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DETERMINATION OF BOUNDS SIMILAR TO THE LEBESGUE CONSTANTS

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Bounds for the Nörlund transformation of a sequence associated with Fourier series are determined. These are applied to obtain a necessary and sufficient condition for the convergence of the Nörlund transformation of Fourier series when the generating function satisfies a condition lighter than the continuity requirement.

1. Introduction. It is well known that the unboundedness of Lebesgue constants implies the existence of a function whose Fourier series diverges at a point of continuity (e.g., see [5], § 6). Considering a class of transformed sequences of Fourier series at a point at which the generating function satisfies a lighter assumption than the continuity, we first obtain bounds for the sequence. An interesting application of such a result gives a necessary and sufficient condition for the convergence of the transformed sequence.

In the present paper, we consider the Nörlund transformation (N, p_n) associated with a given sequence of numbers $\{p_n\}$ such that $P_n = \sum_{k=0}^{n} p_k \neq 0$ and $p_{-1} = 0$. The (N, p_n) transformation of a series $a = \sum_{k=0}^{\infty} a_k$ or the sequence of its partial sums $\{s_n\}$, is defined by the sequence $\{t_n(a)\}$ where

$$t_n(a) = \sum_{k=0}^{n} P_{n-k} a_k / P_n = \sum_{k=0}^{n} p_{n-k} s_k / P_n \, .$$

Suppose $f(t)$ is a periodic function with period 2π and $f(t) \in L(0, 2\pi)$. Let $F = \sum_{n=1}^{\infty} A_n(x)$ denotes the Fourier series of $f(t)$, at $t = x$. We introduce the following notations for convenience. For a given number s

$$\varphi(t) = f(x+t) + f(x-t) - 2s; \ \ t\varphi_1(t) = \int_0^t \varphi(u)du \, ,$$

$$\Delta_n s_n = \Delta s_n = s_n - s_{n+1}, \ \Delta^2 s_n = \Delta(\Delta s_n); \ R_n = np_n/P_n \, ;$$

$$P_n V_n(r) = n^{r-1} \sum_{k=1}^{n} k \, |\Delta^r p_{k-r}|; \ P_n S_n(r) = n^{r-1} \sum_{k=1}^{n} P_k k^{-r} \, ,$$

where $r = 1, 2$.

For sequences $\{a_n\}$ and $\{b_n\}$, $a_n \asymp b_n$ means that a_n lies between two positive constant multiples of b_n.

K denotes a positive constant not necessarily the same at each occurrence and $[x]$ denotes the greatest integer not greater than x, in particular $w = [1/t]$. For any sequence $\{a_n\}$, $a(x) = a_{[x]}$.

2. The main results. The following theorem which provides bounds of Lebesgue constants for Nörlund method is essentially due to Hille and Tamarkin ([4], Theorem 7).

THEOREM A. *Suppose that* $\{p_n\}$ *is a positive sequence such that* $\{V_n(1)\} \in B$, i.e., $\{V_n(1)\}$ *is a bounded sequence, then*

$$\int_0^\pi |N(n, t)|\, dt \asymp S_n(1)\ , \tag{2.1}$$

where $\{N(n, t)\}$ *is the sequence of* (N, p_n) *transformation of* $1/2 + \sum_{k=1}^\infty \cos kt$.

The original version of Theorem A as given in [4] contains the additional hypotheses that $\{R_n\} \in B$ and (N, p_n) is regular. However, we observe that

$$np_n = -\sum_{k=0}^{n-1} \Delta(kp_k) = P_{n-1} - \sum_{k=1}^{n} k(\Delta p_{k-1}) \tag{2.2}$$

and, therefore, $\{V_n(1)\} \in B$ implies $\{R_n\} \in B$ and the latter implies that (N, p_n) is regular.

As an interesting application of (2.1), Hille and Tamarkin ([4], Theorem II) proved the following result.

THEOREM B. *Suppose that* $p_n > 0$, $\{V_n(1)\} \in B$. *Then in order that the Fourier series* F *should be summable* (N, p_n) *to* s *whenever* $\varphi(t) = o(1)$ *as* $t \to 0$, *it is necessary and sufficient that* $\{S_n(1)\} \in B$.

Under a less restrictive condition on $f(t)$ viz., $\varphi_1(t) = o(1)$, $t \to 0$, Astrachan ([1], Theorem I; see also Dikshit [3]) has obtained only a set of sufficient conditions for the (N, p_n) summability of the series F.

In the present paper, we first prove the following and then deduce a necessary and sufficient condition for the (N, p_n) summability of the series F under the assumption: $\varphi_1(t) = o(1)$, $t \to 0$.

THEOREM 1. *Suppose that* $\{p_n\}$ *is a positive sequence such that* $\{V_n(2)\} \in B$, *then*

$$\int_0^\pi |M(n, t)|\, dt \asymp S_n(2)\ , \tag{2.3}$$

where $\{M(n, t)\}$ *is the* (N, p_n) *transformation of* $\{k \cos kt\}$.

Using the result (2.3), we shall prove the following:

THEOREM 2. *Suppose that* $\{p_n\}$ *satisfies the hypotheses of Theorem* 1. *Then in order that the Fourier series* F *should be summable*

(N, p_n) to s whenever $\varphi_1(t) = o(1)$, $t \to 0$ it is necessary and sufficient that $\{S_n(2)\} \in B$.

3. Preliminary results. We use the following lemmas for the proof of our theorems.

LEMMA 1. *Let $\{a_n\}$ be a given sequence, then for any $x \neq 1$, we have*

$$\sum_{k=r}^{s} a_k x^k = x(1-x)^{-2}\left\{\sum_{k=r}^{s-2}(\Delta^2 a_k)(x^{k+1} - x^r) + (\Delta a_{s-1})(x^s - x^r)\right\} + (1-x)^{-1}(a_r x^r - a_s x^{s+1}) ,$$

where r and s are integers such that $s - 2 \geqq r \geqq 0$.

The proof of Lemma 1 is direct.

LEMMA 2. *Suppose a sequence $\{s_n\}$ satisfies the conditions:*

$$\sum_{k=1}^{n} |s_k| \leqq KT_n \text{ and } \sum_{k=1}^{n} k^2 |\Delta^2 s_{k-2}| \leqq KT_n$$

for some sequence of positive numbers $\{T_n\}$. Then

$$\sum_{k=1}^{n} k |\Delta s_{k-1}| \leqq KT_n .$$

Lemma 2 is a particular case of a more general result given in ([3], Lemma 1).

LEMMA 3. *If $\{p_n\}$ is a nonnegative sequence and $\{V_n(2)\} \in B$, then* (i) *$\{V_n(1)\} \in B$, and* (ii) *$n = 0(P_n)$.*

Proof. It follows trivially from the assumption $\{V_n(2)\} \in B$ that

$$\sum_{k=1}^{n} k^2 |\Delta^2 p_{k-2}| = 0(|P_n|)$$

and (i) therefore follows from Lemma 2.

In order to show (ii), we observe that if, for any k, $\Delta^2 p_{k-2} \neq 0$ then, for all sufficiently large n, $KP_n \geqq n$. Otherwise, if $\Delta^2 p_{k-2} = 0$ for all $k \geqq 1$, then Δp_{k-2} is a constant which is obtained by putting $k = 1$. Thus, $KP_n \geqq -\sum_{k=1}^{n+1} \Delta p_{k-2} = (n+1)p_0$ and (ii) follows.

The next lemma follows from a result due to Hille and Tamarkin ([4], Lemma 9) when we observe that $\{V_n(1)\} \in B$ implies that $\{R_n\} \in B$.

LEMMA 4. *If $\{p_n\}$ is a positive sequence and $\{V_n(1)\} \in B$, then $0 < \varepsilon \leqq v/u \leqq 1/\varepsilon$ implies the existence of an a such that $0 < a \leqq P(v)/P(u) \leqq 1/a$.*

LEMMA 5. *If* $\{p_n\}$ *is a positive sequence and* $\{V_n(1)\} \in B$, *then for any positive* $\delta \leqq \pi$,

$$\int_{1/n}^{\delta} t^{-1} \left| \sum_{k=w+1}^{n} p_k \exp ikt \right| dt = 0(P_n) .$$

The proof of Lemma 5 is essentially included in ([4], see (6.07) and Lemma 7 with $m = 2, 3$ or 4 for which the condition: $\{S_n(1)\} \in B$ is not used).

4. Proof of Theorem 1.

We first write

$$\begin{aligned} P_n M(n, t) &= \sum_{k=0}^{w-1} p_k(n-k) \cos (n-k)t + \operatorname{Re} \sum_{k=w}^{n} p_k(n-k) \exp i(n-k)t \\ &= \sum_1 + \operatorname{Re} \sum_2 , \end{aligned}$$

say. Applying Lemma 1 to $\sum_2$, we obtain

$$\sum_2 = (1 - \exp(-it))^{-1} p_w(n-w) \exp i(n-w)t - X(n, t)$$

where

$$|X(n, t)| \leqq Kt^{-2}\left\{p_{n-1} + \sum_{k=w}^{n-2} |\Delta_k^2(p_k(n-k))|\right\} . \tag{4.1}$$

Thus, we have

$$P_n M(n, t) = \sum_1 + \sum_3 - \operatorname{Re} X(n, t) , \tag{4.2}$$

where $\sum_3 = p_w(n-w) \sin(n - w + 1/2)t/\{2 \sin t/2\}$.

We now introduce the intervals $I_r = ((2r+1/3)\pi/n, (2r+4/9)\pi/n)$ for $r = 1, 2, \cdots, [n/4\pi] - 1$, which are all disjoint subintervals of $(2/n, 1)$. Considering $\sum_1$, we observe that the restriction $0 \leqq k < w$ implies $0 \leqq kt < 1$ for all $t \in (2/n, 1)$, so that whenever $t \in I_r$, $(n-k)t \in J_r = ((2r+1/3)\pi - 1, (2r+4/9)\pi)$. Thus, for $t \in I_r$, $\cos(n-k)t$ is not less than $\cos(4\pi/9)$. We also see that for $t \in (2/n, 1/2)$, $0 < t(w - 1/2) < 1$ and, therefore, $(n - w + 1/2)t \in J_r$ whenever $t \in I_r$. Thus $\sin(n - w + 1/2)t$ is not less than $\sin(\pi/3 - 1) = 2C_0$, say. In view of these observations, if we write $E = UI_r$ and $Y_n = \int_{1/n}^{\pi} |X(n, t)| dt$, where n is sufficiently large, then

$$\begin{aligned} P_n \int_{1/n}^{\pi} |M(n, t)| dt + Y_n &\geqq \int_{2/n}^{1/2} \left|\sum_1 + \sum_3\right| dt \\ &\geqq 2C_0 \int_E \left\{ \sum_{k=0}^{w-1} p_k(n-k) + p_w(n-w)\left(2 \sin \frac{1}{2} t\right)^{-1} \right\} dt \\ &> C_0 n \int_E \left\{ \sum_{k=0}^{w} p_k \right\} dt = C_0 n \int_E P(1/t) dt , \end{aligned} \tag{4.3}$$

since for $t \in E \subset (2/n, 1/2)$, $n - w > n/2$.

Writing $d = \pi/9n$, we observe that each interval I_r is of length d and any two consecutive intervals I_r, I_{r+1} are separated by a distance $17d$. Now we move the intervals I_r to the left by taking $s = t - 17(r-1)d$ so that all the intervals I_r abut upon each other. Suppose the shifted interval I_r is denoted by I_r^*, then we see that for $s \in I_r^*$ and $t \in I_r$ $18\ s \geqq t$ so that $P(1/t) \geqq P(1/18s)$ and $P(1/t) \geqq c'P(7\pi/3s)$ for some $c' > 0$, by virtue of Lemma 4. Thus, we have from (4.3)

$$\begin{aligned} P_n\int_{1/n}^{\pi} |M(n, t)|\,dt + Y_n &\geqq C_0 c' n \int_{7\pi/3n}^{7\pi/3b_n} P(7\pi/3s)ds \\ &\geqq c'' n \int_{b_n}^{n} u^{-2} P(u)du \geqq cP_n S_n(2)\ , \end{aligned} \tag{4.4}$$

where $b_n \to 14\pi/3$ as $n \to \infty$, and c, c', c'' are some positive constants.

In order to obtain the lower bound in (2.3), we assume for the moment that for some fixed K,

$$Y_n = \int_{1/n}^{\pi} |X(n, t)|\,dt \leqq KP_n \tag{4.5}$$

and deal with the cases $S_n(2) \geqq 2K/c$ and $S_n(2) < 2K/c$ separately. In the former case, (4.5) gives that $Y_n \leqq (1/2)cP_nS_n(2)$ so that we have from (4.4),

$$\int_{1/n}^{\pi} |M(n, t)|\,dt \geqq \frac{1}{2} cS_n(2)\ . \tag{4.6}$$

For the other case, we first observe that if $t \leqq \pi/3n$, then for all k with $0 \leqq k \leqq n$, $\cos kt \geqq 1/2$. Hence under the hypothesis: that $p_n > 0$, we have

$$\begin{aligned} \int_0^{\pi} |M(n, t)|dt &> \int_0^{\pi/3n} |M(n, t)|\,dt \\ &\geqq \frac{1}{2}\,\frac{\pi}{3n}\,\frac{1}{P_n}\sum_{k=1}^{n} kp_{n-k} \geqq \frac{\pi}{12P_n}\sum_{k=r(n)}^{n} p_{n-k} \end{aligned}$$

where $2r(n) = n$ or $n+1$ according as n is even or odd. Now using Lemmas 3 and 4, we have

$$\int_0^{\pi} |M(n, t)|\,dt \geqq \pi P\Big(\frac{n-1}{2}\Big)\Big/ 12P_n \geqq c^*\ ,$$

where c^* is some positive constant. Thus, in view of the condition $S_n(2) < 2K/c$, we have

$$\int_0^{\pi} |M(n, t)|\,dt > (cc^*/2K)S_n(2)\ . \tag{4.7}$$

In view of (4.6) and (4.7), we have in either case

$$\int_0^\pi |M(n, t)|\,dt \geqq AS_n(2) \tag{4.8}$$

where $A = \min(c/2,\ cc^*/2K)$.

We now complete the proof of the lower bound in (2.3) by showing (4.5). Substituting $t^{-1} = u$ in (4.5) and observing that

$$\{\Delta_k^2(n-k)p_k\} = (n-k-2)\Delta^2 p_k + 2\Delta p_k\ ,$$

we have

$$\int_{1/n}^\pi |X(n, t)|\,dt \leqq Knp_{n-1} + KL(n) \tag{4.9}$$

where

$$L(n) = \int_{1/\pi}^n \sum_{k=[u]}^n \{(n-k)|\Delta^2 p_{k-2}| + |\Delta p_{k-2}|\}du\ .$$

But

$$\begin{aligned} L(n) &\leqq Kn\sum_{r=1}^n \sum_{k=r}^n |\Delta^2 p_{k-2}| + K\sum_{r=1}^n \sum_{k=r}^n |\Delta p_{k-2}| + Knp_0 \\ &= Kn\sum_{k=1}^n k|\Delta^2 p_{k-2}| + K\sum_{k=1}^n k|\Delta p_{k-2}| + Kn \leqq KP_n\ , \end{aligned} \tag{4.10}$$

by virtue of the hypothesis $\{V_n(2)\} \in B$ and Lemma 3. Combining (4.9) and (4.10), we prove (4.5), when we observe that $\{R_n\} \in B$ by Lemma 3.

It follows from the proof of Theorem I in ([1], pp. 551–553) that under the hypotheses of Theorem 1

$$\int_0^\pi |M(n, t)|\,dt = o(1) + o(S_n(2))\ . \tag{4.11}$$

Writing $m = [n/2]$ and using the hypotheses of Theorem 1 it follows from Lemmas 3 and 4 that there is a positive number K such that

$$S_n(2) \geqq \frac{n}{P_n}\sum_{k=m}^n \frac{P_k}{k^2} \geqq Kn\sum_{k=m}^n \frac{1}{k^2}\ . \tag{4.12}$$

We thus obtain (2.3) from (4.11) when we observe that the lower bound in (4.12) tends to K as $n \to \infty$.

This completes the proof of Theorem 1.

5. Proof of Theorem 2.

We first observe that in view of

Lemma 3, the hypotheses of Theorem 2, imply the regularity of the (N, p_n) method. Thus, if $\{t_n(F)\}$ is the sequence of (N, p_n) transformation of the series F, then

$$\begin{aligned} t_n(F) - s &= o(1) + \frac{1}{\pi P_n}\int_0^\pi \varphi_1(t)t^{-1}\Big(\sum_{k=0}^n p_{n-k}\sin kt\Big)dt \\ &\quad - \frac{1}{\pi P_n}\int_0^\pi \varphi_1(t)\Big(\sum_{k=0}^n p_{n-k}k\cos kt\Big)dt \\ &= o(1) + T_1 - T_2, \text{ say} . \end{aligned}$$

In order to prove the necessity part, we first observe that if the Fourier series is (N, p_n) summable whenever $\varphi_1(t) = o(1)$, $t \to 0$ then it is certainly summable (N, p_n) whenever $\varphi(t) = o(1)$. The latter implies that $\{S_n(1)\} \in B$, when we appeal to Lemma 2 and a result due to Hille and Tamarkin ([4], Theorem II). Further, $\{V_n(1)\} \in B$ by virtue of Lemma 2 and, therefore, following the proof of Theorem 1 in ([4], pp. 769–770), we see that $T_1 = o(1)$ as $n \to \infty$, whenever $\varphi_1(t) = o(1)$, $t \to 0$. Thus, the (N, p_n) summability of F to s implies that as $n \to \infty$

$$\int_0^\pi \varphi_1(t)M(n, t)dt = o(1) . \tag{5.1}$$

We now claim that a necessary condition for (5.1) is that

$$\limsup_{n\to\infty} \int_0^\pi |M(n, t)|dt < \infty . \tag{5.2}$$

Assuming that (5.2) fails, that is, that

$$\limsup_{n\to\infty} \int_0^\pi |M(n, t)|dt = \infty , \tag{5.2$'$}$$

we construct a function $\varphi_1(t)$ such that (5.1) fails.

In view of the hypothesis $\{V_n(2)\} \in B$ and Lemma 2, we have from (4.1)–(4.2) that

$$\int_z^\pi |M(n, t)|dt = C(z) = O(1) \tag{5.3}$$

for any fixed $z > 0$. Taking $x_0(0) = \pi$, we observe that in view of (5.2′), we can find an increasing sequence of positive integers $\{n(r)\}_{r=1}^\infty$ and a decreasing sequence of numbers $\{x_0(r)\}_{r=1}^\infty$ such that

$$\int_{Q_r} |M(n(r), t)|dt > r^2[r + C(x_0(r-1))] \tag{5.4}$$

and

$$\int_0^{x_0(r)} |M(n(r), t)|\,dt < r^{-2}\,, \tag{5.5}$$

where $Q_r = [x_0(r),\ x_0(r-1)]$.

By choosing

$$\varphi_1(t) = r^{-2} \operatorname{sgn} M(n(r),\ t)$$

everywhere in Q_r except in sufficiently small neighborhoods of $x_0(r)$, $x_0(r-1)$ and those points of Q_r at which $M(n(r), t)$ changes sign, it is clear that we can define $\varphi_1(t)$ in each Q_r in such a way that its derivative exists and is bounded everywhere, that it vanishes at $x_0(r)$, $x_0(r-1)$, that $|\varphi_1(t)| \leqq r^{-2}$, and that

$$\int_{Q_r} M(n(r),\ t)\varphi_1(t)dt$$

is arbitrary near to

$$r^{-2}\int_{Q_r} |M(n(r),\ t)|\,dt\ .$$

Thus using (5.4)–(5.5), we have

$$\begin{aligned}
&\left|\int_0^{\pi}\varphi_1(t)M(n(r),\ t)dt\right| \\
&\quad \geqq \left|\int_{Q_r}\varphi_1(t)M(n(r),\ t)dt\right| - \left|\int_0^{x_0(r)}\varphi_1(t)M(n(r),\ t)dt\right| \\
&\qquad - \left|\int_{x_0(r-1)}^{\pi}\varphi_1(t)M(n(r),\ t)dt\right| \\
&\quad \geqq -O(r^{-2}) + r^{-2}\int_{Q_r}|M(n(r),\ t)|\,dt - C(x_0(r-1)) \\
&\quad \geqq -O(r^{-2}) + r\ .
\end{aligned}$$

This contradicts (5.1) and hence, we have shown that (5.2) is a necessary condition for (5.1). The necessity part of Theorem 2 now follows when we appeal to Theorem 1.

For the sufficiency part of Theorem 2, reference may be made to [1] and [3].

REMARKS. A simple example of a function $\varphi_1(t)$ meeting the requirements of the construction given after (5.5) is a piecewise quintic polynomial function or more precisely a deficient quintic spline function. For the definition of such functions reference may be made to [2].

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