem 1.2 the spectral sequences mentioned earlier.

There is an analogous spectral sequence for contravariant functors.

THEOREM 1.3. *Suppose given a sequence $\{h^n: n \in Z\}$ of contravariant functors from some category of C^*-algebras to abelian groups which satisfies the exactness axiom. Let A be a filtered C^*-algebra in the category. Then there is a spectral sequence $\{E_r, d_r\}$ with $d_r: E_r^{p,q} \to E_r^{p+r,q-r+1}$ which converges to the graded object associated to the natural filtration on $\varprojlim_n h^*(A_n)$, and with*

$$E_1^{p,q} = h^{p+q}(A_p/A_{p-1}) .$$

There is some evidence to indicate that frequently $h^*(A)$ maps onto $\varprojlim_n h^*(A_n)$ with kernel equal to $\varprojlim_n{}^1 h^*(A_n)$. The matter is discussed in § 3.

We produce these spectral sequences by the method of exact couples due to Massey [12]. The exposition of Liulevicius [10] of the construction of the Atiyah-Hirzebruch spectral sequence in the stable homotopy category via exact couples is so clear and lucid that we have followed it as closely as possible in § 6. It is an exercise to deduce the classical Atiyah-Hirzebruch spectral sequence converging to $K_*(X)$ and with $E^2_{p,q} = H_p(X; K_q(pt))$ from our results, where X is a finite complex.

Nowhere in this paper do we try to identify the E^2-terms of the spectral sequences. In the classical Atiyah-Hirzebruch spectral sequence mentioned above one shows that $E^1_{p,*} \cong C_p(X; K_q(pt))$ and hence $E^2 = H_*(X; K_*(pt))$. For C^*-algebras we have at present no notion of "chains" or of "ordinary" homology. Perhaps one should *define* $H_p(A; \boldsymbol{Z})$ by putting conditions upon allowable filtrations and then setting

$$H_p(A; \boldsymbol{Z}) = E^2_{p,0}$$

in the spectral sequence converging to K_*A. The definition makes sense only for filtered C^*-algebras and would seem to depend upon the filtration. This is analogous to the difficulty with simplicial homology; it too is only defined for certain spaces and seems to depend on the choice of simplicial structure on the space. More examples are needed to clarify the situation.

The remainder of the paper is organized as follows. Section 2 deals with the K-theory spectral sequence for filtered C^*-algebras and for solvable C^*-algebras. Section 3 is devoted to consideration of the spectral sequences which arise from the contravariant functors of Brown-Douglas-Fillmore, Pimsner-Popa-Voiculescu, and Kasparov. In § 4 we state two spectral sequence comparison theorems. These illustrate other types of information which may be gleaned from spectral sequences. In § 5 we construct a plethora of "homology" theories for which our results hold. In particular we study the effect of introduction of coefficients into K-theory and we analyze $K_*(A \otimes \mathcal{O}_n)$ where $\mathcal{O}_n$ is the Cuntz C^*-algebra. Having postponed the hard work as long as possible, we conclude in §§ 6, 7 and 8 by the construction of the spectral sequences, the identification of the E^1 and E^∞ terms, the identification of the d^1 differential, and the introduction of module structures.

It is customary at this point to acknowledge help received from others. In this instance I should like to thank Arunas Liulevicius, Saunders MacLane, and J. Peter May of the University of Chicago who taught a graduate student that homological algebra is a tool to be used, not feared, and that it may be used throughout mathematics.

2. The spectral sequence for K-theory. In this section we specialize the spectral sequence of Theorem 1.2 (which will be constructed in § 6) to K-theory. The basic observation to be made is that if we regard K_* as a Z-graded theory via

$$K_n(A) = \begin{cases} K_0(A) & n \text{ even} \\ K_1(A) & n \text{ odd} \end{cases}$$

then K_* satisfies the exactness axiom (cf. Taylor [20]). Thus from each filtered C^*-algebra A we obtain a spectral sequence with

$$E^1_{p,q} = K_{p+q}(A_p/A_{p-1})$$

which converges to $\varinjlim_n K_*(A_n)$. However, K_* commutes with direct limits:

$$K_*(A) \cong \varinjlim_n K_*(A_n)$$

and hence we have established the following theorem.

THEOREM 2.1. *Suppose given a filtered C^*-algebra*

$$A_0 \subset A_1 \subset \cdots \subset A_n \subset \cdots \subset A$$

with $\overline{\bigcup_n A_n} = A$. Then there is a spectral sequence $\{E^r(A), d^r\}$ with $d^r\colon E^r_{p,q} \to E^r_{p-r,q+r-1}$ which converges to K_*A and has

$$E^1_{p,q} = K_{p+q}(A_p/A_{p-1}) \ .$$

If the filtration is finite with $A_n = A$ for $n \geqq N$ then $E^1_{p,q} = 0$ for $p \geqq N+1$ and $E^N = E^\infty$. The spectral sequence is natural with respect to filtration-respecting maps of C^*-algebras $\varphi\colon A \to A'$.

Those who prefer to regard K_* as a $Z/2$-graded theory may regard all indices involving q as elements of $Z/2$. In that case it is good to remember that d^{2r} preserves total degree $p+q$ while d^{2r+1} changes total degree.

The simplest case of a filtration comes from considering an extension of the form

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0 \ .$$

Set $A_0 = 0$, $A_1 = J$ and $A_2 = A$. The resulting spectral sequence degenerates rather thoroughly. Working with $q \in Z/2$, we have

$$E^1_{1,q} \cong K_{q+1}(A_1/A_0) = K_{q+1}(J)$$
$$E^1_{2,q} \cong K_{q+2}(A_2/A_1) = K_{q+2}(A/J)$$
$$E^1_{p,q} = 0 \text{ for } p \neq 1, 2 \ ,$$

and thus $E^2 = E^\infty$ (since d^r changes the homological degree p by r). The d^1 differential corresponds to the boundary map in the long exact sequence of the extension via the following commutative diagram.

$$\begin{array}{ccc} E^1_{1,q} & \xleftarrow{d^1} & E^1_{2,q} \\ \downarrow\cong & & \downarrow\cong \\ K_{q+1}(J) & \xleftarrow{\partial} & K_{q+2}(A/J) \end{array}$$

There are many examples of extensions where the connecting homomorphism is nontrivial; these demonstrate that $E^1 \neq E^2$ frequently. A simple but interesting example is the extension

$$0 \longrightarrow \mathscr{K}(\mathscr{H}) \longrightarrow \mathscr{L}(\mathscr{H}) \longrightarrow \mathscr{L}(\mathscr{H})/\mathscr{K}(\mathscr{H}) \longrightarrow 0$$

where $\partial\colon K_1(\mathscr{L}/\mathscr{H})/(\mathscr{K}(\mathscr{H})) \to K_0(\mathscr{K}(\mathscr{H}))$ is an isomorphism closely related to the Fredholm index.

A more interesting example arises from the work of Brown, Douglas, and Fillmore [1]. If $T \in \mathscr{L}(\mathscr{H})$ with $T^*T - TT^* \in \mathscr{K}(\mathscr{H})$, then T is of the form (Normal) + (Compact) if and only if the K-theory boundary homomorphism from the extension

$$0 \longrightarrow \mathscr{K}(\mathscr{H}) \longrightarrow C^*\{I, T, \mathscr{K}(\mathscr{H})\} \longrightarrow C(\sigma(\pi T)) \longrightarrow 0$$

is nontrivial.

More generally, boundary homomorphisms correspond to the γ-invariant which J. Rosenberg has shown [15] to detect various nonsplitting phenomena. Perhaps the most striking of these is his proof that the canonical extension

$$0 \longrightarrow C_o(\boldsymbol{R} - \{0\},\ \mathscr{K}(\mathscr{H})) \longrightarrow C^*(G_3) \longrightarrow C_o(\boldsymbol{R}^2) \longrightarrow 0$$

associated to the Heisenberg group G_3 does not split.

We conclude this section by specializing to solvable C^*-algebras.

THEOREM 2.2. *Suppose given a solvable C^*-algebra A with*

$$A_p/A_{p-1} \cong C_o(X_p, \mathscr{K}(\mathscr{H}_p))$$

for locally compact spaces X_p. Then there is a spectral sequence (indexed as in (2.1)) which converges to K_*A and has

$$E^1_{p,q} = K^{p+q}(X_p)\ .$$

The spectral sequence collapses if $A_n = A$ for $n \geqq N$ and is natural with respect to morphisms of solvable C^*-algebras.

3. The spectral sequence for $\mathscr{E}xt$. There is some awkwardness in applying the spectral sequence of Theorem 1.3 to the Brown-Douglas-Fillmore functors $\mathscr{E}xt_n(-)$ [1] or the Pimsner-Popa-Voiculescu functors $\mathscr{E}xt_n(Y; -)$ [13], since it is not completely understood for which C^*-algebras these functors satisfy the exactness axiom. Properly restricted so that exactness is satisfied, Theorem 1.3 does apply. The most general "cohomology" theory arises from the Kasparov functor [9], [16] $\mathscr{E}xt(A, B)$ which classifies extensions of the form

$$0 \longrightarrow B \otimes \mathscr{K}(\mathscr{H}) \longrightarrow \mathscr{E} \longrightarrow A \longrightarrow 0$$

(and agrees with $\mathscr{E}xt(A)$ when $B = \boldsymbol{C}$). For B fixed the sequence of functors $\mathscr{E}xt_n(A, B)$ (contravariant in the A-variable) satisfies exactness provided that B has a countable approximate unit and we restrict A to be separable and nuclear. We thus obtain the following theorem.

THEOREM 3.1. *Fix B and let A be a C^*-algebra which is filtered in the category of separable nuclear C^*-algebras. Then there is a spectral sequence which converges to $\varprojlim_n \mathscr{E}xt_*(A_n, B)$ and has*

$$E_1^{p,q} = \mathscr{E}xt_{p+q}(A_p/A_{p-1}, B) .$$

The spectral sequence is natural in A in the usual manner and is graded so that $d_r\colon E_r^{p,q} \to E_r^{p+r,q-r+1}$.

One obvious problem, then, is to determine the relationship between $\varinjlim h^*(A_n)$ and $h^*(A)$ for functors such as $\mathscr{E}xt(-, B)$. Examples demonstrate that the groups need not agree. Under favorable circumstances one hopes for a Milnor $\lim^1$ sequence of the form

$$(3.2) \qquad 0 \longrightarrow {\varprojlim_n}^1 h^{q+1}(A_n) \longrightarrow h^q(A) \longrightarrow \varprojlim_n h^q(A_n) \longrightarrow 0 .$$

This holds for Steenrod homology theories [8] on compact metric spaces, such as $\mathscr{E}xt_n(-)$ restricted to commutative C^*-algebras. The standard proofs of (3.2) all require some homotopy invariance property. It is not known for which C^*-algebras the various functors under discussion have "good" homotopy behavior. (Indeed, this would seem to be the main stumbling block in the path of developing a general theory of "homology" and "cohomology" theories for C^*-algebras a la Eilenberg-Steenrod.)

In any event, the homotopy issue is separate from that of the existence and convergence of the spectral sequences.

4. Comparison theorems. In this section we establish two comparison theorems for our machinery. These are standard and appear here only for the convenience of the reader. Throughout, $\{h_n\}$ and $\{h^n\}$ are understood to be sequences of covariant and contravariant functors respectively satisfying the exactness axioms, all C^*-algebras are understood to be filtered in a suitable category, and maps $\varphi: A \to A'$ are understood to be filtration preserving.

THEOREM 4.1. (a) *Suppose that* $\varphi: A \to A'$ *induces isomorphisms*

$$\varphi_*: h_*(A_p/A_{p-1}) \longrightarrow h_*(A'_p/A'_{p-1})$$

for all p. *Then*

$$\varphi_*: \varinjlim_n h_*(A_n) \longrightarrow \varinjlim_n h_*(A'_n)$$

is an isomorphism.

(b) *Suppose that* $\varphi: A \to A'$ *induces isomorphisms*

$$\varphi^*: h^*(A'_p/A'_{p-1}) \longrightarrow h^*(A_p/A_{p-1})$$

for all p. *Then*

$$\varphi^*: \varprojlim_n h^*(A'_n) \longrightarrow \varprojlim_n h^*(A_n)$$

is an isomorphism.

Proof. We prove part (a); part (b) follows by a dual argument. The map φ induces a morphism of spectral sequences $\varphi^r: E^r(A) \to E^r(A')$. By (6.11), our assumption implies that φ^1 is an isomorphism of differential groups. Then φ^r is an isomorphism of differential groups for all $r \geqq 1$ and $\varphi^\infty: E^\infty(A) \to E^\infty(A')$ is an isomorphism. By (6.12), φ_* is an isomorphism. □

THEOREM 4.2. *Suppose that* $\nu: \{h_n\} \to \{\tilde{h}_n\}$ *is a natural transformation respecting boundary homomorphisms and that*

$$\nu_{(A_p/A_{p-1})}: h_*(A_p/A_{p-1}) \longrightarrow \tilde{h}_*(A_p/A_{p-1})$$

is an isomorphism. Then the induced map

$$\nu_*: \varinjlim_n h_*(A_n) \longrightarrow \varinjlim_n \tilde{h}_*(A_n)$$

is an isomorphism. The dual result holds for $\nu: \{h^n\} \to \{\tilde{h}^n\}$.

Proof. Formula (6.13) implies that

$$\nu^1: E^1(A) \longrightarrow \tilde{E}^1(A)$$

is an isomorphism, and then v^r is an isomorphism for all $r \geqq 1$, ν^∞ is an isomorphism, and (6.14) implies that ν_* is an isomorphism. □

5. Some "homology" theories. If h_* is a generalized homology theory on CW-complexes or spectra then it is well-known that for any fixed complex or spectrum Y there is a new theory defined by

$$X \rightsquigarrow h_*(X \wedge Y) . \tag{5.1}$$

Starting with stable homotopy theory $\pi_*^S(X)$ one obtains **all** homology theories by the construction

$$X \rightsquigarrow \pi_*^S(X \wedge Y) .$$

In our context there is as yet no analogue to stable homotopy theory, but the construction (5.1) does generalize.

PROPOSITION 5.2. *Let $\{h_n\}$ be a sequence of functors from a category of C^*-algebras to abelian groups satisfying the exactness axiom and let N be a fixed nuclear C^*-algebra. Then*

$$A \rightsquigarrow h_n(A \otimes N)$$

is a sequence of functors which also satisfies the exactness axiom.

Proof. The only point to verify is the exactness axiom. If

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0$$

is a short exact sequence then

$$0 \longrightarrow J \otimes N \longrightarrow A \otimes N \longrightarrow (A/J) \otimes N \longrightarrow 0 \tag{5.3}$$

is exact, since N is nuclear. Apply h_* to (5.3) and we obtain an exact triangle

$$\begin{array}{ccc} h_*(J \otimes N) & \longrightarrow & h_*(A \otimes N) \\ {\scriptstyle\partial}\nwarrow & & \swarrow \\ & h_*((A/J) \otimes N) & \end{array}$$

and thus $h_*(-\otimes N)$ satisfies the exactness axiom. □

It seems plausible to define "K-theory with coefficients in $K_*(N)$" for a fixed nuclear C^*-algebra N by

$$K_q(A; K_*(N)) \equiv K_q(A \otimes N) , \tag{5.4}$$

although this depends a priori upon N and not only upon $K_*(N)$. Proposition 5.2 shows that

$$A \rightsquigarrow K_*(A; K_*(N))$$

satisfies the exactness axiom. The following theorem shows that the definition has possibilities. (Direct limits are to be taken over countable directed sequences, and the maps $A_\alpha \to A_\beta$ need not be injective).

THEOREM 5.5. *If N is a separable C^*-algebra and A is a direct limit of solvable C^*-algebras then there is a natural short exact sequence of $\boldsymbol{Z}/2$-graded groups*

$$(5.6) \quad 0 \longrightarrow K_*(A) \otimes K_*(N) \xrightarrow{\alpha} K_*(A; K_*(N)) \xrightarrow{\beta} \operatorname{Tor}(K_*(A), K_*(N)) \longrightarrow 0$$

where α has degree 0 and β has degree 1.

Proof. This is immediate from the Künneth formula for the K-theory of C^*-algebras [17, Theorem 4.1], when A is solvable. Each of the functors commutes with direct limits (the only non-obvious matter is $(\varinjlim_n A_\alpha) \otimes N \cong \varinjlim_n (A_\alpha \otimes N)$ which holds since each A_α is nuclear), and the direct limit of short exact sequences is exact. □

For example, take $N = \mathcal{O}_{n+1}$, the Cuntz C^*-algebra [3] with

$$K_0(\mathcal{O}_{n+1}) = \boldsymbol{Z}/n, \quad K_1(\mathcal{O}_n) = 0 .$$

Cuntz defines

$$K_q(A; \boldsymbol{Z}/n) = K_q(A \otimes \mathcal{O}_{n+1})$$

(which agrees with our notation) and he shows ([3] II, Remark 2.1) that there is a natural short exact sequence

$$(5.7) \quad 0 \longrightarrow K_q(A) \otimes \boldsymbol{Z}/n \xrightarrow{\alpha} K_q(A; \boldsymbol{Z}/n) \xrightarrow{\beta} \operatorname{Tor}(K_{q-1}(A), \boldsymbol{Z}/n) \longrightarrow 0$$

for arbitrary A. (Our Künneth Theorem [17] implies this result.)

REMARK 5.8. Let G be a dimension group [6], so that there is some separable AF algebra N with $K_0(N) = G$ and $K_1(N) = 0$. Then $K_q(A; G)$ is defined, and

$$K_q(A) \otimes G \xrightarrow{\alpha} K_q(A; G)$$

is an isomorphism for all separable C^*-algebras A, by an easy limit argument. This demonstrates that it is possible to localize. For example, the UHF C^*-algebra N with $K_0(N) = \boldsymbol{Z}[1/2]$ yields $K_q(A; \boldsymbol{Z}[1/2])$.

6. Construction of the spectral sequence in "homology".

As indicated in the introduction, it is convenient and perhaps even useful to work more generally. Let us assume given a sequence $\{h_n \mid n \in \boldsymbol{Z}\}$ of covariant functors from C^*-algebras to abelian groups satisfying the following axiom.

6.1. Exactness axiom. If J is a closed ideal of A then there is a long exact sequence

$$\cdots \longrightarrow h_{n+1}(A/J) \xrightarrow{\partial} h_n(J) \longrightarrow h_n(A) \longrightarrow h_n(A/J) \xrightarrow{\partial} h_{n-1}(J) \longrightarrow \cdots$$

which is natural with respect to (A, J).

Let A be a filtered C^*-algebra via a sequence of closed ideals

$$A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_n \subset A_{n+1} \subset \cdots \subset A$$

with $\overline{\bigcup_n A_n} = A$, and let $A_n = A_0$ for $n < 0$. We have exact triangles (i.e., long exact sequences condensed for convenience)

$$\begin{array}{ccc} h_*(A_{k+1}) & \xrightarrow{h_*(j_{k+1})} & h_*(A_{k+1}/A_k) \\ {\scriptstyle h_*(f_k)}\nwarrow & & \swarrow{\scriptstyle \partial} \\ & h_*(A_k) & \end{array} \tag{6.2}$$

where $f_k \colon A_k \to A_{k+1}$ and $j_{k+1} \colon A_{k+1} \to A_{k+1}/A_k$ are the natural maps and ∂ is the connecting homomorphism $\partial \colon h_n(A_{k+1}/A_k) \to h_{n-1}(A_k)$ of degree -1. The associated exact couple is defined as follows:

$$\begin{aligned} E^1_{p,q} &= h_{p+q}(A_p/A_{p-1}) \\ D^1_{p,q} &= h_{p+q}(A_p) \\ i^1 \colon D^1_{p,q} &\longrightarrow D^1_{p+1,q-1} \end{aligned} \tag{6.3}$$

s given by $h_*(f_p)$,

$$j^1 \colon D^1_{p,q} \longrightarrow E^1_{p,q}$$

is given by $h_*(j_{p+1})$ and

$$\partial^1 \colon E^1_{p,q} \longrightarrow D^1_{p-1,q} ,$$

is given by

$$\partial = \partial(p, p-1) \colon h_{p+q}(A_p/A_{p-1}) \longrightarrow h_{p+q-1}(A_{p-1}) .$$

The associated spectral sequence

$$\{E^r_{p,q}, d^r\} \tag{6.4}$$

is called the "spectral sequence in homology h_* associated with the filtration $A_0 \subset A_1 \subset \cdots \subset A_n \subset \cdots$ of A." Notice that

$$d^r: E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}$$

where d^r is the differential in the $(r-1)-st$ derived exact couple. The identification of $E^r_{p,q}$ is standard—it is a subquotient of $E^1_{p,q}$, namely

$$E^r_{p,q} = \frac{\text{Image}\, h_*(p,\ p-1,\ p-r)}{\text{Image}\, \partial(p+r,\ p,\ p-1)} \tag{6.5}$$

where $(p,\ p-1,\ p-r)$ is the natural map

$$A_p/A_{p-r} \longrightarrow A_p/A_{p-1}$$

and

$$\partial(p+r,\ p,\ p-1): h_*(A_{p+r}/A_p) \longrightarrow h_*(A_p/A_{p-1})$$

is the composite of the boundary

$$\partial: h_*(A_{p+r}/A_p) \longrightarrow h_*(A_p)$$

and the map induced by projection

$$h_*(j_p): h_*(A_p) \longrightarrow h_*(A_p/A_{p-1})\ .$$

For a general exact couple one defines

$$E^\infty_{p,q} = \frac{\partial^{-1}\left(\bigcap_{r=1}^{\infty} \text{Im}\, i^r\right)}{j\left(\bigcup_{r=1}^{\infty} i^{-r+1}(\text{Im}\, \partial)\right)} \tag{6.6}$$

as a subquotient of $E^1_{p,q}$, where $\partial = \partial^1$, $j = j^1$, $i = i^1$ above.

We claim that E^∞ is the graded object associated to the natural filtration of $\varinjlim h_*(A_n)$, where the direct system is given by the maps

$$h_*(f_n): h_*(A_n) \longrightarrow h_*(A_{n+1})\ .$$

The natural filtration of $\varinjlim h_*(A_n)$ is defined as follows:

$$\mathscr{F}_p \varinjlim_n h_*(A_n) = \text{Image}\,(i_p: h_*(A_p) \longrightarrow \varinjlim h_*(A_p))\ .$$

The precise form of the claim is then the following lemma.

LEMMA 6.7. $\mathscr{F}_p/\mathscr{F}_{p-1} \cong E^\infty_{p,*}$.

Proof. Consider the following commutative diagram in which the row is exact:

$$\begin{array}{ccccc} & & \varinjlim h_*(A_n) & & \\ & \overset{i_{p-1}}{\nearrow} & \uparrow i_p & & \\ h_*(A_{p-1}) & \xrightarrow{h_*(f_{p-1})} & h_*(A_p) & \xrightarrow{h_*(j_p)} & h_*(A_p/A_{p-1}) \end{array}$$

so

$$\begin{aligned} \mathscr{F}_p/\mathscr{F}_{p-1} &\equiv (\operatorname{Im} i_p)/(\operatorname{Im} i_{p-1}) \\ &\cong \frac{h_*(A_p)}{\operatorname{Ker} i_p + \operatorname{Im} h_*(f_{p-1})} \\ &\cong \frac{\operatorname{Im} h_*(j_p)}{h_*(j_p)[\operatorname{Ker} i_p]} \,. \end{aligned} \tag{6.8}$$

Now $x \in \operatorname{Ker} i_p$ if and only if for some $r > 0$ the element x goes to zero under the map $h_*(A_p) \to h_*(A_{p+r})$ induced by the inclusion; that is,

$$\operatorname{Ker} i_p = \bigcup_{r>0} \operatorname{Im} \partial(p + r, p)$$

but we have the commutative diagram

$$\begin{array}{ccccc} h_*(A_{p+r}/A_p) & \xrightarrow{\partial(p+r,\, p)} & h_*(A_p) & \xrightarrow{h_*(j_p)} & h_*(A_p/A_{p-1}) \\ \downarrow & & \downarrow i^{-r+1} & & \\ h_*(A_{p+r}/A_{p+r-1}) & \xrightarrow{\partial(p+r,\, p+r-1)} & h_*(A_{p+r-1}) & & \end{array}$$

and so

$$\operatorname{Ker} i_p = \bigcup_{r=1}^{\infty} i^{-r+1}(\operatorname{Im} \partial) \,. \tag{6.9}$$

Since $A_n = 0$ for $n < 0$, we have $i^r\colon h_*(A_{p-1-r}) \to h_*(A_{p-1})$ identically zero for r large, and hence

$$\begin{aligned} \operatorname{Im} h_*(j_p) &= \operatorname{Ker}(\partial\colon h_*(A_p/A_{p-1}) \longrightarrow h_*(A_{p-1})) \\ &= \partial^{-1}(0) \\ &= \partial^{-1}\Big(\bigcap_{r=1}^{\infty} \operatorname{Im} i^r\Big) \,. \end{aligned} \tag{6.10}$$

Thus

$$\begin{aligned} \mathscr{F}_p/\mathscr{F}_{p-1} &\cong \frac{\operatorname{Im} h_*(j_p)}{h_*(j_p)[\operatorname{Ker} i_p]} && \text{by (6.8)} \\ &\cong \frac{\partial^{-1}\Big(\bigcap_{r=1}^{\infty} \operatorname{Im} i^r\Big)}{h_*(j_p)[\operatorname{Ker} i_p]} && \text{by (6.10)} \end{aligned}$$

$$\cong \frac{\partial^{-1}\left(\bigcap_{r=1}^{\infty} \operatorname{Im} i^r\right)}{h_*(j_p)\left[\bigcup_{r=1}^{\infty} \operatorname{Im} i^{-r+1}(\operatorname{Im} \partial)\right]} \qquad \text{by (6.9)}$$

$$= E^{\infty}_{p,*} \qquad \text{by (6.6)}$$

which completes the proof of the lemma. □

It is easy to see that the assignment of the exact couple to a filtered C^*-algebra is natural: if A and A' are filtered C^*-algebras and $\varphi\colon A \to A'$ is a C^*-map with $\varphi(A_p) \subset A'_p$ for all p then φ induces a morphism of exact couples and hence a morphism of spectral sequences

$$\varphi^r\colon E^r(A) \longrightarrow E^r(A')$$

which on the E^1 level corresponds to the natural map

$$h_*(A_p/A_{p-1}) \longrightarrow h_*(A'_p/A'_{p-1}) \tag{6.11}$$

and at the E^∞ level is a graded map associated to the natural map

$$\varinjlim_n h_*(\varphi \mid A_n)\colon \varinjlim_n h_*(A_n) \longrightarrow \varinjlim_n h_*(A'_n)\ . \tag{6.12}$$

Similarly, given two sequences $\{h_n\}$ and $\{\tilde{h}_n\}$ satisfying the exactness axiom, a natural transformation $\nu\colon \{h_n\} \to \{\tilde{h}_n\}$ which commutes with boundary maps, and a filtered C^*-algebra A, then there is induced a morphism of spectral sequences

$$\nu^r\colon \{E^r(A)\} \longrightarrow \{\tilde{E}^r(A)\}$$

which at the E^1 level corresponds to

$$\nu_{(A_p/A_{p-1})}\colon h_*(A_p/A_{p-1}) \longrightarrow \tilde{h}_*(A_p/A_{p-1}) \tag{6.13}$$

and at the E^∞ level is a graded map associated to the natural map

$$\varinjlim_n \nu_{A_n}\colon \varinjlim_n h_*(A_n) \longrightarrow \varinjlim_n \tilde{h}_*(A_n) \tag{6.14}$$

induced by $\{\nu_{A_n}\}$.

We conclude this section by observing that the first differential

$$d^1\colon E^1_{p,q} \longrightarrow E^1_{p-1,q}$$

is precisely the composite

$$\begin{array}{ccccc} h_{p+q}(A_p/A_{p-1}) & \xrightarrow{\partial} & h_{p+q-1}(A_{p-1}) & \xrightarrow{h_*(j_{p-1})} & h_{p+q-1}(A_{p-1}/A_{p-2}) \\ \| & & & & \| \\ E^1_{p,q} & & \xrightarrow{\quad d^1 \quad} & & E^1_{p-1,q}\ . \end{array}$$

7. Construction of the spectral sequence in "cohomology". This construction is very similar to that in homology, so we omit details. Start by assuming given a sequence $\{h^n \mid n \in \boldsymbol{Z}\}$ of contravariant functors from C^*-algebras to abelian groups satisfying the following axiom:

7.1. Exactness axiom: If J is a closed ideal of A then there is a long exact sequence

$$\begin{aligned} \cdots \longleftarrow h^{n+1}(A/J) \overset{\delta}{\longleftarrow} h^n(J) \longleftarrow h^n(A) \\ \longleftarrow h^n(A/J) \overset{\delta}{\longleftarrow} h^{n-1}(J) \longleftarrow \cdots \end{aligned}$$

which is natural with respect to (A, J).

We have exact triangles

$$\begin{array}{ccc} h^*(A_{k+1}) & \xrightarrow{h_*(f_k)} & h^*(A_k) \\ {\scriptstyle h^*(j_{k+1})} \nwarrow & & \swarrow {\scriptstyle \delta} \\ & h^*(A_{k+1}/A_k) & \end{array}$$

and hence an exact couple as in § 6:

$$\begin{aligned} E_1^{p,q} &= h^{p+q}(A_p/A_{p-1}) \\ D_1^{p,q} &= h^{p+q}(A_p) \\ i_1 &: D_1^{p,q} \longrightarrow D_1^{p-1,q+1} \end{aligned}$$

induced by $h^*(f_{p-1})$,

$$j_1 : D_1^{p,q} \longrightarrow E_1^{p+1,q}$$

is the map $\delta = \delta(p+1, p)$, and

$$\partial_1 : E_1^{p,q} \longrightarrow A_1^{p,q}$$

is the map $h^*(j_p)$. The associated spectral sequence has

$$d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1} .$$

Let

$$\mathscr{F}^p = \operatorname{Ker} \left(\varprojlim_n h^*(A_n) \xrightarrow{t_{p-1}} h^*(A_{p-1})\right) .$$

Then

$$\begin{aligned} \mathscr{F}^p/\mathscr{F}^{p+1} &= (\operatorname{Ker} t_{p-1})/(\operatorname{Ker} t_p) \\ &\cong (\operatorname{Im} t_p) \cap \operatorname{Ker} h^*(f_{p-1}) \\ &\cong \operatorname{Im} t_p \cap \operatorname{Im} h^*(j_p) . \end{aligned}$$

But

$$\operatorname{Im} t_p = h^*(A_p) \cap \left[\bigcap_{r=1}^{\infty} \operatorname{Im} i^r \right]$$

and since ∂ in the exact couple is induced by $h^*(j_p)$, we have

$$E_\infty^{p,*} \cong \mathscr{F}^p / \mathscr{F}^{p+1} .$$

Thus the cohomology spectral sequence converges to a graded object associated to $\varprojlim_n h^*(A_n)$.

The spectral sequence has the same naturality properties with respect to A and h^* as does the homology spectral sequence.

8. **Module structures.** Recall that a *graded ring* is a ($\boldsymbol{Z}$ or $\boldsymbol{Z}/2$) graded abelian group $R = \{R_q\}$ with an associative multiplication

$$R_q \otimes R_{\bar{q}} \longrightarrow R_{q+\bar{q}} .$$

A graded abelian group M is a *left graded R-module* if there is a homomorphism

$$R \otimes M \longrightarrow M$$

of degree zero satisfying the usual module properties. Let $\mathbf{mod}(R)$ denote the category of left graded R-modules.

DEFINITION 8.1. A sequence of covariant functors $\{h_n\}$ on C^*-algebras is a *sequence of $h_*(\boldsymbol{C})$-modules* if

(1) $h_*(\boldsymbol{C})$ is a graded ring,

(2) each h_n is a covariant functor to $\mathbf{mod}(h_*(\boldsymbol{C}))$,

(3) the exactness axiom is satisfied and the boundary homomorphisms take values in $\mathbf{mod}(h_*(\boldsymbol{C}))$.

The theory K_* satisfies these conditions. Regarded as a $\boldsymbol{Z}$-graded theory, we have

$$K_*(\boldsymbol{C}) = Z[t, t^{-1}]$$

where t corresponds to the Bott periodicity element, and the groups $K_q(A)$ are $K_*(\boldsymbol{C})$-modules.

For a more exotic example, suppose that N is a nuclear C^*-algebra with $N \otimes N \otimes \mathscr{K} \cong N \otimes \mathscr{K}$. Let $h_q(A) = K_q(A \otimes N)$. The Künneth pairing α [17] yields a graded ring structure on $h_*(A)$:

$$\begin{array}{ccccc}
h_q(\boldsymbol{C}) \otimes h_{\bar{q}}(\boldsymbol{C}) & & \longrightarrow & & h_{q+\bar{q}}(\boldsymbol{C}) \\
\Vert & & & & \Vert \\
K_q(N) \otimes K_{\bar{q}}(N) & \xrightarrow{\alpha} & K_{q+\bar{q}}(N \otimes N) & \xrightarrow{\simeq} & K_{q+\bar{q}}(N) .
\end{array}$$

Similarly, $h_q(A)$ is a $h_*(\boldsymbol{C})$-module.

THEOREM 8.2. *Let $\{h_n\}$ be a sequence of $h_*(C)$-modules and let A be a filtered C^*-algebra. Then the spectral sequence of Theorem* 1.2 *has the following additional structure*:

(1) *Each $E^r_{p,q}$ is a left $h_*(C)$-module, d^r is a module homomorphism, and*

$$E^{r+1} \cong H(E^r; d^r)$$

as $h_*(C)$-modules.

(2) *$E^1_{p,q} \cong h_{p+q}(A_p/A_{p-1})$ as $h_*(C)$-modules*; i.e., *each $\mathscr{F}_p$ is a $h_*(C)$-module and $E^\infty_{p,*} \cong \mathscr{F}_p/\mathscr{F}_{p-1}$ as $h_*(C)$-modules.*

The analogous theorem holds for sequences of contravariant functors. In both cases there is virtually nothing to prove, since the exact couple stores all of the information required for the spectral sequence and it lies in the category of $h_*(C)$-modules.

REFERENCES

1. L. G. Brown, R. G. Douglas, and P. A. Fillmore, *Extensions of C^*-algebras and K-homology*, Annals of Math., (2) **105** (1977), 265–324.

2. A. Connes, *An analogue of the Thom isomorphism for crossed products of a C^*-algebra by an action of R*, Advances in Math., **39** (1981), 31–55.

3. J. Cuntz, *K-theory for certain C^*-algebras*, I, Annals of Math., (2) **113** (1981), 181–197. II, J. Operator Theory, to appear.

4. J. Dixmier, *Les C^*-algèbres et leur représentations*, Gauthier-Villars, Paris, 1969.

5. A. Dynin, *Inversion problem for singular integral operators: C^*-approach*, Proc. Nat. Acad. Sci., **75** (1978), 4668–4670.

6. E. Effros, *Dimensions and C^*-algebras*, CBMS Regional Conf. Ser. in Math., Amer. Math. Soc., 1981 (to appear).

7. D. S. Kahn, J. Kaminker, and C. Schochet, *Generalized homology theories on compact metric spaces*, Michigan Math. J., **24** (1977), 203–224.

8. J. Kaminker and C. Schochet, *K-theory and Steenrod homology: applications to the Brown-Douglas-Fillmore theory of operator algebras*, Trans. Amer. Math. Soc., **227** (1977), 63–107.

9. G. G. Kasparov, *K-functors in the extension theory of C^*-algebras*, Funkcional Anal. i Priložen., **13**, no. 4 (1979), 73–74. Functional Anal. Appl., **13** (1979), 296–297.

10. A. Liulevicius, *Cobordism and homology*, Univ. de Sao Paulo, 1969.

11. S. MacLane, *Homology*, Springer-Verlag, New York, 1963.

12. W. Massey, *Exact couples in algebraic topology I-V*, Annals of Math., **56** (1953), 363–396 and **57** (1953), 248–286.

13. M. Pimsner, S. Popa, and D. Voiculescu, *Homogeneous C^*-extensions of $C(X)\otimes K(H)$*, Part I, J. Operator Theory **1** (1979), 55–108; Part II, J. Operator Theory, to appear.

14. M. Pimsner and D. Voiculescu, *Exact sequences for K-groups and Ext groups of certain cross product C^*-algebras*, preprint INCREST.

15. J. Rosenberg, *Homological invariants of extensions of C^*-algebras*, in Operator Algebras and Applications, R. V. Kadison, ed., Proc. Symp. Pure Math., vol. 38, Amer. Math. Soc., (1981), to appear.

16. J. Rosenberg and C. Schochet, *Comparing functors classifying extensions of C^*-algebras*, J. Operator Theory, to appear.

17. C. Schochet, *Topological methods for C^*-algebras II: Geometric resolutions and the Künneth formula,* Pacific J. Math., to appear.
18. ———, *Homogeneous extensions of C^*-algebras and K-theory I,* Bull. Amer. Math. Soc., (N, S) **3** (1980), 715–718.
19. C. Schochet, *Homogeneous extensions of C^*-algebras and K-theory II,* preprint.
20. J. Taylor, *Banach algebras and topology,* in Algebras in Analysis (Proc. Conf. at Univ. of Birmingham), J. H. Williamson, ed., Academic Press, 1975, 118–186.

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