PACIFIC JOURNAL OF MATHEMATICS
Vol. 97, No. 1, 1981

CONVERSE MEASURABILITY THEOREMS FOR YEH-WIENER SPACE

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Cameron and Storvick established a theorem for evaluating in terms of a Wiener integral the Yeh-Wiener integral of a functional of x which depends on the values of x on a finite number of horizontal lines. Skoug obtained the converse of the theorem in case of one horizontal line. In this paper we extend Skoug's result to the case of a finite number of horizontal lines.

1. Introduction. Let $C_1[a, b]$ denote the Wiener space of functions of one variable, i.e., $C_1[a, b] = \{x(\cdot) \mid x(a) = 0 \text{ and } x(s) \text{ is continuous on } [a, b]\}$. Let $R = \{(s, t) \mid a \leqq s \leqq b, \alpha \leqq t \leqq \beta\}$ and let $C_2[R]$ be Yeh-Wiener space (or 2 parameter Wiener space), i.e., $C_2[R] = \{x(\cdot, \cdot) \mid x(a, t) = x(s, \alpha) = 0, x(s, t) \text{ is continuous on } R\}$. Let ν be Wiener measure on $C_1[a, b]$ and let m be Yeh-Wiener measure on $C_2[R]$. For a discussion of Yeh-Wiener measure see [1], [3] and [4]. $\boldsymbol{R}$ will denote the real numbers and $\boldsymbol{C}$ the complex numbers. We shall use the following notation for the Cartesian product of n Wiener spaces $\overset{n}{\times} C_1[a, b] = C_1[a, b] \times \overset{(n)}{\cdots} \times C_1[a, b]$ and $\overset{n}{\times} \nu = \nu \times \overset{(n)}{\cdots} \times \nu$ will denote the product of n Wiener measures on $\overset{n}{\times} C_1[a, b]$.

Let $\alpha = t_0 < t_1 < \cdots < t_n = \beta$ be a subdivision of $[\alpha, \beta]$. Define $\varphi: \overset{n}{\times} C_1[a, b] \to \overset{n}{\times} C_1[a, b]$ by

$$\begin{aligned}\varphi(y_1, y_2, \cdots, y_n) &= \left(\sqrt{\frac{t_1 - t_0}{2}}y_1, \sqrt{\frac{t_1 - t_0}{2}}y_1 + \sqrt{\frac{t_2 - t_1}{2}}y_2, \cdots, \sqrt{\frac{t_1 - t_0}{2}}y_1 + \cdots \right.\\ &\qquad \left. + \sqrt{\frac{t_n - t_{n-1}}{2}}y_n\right).\end{aligned}$$

Then φ is $1 - 1$, onto and continuous with respect to the uniform topology. Let $G: C_2[R] \to \overset{n}{\times} C_1[a, b]$ be defined by $G(x) = (x(\cdot, t_1), x(\cdot, t_2), \cdots, x(\cdot, t_n))$. Then G is a continuous function from $C_2[R]$ onto $\overset{n}{\times} C_1[a, b]$.

In [1] Cameron and Storvick evaluated certain Yeh-Wiener integrals in terms of Wiener integrals. In particular they obtained the following theorem;

THEOREM A (*n-parallel lines theorem*). *Let $f(y_1, y_2, \cdots, y_n)$ be a real or complex valued functional defined on $\overset{n}{\times} C_1[a, b]$ such that*

$f \circ \varphi$ is a Wiener measurable functional of $(y_1, y_2, \cdots, y_n)$ on $\overset{n}{\times} C_1[a, b]$. Then $f \circ G$ is a Yeh-Wiener measurable functional of x on $C_2[R]$ and

$$\int_{C_2[R]} f \circ G(x)dx = \int_{\overset{n}{\times} C_1[a,b]} f \circ \varphi(y_1, y_2, \cdots, y_n)d(y_1 \times \cdots \times y_n)$$

where the existence of either integral implies the existence of the other and their equality.

We note that Theorem A, in the case $n=1$, is called the one line theorem. Now we explicitly state and prove the following corollary of Theorem A which plays a key role in the proof of Lemma 3 in §2.

COROLLARY. *Let A be any subset of $\overset{n}{\times} C_1[a, b]$. If $\varphi^{-1}A$ is $\overset{n}{\times}\nu$-measurable, then $G^{-1}A$ is Yeh-Wiener measurable and $\overset{n}{\times}\nu(\varphi^{-1}A) = m(G^{-1}A)$.*

Proof. Let $f(y_1, y_2, \cdots, y_n) = \chi_A(y_1, y_2, \cdots, y_n)$. Then

$$f \circ \varphi(y_1, y_2, \cdots, y_n) = \chi_A(\varphi(y_1, y_2, \cdots, y_n)) = \chi_{\varphi^{-1}A}(y_1, y_2, \cdots, y_n) .$$

If $\varphi^{-1}A$ is $\overset{n}{\times}\nu$-measurable, then $f \circ \varphi$ is $\overset{n}{\times}\nu$-measurable. Hence by Theorem A, $f \circ G$ is a Yeh-Wiener measurable functional of x on $C_2[R]$. But $f \circ G(x) = \chi_A(G(x)) = \chi_{G^{-1}A}(x)$. Thus $G^{-1}A$ is Yeh-Wiener measurable.

$$\begin{aligned}
\overset{n}{\times}\nu(\varphi^{-1}A) &= \int_{\overset{n}{\times} C_1[a,b]} \chi_{\varphi^{-1}A}(y_1, y_2, \cdots, y_n)d(y_1 \times \cdots \times y_n) \\
&= \int_{\overset{n}{\times} C_1[a,b]} f \circ \varphi(y_1, y_2, \cdots, y_n)d(y_1 \times \cdots \times y_n) \\
&= \int_{C_2[R]} f(x(\cdot, t_1), \cdots, x(\cdot, t_n))dx \\
&= \int_{C_2[R]} f \circ G(x)dx = \int_{C_2[R]} \chi_{G^{-1}A}(x)dx = m(G^{-1}A) .
\end{aligned}$$

It has long been known that measurability questions in Wiener space and Yeh-Wiener space are often rather delicate. In [3] Skoug established some relationships between Yeh-Wiener measurability and Wiener measurability of certain sets and functionals. Furthermore he obtained the converse of the one line theorem. In this paper we extend his result to the n-parallel lines theorem. In particular we show that if A is any subset of $\overset{n}{\times} C_1[a, b]$, then $G^{-1}A$ is a Yeh-Wiener measurable subset of $C_2[R]$ if and only if $\varphi^{-1}A$ is a Wiener measurable subset of $\overset{n}{\times} C_1[a, b]$.

2. Lemmas. The converse measurability theorems in §3 will follow quite readily once we establish three lemmas.

DEFINITION. Let δ be a fixed constant satisfying $0 < \delta < 1/2$ and let $\lambda > 0$ be given. Let

$$A_\lambda \equiv A_\lambda(\delta) \equiv \{x \in C_2[R]: |x(s_2, t_2) - x(s_1, t_1)| \leqq \lambda[(s_2 - s_1)^2 + (t_2 - t_1)^2]^{\delta/2} \text{ for all } s_1, s_2 \in [a, b] \text{ and } t_1, t_2 \in [\alpha, \beta]\} .$$

Our first lemma is taken from [3]. We state it without proof.

LEMMA 1. (a) *For any* $\varepsilon > 0$, *there exists* $\lambda_0 > 0$ *such that* $m(A_\lambda^c) < \varepsilon$ *for all* $\lambda \geqq \lambda_0$. *In fact* $m(\bigcup_{n=1}^\infty A_n) = 1$. (b) *For each* $\lambda > 0$, A_λ *is compact in the uniform topology in* $C_2[R]$.

LEMMA 2. *Let* A *be any subset of* $\overset{n}{\times} C_1[a, b]$ *and let* V *be any open set in* $C_2[R]$ *containing* $G^{-1}A$. *Let* $\lambda > 0$ *be given. Then there exists an open set* U *in* $\overset{n}{\times} C_1[a, b]$ *such that* $A \subseteqq U$ *and* $A_\lambda \cap G^{-1}U \subseteqq V$.

Proof. Case 1. Assume that A consists of just one point, say, $(y_1, \cdots, y_n)$. Suppose that Lemma 2 is false. For $n = 1, 2, 3, \cdots$, let U_n be open sphere of radius $1/n$ about $(y_1, y_2, \cdots, y_n)$. Then there exists a sequence of points $\{x_n\}_{n=1}^\infty$ in $(A_\lambda \cap G^{-1}U_n)\backslash V$. Hence $\{x_n\}_{n=1}^\infty \subseteqq A_\lambda$ and $\|Gx_n - (y_1, y_2, \cdots, y_n)\| < 1/n$ where $\|\cdot\|$ is a product norm in $\overset{n}{\times} C_1[a, b]$. Since A_λ is compact in the uniform topology for $C_2[R]$, there exists a subsequence $\{x_{n_k}\}_{k=1}^\infty$ which converges uniformly on R to some element, say x_0, of $C_2[R]$. By continuity of G, $(y_1, \cdots, y_n) = \lim_{k\to\infty} Gx_{n_k} = Gx_0$. Thus Gx_0 is in A and x_0 is in $G^{-1}A$. But V^c is closed and so x_0 is in $V^c \subseteqq (G^{-1}A)^c$ which is contrary to $x_0 \in G^{-1}A$.

Case 2. General case. Let A be any set in $\overset{n}{\times} C_1[a, b]$. By Case 1, we see that for each point z in A there exists an open set U_z in $\overset{n}{\times} C_1[a, b]$ such that $z \in U_z$ and $A_\lambda \cap G^{-1}U_z \subseteqq V$. Then $U \equiv \bigcup_{z \in A} U_z$ is an open set in $\overset{n}{\times} C_1[a, b]$ containing A and

$$A_\lambda \cap G^{-1}U = A_\lambda \cap \left(G^{-1}\left(\bigcup_{z \in A} U_z\right)\right) = \bigcup_{z \in A} (A_\lambda \cap G^{-1}U_z) \subseteqq V .$$

LEMMA 3. *Let* A *be any subset of* $\overset{n}{\times} C_1[a, b]$. *Then* $m^*G^{-1}A = (\overset{n}{\times}\nu)^*(\varphi^{-1}A)$ *where* m^* *and* $(\overset{n}{\times}\nu)^*$ *are outer Yeh-Wiener and product Wiener measures respectively.*

Proof. First we will show that $m^*G^{-1}A \leqq (\overset{n}{\times}\nu)^*(\varphi^{-1}A)$. Let $\widetilde{A}$ be a subset of $\overset{n}{\times} C_1[a, b]$ such that $A \subseteqq \widetilde{A}$, $\varphi^{-1}\widetilde{A}$ is $\overset{n}{\times}\nu$-measurable and $\overset{n}{\times}\nu(\varphi^{-1}\widetilde{A}) = (\overset{n}{\times}\nu)^*(\varphi^{-1}A)$. Note that such $\hat{A}$ exists since there exists a subset B of $\overset{n}{\times} C_1[a, b]$ such that B is $\overset{n}{\times}\nu$-measurable and

$\overset{n}{\times}\nu(B) = (\overset{n}{\times}\nu)^*(\varphi^{-1}A)$ and $\varphi^{-1}A \subseteq B$. Let $\tilde{A} = \varphi(B)$. Then $A = \varphi(\varphi^{-1}A) \subseteq \varphi(B) = \tilde{A}$ and $\varphi^{-1}(\tilde{A}) = \varphi^{-1}(\varphi(B)) = B$. By Corollary of Theorem A, $G^{-1}\tilde{A}$ is Yeh-Wiener measurable and $m^*G^{-1}A \leqq m^*G^{-1}\tilde{A} = mG^{-1}\tilde{A} = \overset{n}{\times}\nu(\varphi^{-1}\tilde{A}) = (\overset{n}{\times}\nu)^*(\varphi^{-1}A)$.

To show $(\overset{n}{\times}\nu)^*(\varphi^{-1}A) \leqq m^*G^{-1}A$, it suffices to show that for given $\varepsilon > 0$, $(\overset{n}{\times}\nu)^*(\varphi^{-1}A) \leqq m^*G^{-1}A + \varepsilon$. Now choose a Yeh-Wiener measurable set H such that $G^{-1}A \subseteq H$ and $m^*G^{-1}A = mH$. Next we choose $n > 0$ so large that $m(A_n^c) < \varepsilon/2$ [Lemma 1]. Then

$$(1) \qquad m(H \cup A_n^c) \leqq mH + m(A_n^c) < m^*G^{-1}A + \varepsilon/2 \, .$$

Let V be an open subset of $C_2[R]$ such that $H \cup A_n^c \subseteq V$ and $m(V \backslash [H \cup A_n^c]) < \varepsilon/2$ [2, Theorem 1.2, p. 27]. Then

$$(2) \qquad mV < m(H \cup A_n^c) + \varepsilon/2 \, .$$

By Lemma 2 (note that $G^{-1}A \subseteq H \subseteq H \cup A_n^c \subseteq V$ and V is open), there exists an open set $U \subseteq \overset{n}{\times}C_1[a, b]$ such that $A \subseteq U$ and $A_n \cap G^{-1}U \subseteq V$. But $(G^{-1}U) \cap A_n^c \subseteq A_n^c \subseteq H \cup A_n^c \subseteq V$. Hence

$$(3) \qquad G^{-1}U = (G^{-1}U \cap A_n) \cup (G^{-1}U \cap A_n^c) \subseteq V \, .$$

Since U is open and φ is continuous, $\varphi^{-1}U$ is open. Hence $\varphi^{-1}U$ is $\overset{n}{\times}\nu$-measurable. By continuity of G, $G^{-1}U$ is Yeh-Wiener measurable and $m(G^{-1}U) = \overset{n}{\times}\nu(\varphi^{-1}U)$. By (1), (2) and (3) we obtain $(\overset{n}{\times}\nu)^*(\varphi^{-1}A) \leqq \overset{n}{\times}\nu(\varphi^{-1}U) = m(G^{-1}U) \leqq mV < m(H \cup A_n^c) + \varepsilon/2 < m^*G^{-1}A + \varepsilon$.

3. Converse measurability theorems. Our first theorem in this section establishes a relationship between Yeh-Wiener measurability and product Wiener measurability of certain related sets. In Theorem 2 we obtain the converse of Theorem A.

THEOREM 1. *Let A be any subset of $\overset{n}{\times}C_1[a, b]$. Then $G^{-1}A$ is Yeh-Wiener measurable if and only if $\varphi^{-1}A$ is $\overset{n}{\times}\nu$-measurable. Furthermore $m(G^{-1}A) = \overset{n}{\times}\nu(\varphi^{-1}A)$.*

Proof. We only need to show that if $G^{-1}A$ is Yeh-Wiener measurable then $\varphi^{-1}A$ is $\overset{n}{\times}\nu$-measurable. So assume that $G^{-1}A$ is Yeh-Wiener measurable. By Lemma 3, $m^*G^{-1}A = (\overset{n}{\times}\nu)^*(\varphi^{-1}A)$. Another application of Lemma 3 yields $(\overset{n}{\times}\nu)^*(\varphi^{-1}A)^c = (\overset{n}{\times}\nu)^*(\varphi^{-1}A^c) = m^*G^{-1}A^c = m^*(G^{-1}A)^c = m(G^{-1}A)^c$. Thus we obtain that

$$\begin{aligned}(\overset{n}{\times}\nu)^*(\varphi^{-1}A)^c + (\overset{n}{\times}\nu)^*(\varphi^{-1}A) &= m^*(G^{-1}A)^c + m^*(G^{-1}A) \\ &= m(G^{-1}A) + m(G^{-1}A)^c = 1\end{aligned}$$

from which it follows that $\varphi^{-1}A$ is $\overset{n}{\times}\nu$-measurable.

THEOREM 2. *Let $\alpha = t_0 < t_1 < \cdots < t_n = \beta$ and let $f(y_1, y_2, \cdots, y_n)$ be a real or complex valued functional defined on $\overset{n}{\times}C_1[a, b]$. Then $f \circ \varphi$ is a Wiener measurable functional of $(y_1, y_2, \cdots, y_n)$ on $\overset{n}{\times}C_1[a, b]$ if and only if $f \circ G$ is a Yeh-Wiener measurable functional of x on $C_2[R]$. In this case,*

$$\int_{C_2[R]} f \circ G(x)dx = \int_{\overset{n}{\times}C_1[a,b]} f \circ \varphi(y_1, y_2, \cdots, y_n)d(y_1 \times \cdots \times y_n)$$

where the existence of either integral implies the existence of the other and their equality.

Proof. By Theorem A it suffices to show measurability only. Let B be any Borel set in $\boldsymbol{R}$ or $\boldsymbol{C}$. Suppose that $f \circ G$ is Yeh-Wiener measurable. Then $G^{-1}(f^{-1}B) = (f \circ G)^{-1}(B)$ is Yeh-Wiener measurable. By Theorem 1, $\varphi^{-1}(f^{-1}B) = (f \circ \varphi)^{-1}(B)$ is $\overset{n}{\times}\nu$-measurable. Hence $f \circ \varphi$ is $\overset{n}{\times}\nu$-measurable.

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Received November 11, 1980. The author wishes to thank Professor G. W. Johnson for his advice in the writing of this paper.

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