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BARYCENTRIC SIMPLICIAL SUBDIVISION OF INFINITE DIMENSIONAL SIMPLEXES AND OCTAHEDRA

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A K-simplex is a convex set affinely homeomorphic to the positive face of the unit ball of a Kakutani L-space and an octahedron is a convex set affinely homeomorphic to the entire unit ball. It is shown how to barycentrically subdivide K-simplexes and octahedra so that the K-simplexes in the subdivision are affinely homeomorphic to the simplexes of probability measures on closed subsets of $(0, \infty)$ with the weak topology. As a consequence, for any closed subset C of $(0, \infty)$, an apparently new complete metric for the weak topology on $\mathscr{M}_1^+(C)$ is given.

1. Introduction. In [2] it was shown how to barycentrically subdivide the unit cube $\square$ of the infinite dimensional space $L^\infty(X, \Sigma, \mu)$ where (X, Σ, μ) is a positive localizable measure space. The elements of the subdivision were Bauer simplexes (under any locally convex topology on $L^\infty(X, \Sigma, \mu)$ between $\sigma(L^\infty, L^1)$ and the Mackey topology $\tau(L^\infty, L^1)$). The extreme points, or zero-skeleton, of the subdivision were the centers of the centrally symmetric or $\sigma(L^\infty, L^1)$ closed faces of $\square$. The $\sigma(L^\infty, L^1)$ closed faces of $\square$ were ordered by inclusion, hence so were their centers. The Bauer simplexes of the subdivision were the closed convex hulls of maximal chains of centers (which chains are compact in the order topology which agrees with $\sigma(L^\infty, L^1)$ or $\tau(L^\infty, L^1)$). The restriction to the positive unit cube $\square^+$ of this subdivision is a Bauer simplicial subdivision of $\square^+$ whose various reflections yield the barycentric subdivision of $\square$. The extreme points of a subdivision simplex in $\square^+$ are of the form $\{\chi_A: A \in C\}$ where C is a maximal chain in the measure algebra Σ_μ (which is the quotient of Σ modulo μ negligible sets). The $\sigma(L^\infty, L^1)$ closed convex hull S_C of $\{\chi_A: A \in C\}$ was shown to be affinely homeomorphic to $\mathscr{M}_1^+(C)$ the Radon probability measures on the compact C by showing that S_C is the set of $f \in \square^+$ with $\{f \geqq t\} \in C$ for all $0 < t \leqq \|f\|_\infty$. This was shown to be in affine correspondence with the convex set $\mathscr{D}(C)$ of distribution functions on C which in turn is affinely isomorphic to isomorphic to $\mathscr{M}_1^+(C)$.

Here we are concerned with barycentric subdivision in the dual (or rather predual) setting. We wish to barycentricically subdivide the unit *octahedron* $\diamondsuit$ of $L^1(X, \Sigma, \mu)$ which is the unit ball. This will be done by barycentrically subdividing the positive unit ball

$\diamondsuit^+$ and reflecting. The subdivision of $\diamondsuit^+$ will be obtained by subdividing the positive face Δ of $\diamondsuit^+$ in a barycentric fashion and extending to $\diamondsuit^+$ by taking the cone of this subdivision with 0 as vertex using the fact that $\diamondsuit^+ = \text{conv}(0, \Delta)$. Δ in general has no extreme points and is non-compact. There are no symmetric faces and faces compact under most topologies tend to be high in codimension. The natural class of faces to consider are the norm closed faces of Δ which are the same as the $\sigma(L^1, L^\infty)$ closed faces of Δ or the split faces of Δ, (which are the faces F so that there is a unique disjoint face F' with $\Delta = \text{conv}(F \cup F')$, [0], [3], [4]), or the σ-convex faces, [3], [4]. The norm closed faces are in 1–1 correspondence with the elements A of Σ_μ. If $g \in L^1(X, \Sigma, \mu)$ then S_g denotes $\{g \neq 0\}$, $S_g^+ = \{g > 0\}$ and $S_g^- = \{g < 0\}$. If F is a norm closed face of Δ then the $A_F \in \Sigma_\mu$ corresponding to it is $\cup\{S_g: g \in F\}$ where $\cup$ denotes supremum in Σ_μ. If $g \in L^{1+}(X, \Sigma, \mu)$ then $F_g = F_{S_g}$ is the smallest norm closed face of Δ containing g. When F is a norm closed face of the form F_g for some $g \in \Delta$ then we say that g is a *barycenter* of F. If F is a norm closed face of Δ we denote by $\mathscr{SP}_F$ the ensemble of split faces of F. We denote $\mathscr{SP}_\Delta$ by $\mathscr{SP}$ and $\mathscr{SP}_{F_g}$ by $\mathscr{SP}_g$ for any $g \in \Delta$. Each $\mathscr{SP}_F$ is a hyperstonean Boolean algebra isomorphic with the hyperstonean Boolean algebra $\{A \in \Sigma_\mu: A \subset A_F\}$ with supremum A_F, [3]. An $F \in \mathscr{SP}$ has a barycenter iff $\mathscr{SP}_F$ satisfies the countable chain condition for Boolean algebras. Δ has a barycenter iff (X, Σ, μ) is σ-finite iff $X = S_g$ for some $g \in \Delta$.

Any Δ which is the positive face of the unit ball of a Kakutani L-space is affinely isometric with the positive face Δ of the unit ball of $L^1(X, \Sigma, \mu)$ for some positive localizable measure space (X, Σ, μ). All Choquet simplexes are of this form. We shall call such Δ *K-simplexes*. Any norm closed face of a K-simplex is a K-simplex. Any K-simplex considered as Δ of $L^1(X, \Sigma, \mu)$ is a norm closed face of the positive face of the unit ball of the L-space $L^{\infty*}(X, \Sigma, \mu)$ when $L^1(X, \Sigma, \mu)$ is regarded as a subset of $L^{\infty*}(X, \Sigma, \mu)$. $L^\infty(X, \Sigma, \mu)$ is Banach lattice isomorphic to $\mathscr{C}(Z_\mu)$, where Z_μ is the Stone space of the measure algebra Σ_μ, and $L^{\infty*}(X, \Sigma, \mu)$ is isomorphic to $\mathscr{M}(Z_\mu)$. Hence, any K-simplex is isometric with a norm closed face of a Bauer simplex. One particular K-simplex is the space $\mathscr{M}_1^+(Y)$ of Radon probabilities on a locally compact space Y which is isometric with the norm closed face of $\mathscr{M}_1^+(Y \cup \{\infty\})$ (Radon probabilities on the one point compactification $Y \cup \{\infty\}$ of Y) of probabilities assigning measure 0 to ∞. Our simplicial subdivisions will turn out to have as elements K-simplexes affinely isomorphic to $\mathscr{M}_1^+(Y)$ for certain locally compact metric spaces Y.

2. The σ-finite case. Let (X, Σ, μ) be a σ-finite positive measure space. Let $g \in \Delta$ have $S_g = X$ so $\Delta = F_g$. Let Chain(Σ_μ) denote all chains in $\Sigma_\mu \backslash \{\varnothing\}$, C-Chain(Σ_μ) denote all complete chains in $\Sigma_\mu \backslash \{\varnothing\}$ and M-Chain(Σ_μ) all maximal chains in $\Sigma_\mu \backslash \{\varnothing\}$. If $A \in \Sigma_\mu \backslash \{\varnothing\}$ let $g\chi_A\left[\int_A g d\mu\right]^{-1} = g^A$ and let $\mu(g, A) = \int_A g d\mu$. For any $C \in$ Chain(Σ_μ) let $C(g) = \{g^A: A \in C\}$. For $C \in$ Chain(Σ_μ) let $S(C, g)$ denote the norm closed convex hull of $C(g)$. Let $\mathscr{S}_g$ denote $\{S(C, g): C \in M\text{-Chain}(\Sigma_\mu)\}$.

LEMMA 2.1. *Let* $C \in$ Chain(Σ_μ).

(a) *The mapping* $A \to g^A$ *is a homeomorphism from* C *with the order topology into* Δ *with the norm topology or any coarser Hausdorff topology.*

(b) *The mapping* $A \to \mu(g, A)$ *is a homeomorphism from* C *into* $(0, 1]$.

(c) C *is compact* iff *it is in* C-Chain(Σ_μ) *and* $\inf(C) \neq \varnothing$.

Proof. It suffices to consider only $C \in C$-Chain(Σ_μ). (b) is immediate since $(0, 1]$ has its order topology. (c) is also immediate. To establish (a) one notes that $A \to g^A$ is an order continuous injection on any chain C into Δ with the norm topology. If C is compact this map is a homeomorphism. Since any complete chain C is locally compact with every compact subset is contained in a compact subchain of the form $C_{A_0} = \{A \in C: A_0 \subset A\}$ for some $A_0 \in C$ the mapping must be a homeomorphism on any $C \in C$-Chain(Σ_μ). □

LEMMA 2.2. *Let* C *be in* Chain(Σ_μ) *and let* $\bar{C}$ *be its closure in* $\Sigma_\mu \backslash \{\varnothing\}$.

(a) $S(C, g) = S(\bar{C}, g)$.

(b) *The extreme points of* $S(C, g)$, $\xi(S(C, g))$, *form a subset of* $\bar{C}(g)$.

(c) *If* C *is compact then* $S(C, g)$ *is a norm compact subset of* Δ.

Proof. Immediate. □

For $h \in L^{1+}(X, \Sigma, \mu)$ let $C(g, h) \in C$-Chain(Σ_μ) denote the complete chain generated by $\{h/g \geqq t\}$ as t varies over $[0, \|h/g\|_\infty)$. If $C \in$ Chain(Σ_μ) let $\tilde{S}(C, g)$ denote those $h \in \Delta$ with $C(g, h) \subset C$. Of course $\tilde{S}(C, g) \neq \varnothing$ iff $X \in C$ iff C is an intersection of chains in M-Chain(Σ_μ). For any $C \in C$-Chain(Σ_μ), $\tilde{S}(C, g)$ is a base for cone$(0, \tilde{S}(C, g)) = \{h \in L^{1+}(X, \Sigma, \mu): C(g, h) \subset C\}$. For $C \in C$-Chain(Σ_μ) cone$(0, \tilde{S}(C, g))$ is closed under taking arbitrary norm bounded infima and suprema and under almost sure sequential convergence. Thus, $\tilde{S}(C, g)$ is closed under almost sure sequential convergence in Δ hence is norm closed. Any element h of conv$(g^A: A \in C)$ is easily verified to lie in $\tilde{S}(C, g)$ hence $S(C, g) \subset \tilde{S}(C, g)$. On the other hand any $h \in$ cone$(0, \tilde{S}(C, g))$ is an increasing limit of positive linear combinations $\sum_{i=1}^n \lambda_i g^{A_i}$ hence any h in $\tilde{S}(C, g)$ is a limit, in norm, of a sequence from conv $(g^A: A \in C)$. Thus, $\tilde{S}(C, g) \subset S(C, g)$.

LEMMA 2.3. (a) *If* $C \in C$-Chain (Σ_μ) *then* $\tilde{S}(C, g) = S(C, g)$.

(b) $C \in C$-Chain (Σ_μ) *is a compact chain* iff $\| h/g \|_\infty < \infty$ *for all* $h \in S(C, g)$.

Proof. (a) has been established.

(b) C is compact iff $\mu(g, A_0) > 0$ where $A_0 = \inf(C)$. In this case $\| g^A/g \|_\infty = \| (g\chi_A/g)(\mu(g, A))^{-1} \|_\infty = \mu(g, A)^{-1} \leqq \mu(g, A_0)^{-1}$. As a result $\| h/g \|_\infty \leqq \mu(g, A_0)^{-1}$ for all $h \in \text{conv}\{g^A: A \in C\}$. By continuity, $\| h/g \|_\infty \leqq \mu(g, A_0)^{-1}$ for all $h \in S(C, g)$. Conversely, if $\mu(g, A_0) = 0$ then $\| g^A/g \|_\infty \to \infty$ as A decreases in C. Choose a decreasing sequence $\{A_i\}$ in C with $0 < \mu(g, A_i) \leqq 2^{-2i}$ for all i. Set $h = \sum_{i=1}^\infty 2^{-i} g^{A_i} \in S(C, g)$. One may verify that $h/g \geqq 2^i$ on A_i for all i so $\| h/g \|_\infty = \infty$. □

Let $C \in C$-Chain (Σ_μ) be compact with infimum A_0 and supremum X. The map $\Phi_1: h \to h/g$ is 1-1 from cone $(0, S(C, g))$ into $L^\infty(X, \Sigma, \mu)$. The image of cone $(0, S(C, g))$ consists precisely of those $f \in L^{\infty+}(X, \Sigma, \mu)$ so that $\{f \geqq t\} \in C$ for all $0 \leqq t \leqq \| f \|_\infty$. In [2] it was shown that the $\sigma(L^\infty, L^1)$ closed convex hull S_C of $\{\chi_A: A \in C\}$ consists precisely of those $f \in \square^+$ with $\{f \geqq t\} \in C$ for $0 < t \leqq \| f \|_\infty$ hence for $0 \leqq t \leqq \| f \|_\infty$. From this it follows that $\Phi_1(\text{cone}(0, S(C, g)))$ is cone $(0, S_C)$. The map Φ_1^{-1} is easily seen to be continuous for $\sigma(L^\infty, L^1)$ and $\sigma(L^1, L^\infty)$. There is an affine homeomorphism Φ_2 from $\mathscr{M}^+(C)$ with the topology $\sigma(\mathscr{M}(C), \mathscr{C}(C))$ to cone $(0, S_C)$ with the topology $\sigma(L^\infty, L^1)$, [2] Proposition 3.2. The function $A \to [\mu(g, A)]$ is an element of $\mathscr{C}(C)$ which is never 0 by Lemma 2.1. The map $\Phi_3: \nu \to \mu(g, A)\nu$ is a homeomorphism of $\mathscr{M}^+(C)$ for $\sigma(\mathscr{M}(C), \mathscr{C}(C))$. The mapping $\psi = \Phi_1^{-1} \circ \Phi_2 \circ \Phi_3$ maps $\mathscr{M}^+(C)$ in a 1-1 continuous fashon onto cone $(0, S(C, g))$. If $A \in C$ then $\psi(\delta_A)$ is easily verified to be g^A. Consequently, $\psi(\mathscr{M}_1^+(C))$ is the $\sigma(L^1, L^\infty)$ closed convex hull of $C(g)$ which is $S(C, g)$ since $\sigma(L^1, L^\infty)$ and the norm topology agree on $S(C, g)$. Since ψ is 1-1 and continuous it is a homeomorphism from $\mathscr{M}_1^+(C)$ to $S(C, g)$. $\xi(S(C, g)) = \psi(\mathscr{M}_1^+(C)) = C(g)$.

PROPOSITION 2.4. *If C is a compact chain in* C-Chain (Σ_μ) *then* $S(C, g)$ *is a Bauer simplex under the norm topology and* $\xi(S(C, g)) = C(g)$.

Proof. When $X \in C$ this has been established. Otherwise C is a closed subset of the compact chain $C \cup \{X\}$ hence $S(C, g)$ is a closed face of the Bauer simplex $S(C \cup \{X\}, g)$. □

The mapping Φ_2^{-1} assigns to each $f \in S_C$ the $p_f \in \mathscr{M}_1^+(C)$ defined by $p_f\{A: A \in C, \{f \geqq t\} \subset A\} = t = d_{p_f}(\{f \geqq t\})$ where d_{p_f} is the left continuous distribution function of p_f on C. One may deduce that if $A \in C$ then the essential infinum, $\text{ess inf}_A (f) = d_f(A)$ of f on A is $d_{p_f}(A)$. These remarks extend to the case where $f \in \text{cone}(0, S_C)$ where $p_f \in \mathscr{M}^+(C)$. If $A_1 \in C$ one may consider the order interval $C_{A_1} = \{A \in C: A_1 \subset A\}$. The restriction of p_f to C_{A_1} has distribution function which is the restriction of d_f to C_{A_1}. The measure $p_f|_{C_{A_1}}$ may considered as an element of $\mathscr{M}^+(C)$ in the usual manner. Its distribution function is the extension $d_f^{A_1}$ of $d_f|_{C_{A_1}}$ to C described by $d_f^{A_1}(A) = d_f(A)$ if $A_1 \subset A$ and by $d_f^{A_1}(A) = d_f(A_1)$ otherwise. This corresponds to the function $f \wedge d_f(A)$ in $\text{cone}(0, S_{C_{A_1}})) \subset \text{cone}(0, S_C))$. Since $p_f \to p_f|_{C_{A_1}}$ is a continuous linear surjection from $\mathscr{M}^+(C)$ to $\mathscr{M}^+(C_{A_1})$ the map $f \to f \wedge d_f(A_1)$ is a continuous linear surjection from $\text{cone}(0, S_C)$ onto $\text{cone}(0, S_{C_{A_1}})$. As a result the map $h \to h \wedge [gd_{h/g}(A_1)]$ is a continuous linear surjection from $\text{cone}(0, S(C, g))$ to $\text{cone}(0, S(C_{A_1}, g))$.

If C is a noncompact complete chain and $h \in \text{cone}(0, S(C, g))$ we may define $d_{h/g}(A) = \text{ess inf}_A (h/g)$ if $A \in C$. The map $Q_A: h \to h \wedge [gd_{h/g}(A)]$ is again a continuous linear map onto $\text{cone}(0, S(C_{A_1}, g))$. Furthermore, if $A_1 \subset A_2$ are in C then $Q_{A_2} \circ Q_{A_1} = Q_{A_2}$. For any h in $\text{cone}(0, S(C, g))$ $Q_A(h)$ increases to h as A decreases in C. The function $d_{h/g}$ is decreasing and left continuous on C. For each $A \in C$, one assigns to h the measure $\tilde{Q}_A(h) \in \mathscr{M}^+(C_A)$ corresponding to the restriction of $d_{h/g}$ to C_A. The mapping $Q_A^\#: h \to \mu(g, A)\tilde{Q}_A(h)$ is a continuous linear surjection from $\text{cone}(0, S(C, g))$ to $\mathscr{M}^+(C_A)$ such that if $A_1 \subset A_2$, are in C then $Q_{A_2}^\# \circ \psi_{A_1} \circ Q_{A_1}^\# = Q_{A_2}^\#$ where ψ_{A_1} is the affine isomorphism from $\mathscr{M}^+(C_{A_1})$ to $\text{cone}(0, S(C_{A_1}, g))$. The norm of $Q_A^\#(h)$ is equal to the norm of $h \wedge [gd_{h/g}(A)]$. As A decreases in C, $Q_A^\#(h)$ (considered as elements of $\mathscr{M}_b^+(C)$) converges to an element $Q^\#(h)$ of $\mathscr{M}_b^+(C)$ whose restriction to any C_A is $Q_A^\#(h)$. Furthermore $Q^\#(h) \in \mathscr{M}_1^+(C)$ iff $h \in S(C, g)$. If $\mu \in \mathscr{M}_b^+(C)$ one may find, for an $A \in C$, the image $\psi_A^\#(\mu)$ of the restriction of μ to C_A under ψ_A in $\text{cone}(0, S(C_A, g)) \subset \text{cone}(0, S(C, g))$. For any $\mu \in \mathscr{M}_b^+(C)$, $Q_A^\# \circ \psi_A^\#(\mu)$ is the restriction of μ to C_A. As A decreases in C, $\psi_A^\#(\mu)$ converges to an element $\psi^\#(\mu)$ of $\text{cone}(0, S(C, g))$ which satisfies $Q^\#(\psi^\#(\mu)) = \mu$. Conversely if $h \in \text{cone}(0, S(C, g))$ then $\psi^\#(Q^\#(h)) = h$.

PROPOSITION 2.5. *Let* $C \in C\text{-Chain}(\Sigma_\mu)$.

(a) *$S(C, g)$ is affinely isomorphic to $\mathscr{M}_1^+(C)$ under $Q^\#$.*

(b) *$\xi(S(C, g)) = C(g)$.*

(c) *If C' is a complete subchain of C then $S(C', g)$ is a norm closed face of $S(C, g)$.*

(d) *Any norm closed face F of $S(C, g)$ is the σ-convex hull of its* *compact subfaces and is* $S(C^F, g)$ *for some* C^F *a complete subchain of* C.

(e) *If* $\tilde{C} \in C$-Chain (Σ_μ) *then* $S(C, g) \cap S(\tilde{C}, g) = S(C \cap \tilde{C}, g)$.

Proof. (a) has already been established.

(b) is immediate from the fact that the extreme points of $\mathscr{M}_1^+(C)$ are the δ_A with $A \in C$ which correspond to g^A for $A \in C$.

To establish (c) it is only necessary to note that $Q^\#$ assigns to the probabilities on C giving full measure to C' the subset $S(C', g)$. Since these probabilities on C are a face of $\mathscr{M}_1^+(C)$, $S(C', g)$ is a face of $S(C, g)$ which is norm closed.

If F is a compact face of $S(C, g)$ then $\xi(F)$ is a compact set in $C(g)$ of the form $\{g^A: A \in C'\}$ for a compact chain $C' \subset C$ hence $F = S(C', g)$. Conversely, if C' is a compact chain in C then $\mathscr{M}_1^+(C')$ is a face of $\mathscr{M}_1^+(C')$ which corresponds to $S(C', g)$ under $Q^\#$. Hence $S(C', g)$ is a compact face of $S(C, g)$.

Let $h \in S(C, g)$ and let $\{A_n\}$ decrease to $\emptyset$ in C. For any n set $h_n = h \wedge (gd_{h/g}(A_n))$, $\lambda_1 = \|h_1\|_1$, and $\lambda_n = \|h_n - h_{n-1}\|_1$ if $n > 1$. Set $h^1 = h_1\lambda_1^{-1}$ if $\lambda_1 \neq 0$, set $h^n = (h_n - h_{n-1})\lambda_n^{-1}$ if $n > 1$ and if $\lambda_n \neq 0$, and set $h^j = 0$ if $\lambda_j = 0$ for $j \geqq 1$. It is easily verified using Lemma 2.3 that $h^n \in S(C_{A_n}, g)$ for all n if $h^n \neq 0$. We have $h = \sum_{n=1}^\infty \lambda_n h^n$ and $\sum_{n=1}^\infty \lambda_n = 1$. Thus, h is in the σ-convex hull of the union of the compact faces $\{S(C_{A_n}, g): n \in N\}$ of $S(C, g)$.

Let F be a norm closed face of $S(C, g)$ and let $h \in F$. Let $\{A_n\}$ and $\{h^n\}$ be as in the preceding paragraph. The face $F \cap S(C_{A_n}, g)$ of F and $S(C_{A_n}, g)$ is compact and h is in the σ-convex hull of the union of these faces as n ranges over N. For each n, $\xi(F \cap S(C_{A_n}, g))$ is of the form $C^n(g)$ for a compact subset C^n of C_{A_n}. Furthermore $C^n \cap C_{A_{n-1}} = C^{n-1}$ for all $n > 1$. Thus, F is the σ-convex hull of $C^F = \bigcup_{n \in N} C^n$ which is a complete subchain of C. This establishes (d).

(e) is immediate from Lemma 2.3. □

We recall from [2] that a *simplicial subdivision* of Δ is a collection $\mathscr{S}$ of simplexes which cover Δ, so that if $S_1 \neq S_2$ are in $\mathscr{S}$ then $S_1 \cap S_2$ is a proper face of S_1 and of S_2. A *K-simplicial* subdivision, under a topology on Δ, is defined to be one whose elements are K-simplexes. A *simplicial precomplex* on Δ is a collection $\mathscr{S}$ of simplexes covering Δ such that $\{S_1, S_2\} \subset \mathscr{S}$ then $S_1 \cap S_2$ is a face both of S_1 and S_2. A *simplicial complex*, under a topology on Δ, is a simplicial precomplex which if it contains a simplex S also contains all closed faces of S. If $\mathscr{S}$ is a K-simplicial subdivision then the ensemble $\mathscr{C}$ of closed faces of elements of $\mathscr{S}$ is the *associated simplicial complex*.

PROPOSITION 2.6. *$\mathscr{S}_g = \{S(C, g): C \in M\text{-Chain}(\Sigma_\mu)\}$ is a K-simplicial subdivision of Δ whose associated simplicial complex is $\mathscr{C}_g = \{S(C, g): C \in C\text{-Chain}(\Sigma_\mu)\}$.*

Proof. By Proposition 2.5, it is evident that $\mathscr{S}_g$ is a cover of Δ by K-complexes and that $\mathscr{C}_g$ consists of all norm closed faces of $\mathscr{S}_g$. The only condition not immediately apparent to verify that $\mathscr{S}_g$ is a K-simplicial subdivision of Δ is the condition that if $C_1 \neq C_2$ are in M-Chain (Σ_μ) then $S(C_1, g) \cap S(C_2, g)$ is a proper face of both $S(C_1, g)$ and $S(C_2, g)$. This is a consequence of Proposition 2.5. (e) and the maximality of C_1 and C_2. □

The mapping $Q^\sharp$ transfers the metric $\| \;\|_1$ on $S(C, g)$ to a metric D_g on $\mathscr{M}_1^+(C)$ in the natural fashion so that $D_g(Q^\sharp(h_1), Q^\sharp(h_2)) = \|h_1 - h_2\|_1$. Actually, $Q^\sharp$ is extendable so that it is defined on $L(C, g) = \text{cone}(0, S(C, g)), -\text{cone}(0, S(C, g))$ and is a Banach lattice isomorphism from the L-space $L(C, g)$ to the L-space $\mathscr{M}_b(C)$. The norm of the L-space $L(C, g)$ is not $\| \;\|_1$ but the Minkowski functional ρ_g of $\text{conv}(S(C, g) - S(C, g))$ and $\rho_g \geqq \| \;\|_1$ on $L(C, g)$ since $Q^{\sharp-1}$ is a contraction from $\mathscr{M}_b(C)$ into $L^1(X, \Sigma, \mu)$. Since $S(C, g)$ is $\| \;\|_1$-closed $\mathscr{M}_1^+(C)$ is D_g-complete.

If $A_1 \subset \cdots \subset A_n$ are in a complete chain C and $p = \sum_{i=1}^n \lambda_i \delta_{A_i} \in \mathscr{M}^+(C)$ then $d_p(A) = \sum_{i=k}^n \lambda_i$ if A is in the order interval $(A_{k-1}, A_k] \subset C$ where $A_0 = \varnothing$ and $A_{n+1} = X$. If $h = \sum_{i=1}^n \lambda_i g^{A_i}$ so that $Q^\sharp(h) = p$ then $d_{h/g}(A) = \sum_{i=k}^n \lambda_i \mu(g, A_i)^{-1} = \int_{C_A} \mu(g, B)^{-1} p(dB)$ if $A \in (A_{k-1}, A_k]$. For any k, $-\lambda_k = [d_{h/g}(A_{k+1}) - d_{h/g}(A_k)]\mu(g, A_k)$ so $d_p(A) = -\sum_{i=k}^n \mu(g, A_k)[d_{h/g}(A_{k+1}) - d_{h/g}(A_k)] = \int_{C_A} \mu(g, B) d_{h/g}(dB)$ where the latter is a Lebesque-Stieltjes integral. By continuity, whenever $p \in \mathscr{M}_b^+(C)$ is $Q^\sharp(h)$ for $h \in \text{cone}(0, S(C, g))$ one has $d_p(A) = \int_{C_A} \mu(g, B) d_{h/g}(dB)$ and $d_{h/g}(A) = \int_{C_A} \mu(g, B)^{-1} p(dB)$.

PROPOSITION 2.7. *Let C be a complete chain. If $\{p_1, p_2\} \subset \mathscr{M}_b^+(C)$ then $D_g(p_1, p_2) = \int_C \left| \int_{C_A} \mu(g, B)^{-1} p_1(dB) - \int_{C_A} \mu(g, B)^{-1} p_2(dA) \right| \mu(g, dA)$ where the outer integral is Lebesgue-Stieltjes with respect to the monotone function $\mu(g, \cdot)$. D_g yields a complete metrization of vague and weak convergence on $\mathscr{M}_1^+(C)$.*

Proof. Let $\tilde{D}_g(p_1, p_2)$ denote $\int_C \left| \int_{C_A} \mu(g, B)^{-1} p_1(dB) - \int_{C_A} \mu(g, B)^{-1} p_2 \right| \mu(g, dA)$ for the time being. Let us verify that $\tilde{D}_g(p_1, p_2)$ is finite for all $\{p_1, p_2\} \subset \mathscr{M}_b^+(C)$ or that $\tilde{D}_g(p) = \int_C \left| \int_{C_A} \mu(g, B)^{-1} p(dB) \right| \times$

$\mu(g, dA) < \infty$ if $p = p_1 - p_2 \in \mathscr{M}_b(C)$. Suppose that C is compact then $\tilde{D}_g$ is a continuous convex function of p for $\sigma(\mathscr{M}_b(C), \mathscr{C}(C))$. $\tilde{D}_g$ attains its supremum at extreme points $p = \delta_{A_1} - \delta_{A_2}$, say with $A_1 \subset A_2$. Here $\tilde{D}_g(p)$ may easily be computed to be $\mu(g, A_2)^{-1}[\mu(g, A_2) - \mu(g, A_1)] + [\mu(g, A_1)^{-1} - \mu(g, A_2)^{-1}]\mu(g, A_1) = 2(1 - \mu(g, A_1)\mu(g, A_2)^{-1}) \leqq 2$. Thus, $\tilde{D}_g(p) \leqq 2\|p\|$ for all $p \in \mathscr{M}_b(C)$ if C is compact. If C isn't compact $\tilde{D}_g(p)$ is the limit, as A decreases to $\varnothing$ in C, of $\int_{C_A} \left|\int_{C_{A'}} \mu(g, B)^{-1}p(dB)\right| \times \mu(g, dA')$. Hence, $\tilde{D}_g(p) \leqq \|p\|$ even in this case. Since $\mu(g, \cdot)$ is continuous and strictly increasing on C, $\tilde{D}_g(p_1, p_2) = 0$ implies $\int_{C_A} \mu(g, B)^{-1}p_1(dB) = \int_{C_A} \mu(g, B)^{-1}p_2(dB)$ for a dense set of A in C hence that $h_1 = h_2$ where $Q^\sharp(h_j) = p_j$ for $j = 1, 2$. Thus, $p_1 = p_2$. This suffices to show that $\tilde{D}_g$ is a metric on $\mathscr{M}_1^+(C)$. If $\{p_n\}$ is a $\tilde{D}_g$-Cauchy sequence in conv $(0, \mathscr{M}_1^+(C))$ and $A \in C$ one may select a subsequence $\{p'_n\}$ whose restrictions to C_A are $\sigma(\mathscr{M}_b(C_A), \mathscr{C}(C_A))$ convergent to p'_A. Then $\left|\int_{C_A} \mu(g, B)^{-1}p'_n(dB) - \int_{C_A} \mu(g, B)^{-1}p'_A(dB)\right|$ converges to 0 as $n \to \infty$. Since $\{p_n\}$ is $\tilde{D}_g$-Cauchy $\left|\int_{C_A} \mu(g, B)^{-1}p_n(dB) - \int_{C_A} \mu(g, B)^{-1}p_n(dB)\right|$ converges to 0 for $\mu(g, dA)$ almost all A. We deduce that $\left|\int_{C_A} \mu(g, B)^{-1}p_n(dB) - \int_{C_A} \mu(g, B)^{-1}p'_A(dB)\right|$ converges to 0 for $\mu(g, dA)$ almost all A. As a consequence p'_A is the $\sigma(\mathscr{M}_b^+(C_A), \mathscr{C}(C_A))$ limit of the restrictions of $\{p_n\}$ to C_A. Thus, there is a $p' \in$ conv $(0, \mathscr{M}_1^+(C))$ whose restriction to each C_A is p'_A. For this p' we have $\int_C f dp_n \to \int f dp'$ for all continuous f on C with compact support. That is, $\{p^2\}$ converges vaguely to p. Conversely, if $\{p_n\} \subset$ conv $(0, \mathscr{M}_1^+(C))$ is vaguely convergent to p then $\left|\int_{C_A} \mu(g, B)^{-1}p_n(dB) - \int_{C_A} \mu(g, B)^{-1}p(dB)\right| \to 0$ as $n \to \infty$ for all $A \in C$ from which it follows that $\tilde{D}_g(p_n, p) \to 0$ as $n \to \infty$ if $\{p_n\} \subset$ conv $(0, \mathscr{M}_1^+(C))$. Thus, the metric $\tilde{D}_g$ is complete on conv $(0, \mathscr{M}_1^+(C))$ and gives the topology of vague convergence. If $p_j = \sum_{i=1}^n \lambda_i^j \delta_{A_i}$, for $A_1 \subset \cdots \subset A_n$ and for $j = 1, 2$, are in $\mathscr{M}_b^+(C)$ and equal $Q^\sharp(h_j)$ where $h_j = \sum_{i=1}^n \lambda_i^j g^{A_i}$ then $h_j g^{-1}$ is equal to $\sum_{i=k}^n \lambda_i^j \mu(g, A_i)^{-1}$ on $A_k \backslash A_{k-1}$ so $D_g(p_1, p_2) = \|h_1 - h_2\|_1 = \sum_{k=1}^n \int_{A_k \backslash A_{k-1}} |\sum_{i=k}^n \lambda_i^1 \mu(g, A_i)^{-1} - \sum_{i=k}^n \lambda_i^2 \mu(g, A_i)^{-1}| g(d\mu) = \sum_{k=1}^n |\sum_{i=k}^n \lambda_i^1 \mu(g, A_i)^{-1} - \sum_{i=k}^n \lambda_i^2 \mu(g, A_i)^{-1}|(\mu(g, A_k) - \mu(g, A_{k-1})) = \int_C \left|\int_{C_A} \mu(g, B)^{-1}p_1(dB) - \int_{C_A} \mu(g, B)^{-1}p_2(dB)\right| \mu(g, dA) = \tilde{D}_g(p_1, p_2)$. Thus, $h \to Q^\sharp(h)$ is an isometry from cone $(0, S(C, g))$ to $\mathscr{M}_b^+(C)$ with the metric $\tilde{D}_g$ at least on simple functions h. By continuity $Q^\sharp$ is an isometry from conv $(0, S(C, g))$ onto the vaguely complete conv $(0, \mathscr{M}_1^+(C))$. Thus, $\tilde{D}_g = D_g$. Since $S(C, g)$ is norm complete, $\mathscr{M}_1^+(C)$ is D_g complete. That is, D_g is a complete metrization of vague convergence. It is well known that the weak topology $\sigma(\mathscr{M}_b^+(C), \mathscr{C}_b(C))$ and the vague topology agree on $\mathscr{M}_1^+(C)$ so D_g is a complete metrization of the weak topology as well. □

REMARK. If $f(B) = \mu(g, B)^{-1}$ then

$$D_g(p_1, p_2) = \int_C |d_{fp_1}(A) - d_{fp_2}(A)| \mu(g, dA)$$

where d_{fp_j} is the distribution function of $fp_j \in \mathscr{M}^+(C)$.

Under the homeomorphism $H: A \to \mu(g, A)$ of C into $(0, 1]$ the simplex $\mathscr{M}_1^+(C)$ is assigned to the simplex $\mathscr{M}_1^+(H(C))$ under an affine homeomorphism for the weak topologies. The affine homeomorphism is the unique one sending $\delta_A \in \mathscr{M}_1^+(C)$ to $\delta_{\mu(g,A)} \in \mathscr{M}_1^+(H(C))$. The metric D_g on $\mathscr{M}_b^+(C)$ induces a metric $D_g^\sharp$ on $\mathscr{M}_b^+(H(C))$ in the usual fashion.

COROLLARY 2.7.1. *If p_1, p_2 are in $\mathscr{M}_b^+(H(C))$ then $D_g^\sharp(p_1, p_2) = \int_{H(C)} \left| \int_t^1 (1/s)p_1(ds) - \int_t^1 (1/s)p_2(ds) \right| dt = \int_{H(C)} |d_{1/tp_1}(t) - d_{1/tp_2}(t)| dt$ where dt is Lebesgue-Stieltjes integration with respect to the restriction of $f(t) = t$ to $H(C)$.*

PROPOSITION 2.8. (a) *If μ is a non-atomic measure then all of the simplexes in $\mathscr{S}_g$ are affinely isometric.*

(b) *If μ is not non-atomic there are two simplexes in $\mathscr{S}_g$ which aren't affinely homeomorphic.*

Proof. (a) If μ is non-atomic then $H(C) = (0, 1]$ for all $C \in M$-Chain (Σ_μ). $D_g^\sharp(p_1, p_2) = \int_0^1 |d_{1/tp_1}(t) - d_{1/tp_2}(t)| dt$ yields the same metric on $\mathscr{M}_1^+((0, 1])$ for all $C \in M$-Chain (Σ_μ).

(b) If C_1 and C_2 in M-Chain (Σ_μ) were to have $S(C_1, g)$ and $S(C_2, g)$ affinely homeomorphic, then $\mathscr{M}_1^+(C_1)$ and $\mathscr{M}_1^+(C_2)$ would be affinely homeomorphic under the vague topology so C_1 and C_2 would be homeomorphic. In the proofs of Propositions 6.1 and 6.2 of [2] it is shown that if μ isn't non-atomic there are maximal chains in Σ_μ which aren't homeomorphic. The same procedure is applicable to complete chains in $\Sigma_\mu \backslash \{\varnothing\}$. □

REMARK. In the terminology of [2], $\mathscr{S}_g$ is homogeneous iff μ is non-atomic.

COROLLARY 2.8.1. *If g_1 and g_2 are two elements of Δ so that $F_{g_1} = F_{g_2} = \Delta$ and $C \in C$-Chain (Σ_μ) then $S(C, g_1)$ is affinely homeomorphic to $S(C, g_2)$.*

Proof. Both are affinely homeomorphic to $\mathscr{M}_1^+(C)$ with the weak topology. □

REMARK. (1) This affine homeomorphism probably isn't attainable as an affine isometry unless C is connected.

(2) This states a strong equivalence between the simplicial subdivisions $\mathscr{S}_{g_1}$ and $\mathscr{S}_{g_2}$ of Δ.

Of some interest is the question of which Hausdorff locally convex topologies τ on $L^1(X, \Sigma, \mu)$ induce on each simplex in $\mathscr{S}_g$ its norm topology.

PROPOSITION 2.9. *The Hausdorff locally convex topologies on $L^1(X, \Sigma, \mu)$ which induce the nom topology on simplexes in $\mathscr{S}_g$ are precisely those coarser than the norm topology.*

Proof. Let τ be a Hausdorff locally convex topology on $L^1(X, \Sigma, \mu)$ coarser than the norm topology. Let $C \in M\text{-Chain}(\Sigma_\mu)$. Regard the linear span of $S(C, g)$ as being linearly isomorphic to $\mathscr{M}_b(C)$. If $A \in C$ then $S(C_A, g)$ is norm-compact, hence τ is compact. $\mathscr{M}_b(C_A)$ with its weak topology is linearly homeomorphic to the linear span of $S(C_A, g)$ with the topology τ. That is, τ induces on each $\mathscr{M}_b(C_A) \subset \mathscr{M}_b(C)$ the weak topology. The vague topology on $\mathscr{M}_b(C)$ is the coarsest such topology. Thus, τ is finer than the topology induced by the vague topology on $S(C, g)$. On $S(C, g)$ the norm topology is that induced by the vague topology. Thus, τ must be the norm topology on $S(C, g)$.

Conversely, suppose that τ induces on each $S(C, g)$ the norm topology. To show that τ is coarser than the norm topology on $L^1(X, \Sigma, \mu)$ it is only necessary to show that the τ-dual of $L^1(X, \Sigma, \mu)$ is a subspace of $L^\infty(X, \Sigma, \mu)$. Let λ be in the τ-dual of $L^1(X, \Sigma, \mu)$. Define the additive function λ' on Σ_μ by $\lambda'(A) = \lambda(g\chi_A)$ for $A \in \Sigma_\mu$. If $\{A_n\}$ is an increasing sequence in Σ_μ then $\lim_{n\to\infty} \lambda(g\chi_{A_n}) = \lambda(g\chi_{A_\infty}) = \lambda'(A_\infty)$ where $A_\infty = \bigcup_{n=1}^\infty A_n$. Thus, λ is countably additive on Σ_μ. Hence, $\lambda(A) = \int_A h_\lambda du$ for some $h_\lambda \in L^1(X, \Sigma, \mu)$. Let $A^+ = \{h_\lambda \geqq 0\}$ and $A^- = \{h_\lambda < 0\}$. If f is such that $fg \in L^1(X, \Sigma, \mu)$ with $S_f \subset A^+$ then $\lambda(fg) = \int_{A^+} fh_\lambda d\mu$. Thus, if $h \in L^1(X, \Sigma, \mu)$ with $S_h \subset A^+$ one has $\lambda(g) = \int_{A^+} hg^{-1}h_\lambda d\mu$. If it were true that $\operatorname{ess\,sup}_A (h_\lambda g^{-1}) = \infty$ there would exist an $h \in L^{1+}(X, \Sigma, \mu)$ such that $\{h \neq 0\} \subset A^+$ and $\infty = \int_{A^+} hg^{-1}h_\lambda d\mu = \lambda(h)$. Since $\lambda(h) \in (-\infty, \infty)$ $h_\lambda g^{-1}$ must be bounded on A^+. Similarly, $h_\lambda g^{-1}$ must be bounded on A^-. That is, $g_\lambda = h_\lambda g^{-1} \in L^1(X, \Sigma, \mu)$. This establishes the proposition. □

3. The non-σ-finite case. The results obtained here are basically the same as in § 2 with the exception of the fact that if μ isn't σ-finite there is no equivalent probability measure $g\mu$ with $g \in \Delta$. That is $\Delta \neq F_g$ for any $g \in \Delta$. In this case it turns out to be impossible to give a barycentric subdivision of Δ whose zero-skeleton contains a point in each norm closed face of Δ which has a barycenter. The subdivision simplexes we do obtain turn out to be affinely homeomorphic to K-simplexes $\mathscr{M}_1^+(C)$ where C is a closed subset of $(0, \infty)$ rather than of $(0, 1]$, again where these K-simplexes are endowed with their weak (= vague) topologies.

We let $\text{Chain}_f(\Sigma_\mu)$ denote such that $C \in \text{Chain}(\Sigma_\mu)\mu(A) < \infty$ for all $A \in C$. C-$\text{Chain}_f(\Sigma_\mu)$ and M-$\text{Chain}_f(\Sigma_\mu)$ are similarly defined. If $C \in \text{Chain}_f(\Sigma_\mu)$ we let $S(C)$ be the norm closed convex hull of $C(1) = \{\chi_A[\mu(A)]^{-1}: A \in C\}$. If $\sup(C) = A_0$ exists and $g = \chi_{A_0}[\mu(A_0)]^{-1}$ then $S(C) = S(C, g)$.

Proposition 3.1. (a) *If $C \in \text{Chain}_f(\Sigma_\mu)$ and if $\bar{C}$ is the smallest element of C-$\text{Chain}_f(\Sigma_\mu)$ containing C then $S(C) = S(\bar{C})$.*

(b) *If $C \in C$-$\text{Chain}_f(\Sigma_\mu)$ then an $h \in \Delta$ is in $S(C)$* iff *$\{h > t\} \in C$ for all $0 \leqq t < \|h\|_\infty$.*

(c) *If $C \in C$-$\text{Chain}_f(\Sigma_\mu)$ then $S(C)$ is the σ-convex hull of $\{S(C^A)$: $A \in C\}$ where $C^A = \{A' \in C: A' \subset A\}$.*

(d) *If $C \in C$-$\text{Chain}_f(\Sigma_\mu)$ then $\xi(S(C)) = C$.*

(e) *The maps $A \to \chi_A[\mu(A)]^{-1} \to \mu(A)$ are homeomorphisms from C to $C(1)$ to $(0, \infty)$ if $C \in C$-$\text{Chain}_f(\Sigma_\mu)$.*

Proof. The proofs are analogous to those of the corresponding facts in Lemmas 2.1, 2.2, 2.3 and Proposition 2.4. □

From Propositions 2.5 and 2.9, if $C \in C$-$\text{Chain}_f(\Sigma_\mu)$ and $A \in C$ then $S(C^A)$ is affinely homeomorphic to $\mathscr{M}_1^+(C^A)$ equipped with the weak or vague topology under a unique map, say $Q_A^\#$, which assigns to $\delta_{A'} \in \mathscr{M}_1^+(C^A)$ the element $\chi_{A'}\mu(A')^{-1}$ of $S(C^A)$. This remains true if $Q_A^\#$ is regarded as an affine bijection of $\mathscr{M}_b^+(C^A)$ onto cone $(0, S(C^A))$. If $A_1 \subset A_2$ are in C then $\mathscr{M}_b^+(C^{A_1})$ is injected into $\mathscr{M}_b^+(C^{A_2})$ in the natural fashion. The restriction of $Q_{A_2}^\#$ to $\mathscr{M}_b^+(C^{A_1})$ is just $Q_{A_1}^\#$. If $p \in \mathscr{M}_b^+(C)$ then $\{p|_A: A \in C\}$ converges to p in norm as A increases in C. We have $Q_A^\#(p|_A)$ converging in $S(C)$ to an element $h = Q^\#(p)$ with $Q_A^\#(p|_A) = [h - \text{ess inf}_A h] \vee 0$. The map $Q^\#$ is an affine bijection and agrees with $Q_A^\#$ on $\mathscr{M}_b^+(C^A)$ when C^A is regarded as a subset of $\mathscr{M}_b^+(C)$.

Proposition 3.2. (a) *If C is in C-$\text{Chain}_f(\Sigma_\mu)$ then $\mathscr{M}_b^+(C)$ with the weak topology is affinely homeomorphic with $S(C)$ under $Q^\#$.*

(b) *The norm closed faces of $S(C)$ are of the form $S(C')$ where*

$C' \subset C$ *is in* C-Chain$_f(\Sigma_\mu)$ *and* $S(C_1) \cap S(C_2) = S(C_1 \cap C_2)$.

Proof. The proofs are analogous to those of Propositions 2.4 and 2.5. □

PROPOSITION 3.3. $\mathscr{S}^f = \{S(C)\colon C \in M\text{-Chain}_f(\Sigma_\mu)\}$ *is a K-simplicial subdivision of* Δ *whose associated simplicial complex is* $\mathscr{C}^f = \{S(C)\colon C \in \text{Chain}_f(\Sigma_\mu)\}$.

Proof. The only thing to establish, given the result of Proposition 3.2 is that $\mathscr{S}^f$ covers Δ. If $h \in \Delta$, then $C_0 = \{\{h \geqq t\colon 0 < t < \|h\|_\infty\} \in \text{Chain}_f(\Sigma_\mu)$ and $h \in S(C_0)$. Consequently, $h \in S(C)$ for any $C \in M\text{-Chain}_f(\Sigma_\mu)$ with $C_0 \subset C$. □

The metric on $\mathscr{M}_b^+(C)$ induced by $Q^\sharp$ from the norm on $L^1(X, \Sigma, \mu)$ will be denoted by D for a $C \in C\text{-Chain}_f(\Sigma_\mu)$. Below we denote by H the continuous function $A \to \mu(A)$ on C and by $H(dA)$ the measure on C arising by Lebesgue-Stieltjes integration with respect to the continuous function H. With this terminology Proposition 3.4 is an immediate corollary of Proposition 2.7.

PROPOSITION 3.4. *If* $C \in C\text{-Chain}_f(\Sigma_\mu)$ *and* $\{p_1, p_2\} \subset \mathscr{M}_b^+(C)$ *then*

$$D(p_1, p_2) = \int_C \left| \int_{C_A} \frac{1}{H(B)} p_1(dB) - \int_{C_A} \frac{1}{H(B)} p_2(dB) \right| H(dA)$$
$$= \int_C | d_{H(A)^{-1}p_1}(A) - d_{H(A)^{-1}p_2}(A) | H(dA) \, .$$

If one maps C in C-Chain$_f(\Sigma_\mu)$ homeomorphically into $(0, \infty)$ via the map H assigning $\mu(A)$ to A a homeomorphism is established between $\mathscr{M}_b^+(C)$ and $\mathscr{M}_b^+(H(C))$ for vague or weak topologies. The metric $D^\sharp$ on $\mathscr{M}_b^+(H(C))$ is that induced by D. This corollary is analogous to Corollary 2.7.1.

COROLLARY 3.4.1. *If* $\{p_1, p_2\} \subset \mathscr{M}_b^+(H(C))$ *then* $D^\sharp(p_1, p_2) = \int_{H(C)} \left| \int_t^\infty (1/s) p_1(ds) - \int_t^\infty (1/s) p_2(ds) \right| dt$.

PROPOSITION 3.5. (a) *If* μ *is non-atomic the simplexes in* $\mathscr{S}^f$ *are mutually affinely isometric.*

(b) *If* μ *is not non-atomic there are two simplexes in* $\mathscr{S}^f$ *which aren't affinely homeomorphic.*

Proof. (a) is immediate from Corollary 3.4.1 where $H(C)$ is $(0, \infty)$.

(b) is established in the same manner as was (b) of Corollary 2.8.1. □

PROPOSITION 3.6. *The Hausdorff locally convex topologies on $L^1(X, \Sigma, \mu)$ inducing the norm topology on all elements of $\mathscr{S}^f$ are precisely those coarser than the norm topology.*

Proof. If τ is a Hausdorff locally convex topology on $L^1(X, \Sigma, \mu)$ coarser than the norm topology it may be shown, in the same manner as the proof of Proposition 2.9 that τ agree with the norm topology on each element of $\mathscr{S}^f$.

Conversely, suppose that τ is a Hausdorff locally convex topology on $L^1(X, \Sigma, \mu)$ inducing the norm topology on each element of $\mathscr{S}^f$. To show that τ is coarser than the norm topology it suffices to show that each τ-continuous linear functional λ is of the norm $\lambda(h) = \int hg_\lambda d\mu$ for $h \in L^1(X, \Sigma, \mu)$ for some $g_\lambda \in L^\infty(X, \Sigma, \mu)$. If $A \in \Sigma_\mu$ with $0 < \mu(A) < \infty$ regard $L^1(A, \Sigma, \mu)$ as a subspace of $L^1(X, \Sigma, \mu)$. The trace, $\mathscr{S}^f \cap L^1(A, \Sigma, \mu) = \{S \cap L^1(A, \Sigma, \mu)\}$ is the simplicial subdivision $\mathscr{S}_g$ of the positive face of the unit ball of $L^1(A, \Sigma, \mu)$ with $g = \chi_A[\mu(A)]^{-1}$. The norm and τ topologies agree on all elements of $\mathscr{S}_g$. Thus, there is a g_λ^A in $L^\infty(A, \Sigma, \mu)$ so that $\lambda(h) = \int_A hg_\lambda^A d\mu$ if $h \in L^1(X, \Sigma, \mu)$ with $\{h \neq 0\} \subset A$. It must be the case that $g_\lambda^A = g_\lambda^B$ on $A \cap B$ if $\{A, B\} \subset \Sigma_\mu$ with $0 < \mu(A)$, $\mu(B) < \infty$. Thus, there is a $g_\lambda \in L^\infty_{\text{loc}}(X, \Sigma, \mu) = L^\infty(X, \Sigma, \mu)$, $\lambda(h) = \int hg_\lambda d\mu$ if $h \in L^1(X, \Sigma, \mu)$ with $\mu\{h \neq 0\} < \infty$. ($L^\infty_{\text{loc}}(X, \Sigma, \mu)$ consists of functions whose restrictions to sets of finite measure are bounded.) If $\mu(\{h \neq 0\}) = \infty$ then $\lambda(h) = \lim_{\varepsilon\to 0} \lambda(h - h \wedge \varepsilon) = \lim_{\varepsilon\to 0} \lambda(h - h \wedge \varepsilon) = \lim_{\varepsilon\to 0} \int [h - (h \wedge \varepsilon)]g_\lambda d\mu = \int hg_\lambda d\mu$ since τ agrees with the norm topology on elements of $\mathscr{S}^f$. This establishes the proposition. □

PROPOSITION 3.7. (a) *Let (X, Σ, μ) be an infinite measure space. There is no $g \in \Delta$ such that $g^A = \chi_A \mu(A)^{-1}$ if $0 < \mu(A) < \infty$.*

(b) *Let (X, Σ, μ) be non-σ-finite there is no collection $\{g^A$: $\mu(A) > 0$, $A\sigma$-finite$\}$ in Δ such that $g^A = g^B \chi_A \mu(g^B, A)^{-1}$ if $A \subset B$ are σ-finite elements of Σ_μ.*

Proof. (a) is only non-trivial if (X, Σ, μ) is σ-finite. In the σ-finite case the condition on g is that it be constant on any set of finite measure so g is a constant λ on X. In this case we have $1 = \|g\|_1 = \lambda\mu(X) \in \{0, \infty\}$ which is impossible.

(b) Let $\{A_\alpha\}$ be a maximal disjoint collection of σ-finite elements of Σ_μ. Define the measure ν_α on A_α as $g^{A_\alpha}\mu$. Let ν be the positive measure on Σ_μ equal to ν_α on each A_α. The map $f \to \sum f/g^{A_\alpha}$ is an isometry from $L^1(\mu)$ to $L^1(\nu)$ which assigns to each g^{A_α} the function $\chi_{A_\alpha} = \chi_{A_\alpha}(\nu(A_\alpha))^{-1}$. Actually, for all A σ-finite for μ, g^A is assigned to $\chi_A[\nu(A)]^{-1}$. Choosing countably many distinct A_{α_n}, setting $A = \bigcup_{n=1}^\infty A_{\alpha_n}$ and g the image of g^A we are led to a contradiction of (a). $\square$

REMARKS. This proposition shows that it is impossible to have a barycentric subdivision of Δ when (X, Σ, μ) is non-σ-finite using barycenters of all norm closed faces of Δ which have barycenters if the barycenters are to be chosen in the coherent fashion we have used. However this section guarantees barycentric subdivision utilizing barycenters of some norm closed faces of Δ. Even in the σ-finite case the barycentric subdivision $\mathscr{S}^f$ is definable and will not utilize barycenters of all norm closed having barycenters.

4. Barycentric subdivisions of octahedra. By an *octahedron* we mean a unit ball of a Kakutani L-space with its norm topology or any affinely homeomorphic image of such a ball. We will deal with octrahedra represented as the ball $\diamondsuit$ of $L^1(X, \Sigma, \mu)$ where (X, Σ, μ) is a positive localizable measure space. Since $\diamondsuit$ is centrally symmetric its center 0 is natural barycenter of $\diamondsuit$ to use in a barycentric subdivision of $\diamondsuit$. The convex hull of 0 and the positive face Δ of $\diamondsuit$ is the positive unit ball $\diamondsuit^+$ of $L^1(X, \Sigma, \mu)$. With the norm topology $\diamondsuit^+$ is a K-simplex with 0 an extreme point. $\diamondsuit^+$ is affinely homeomorphic to the positive face of the unit ball of $L^1(X', \Sigma', \mu')$ where X' is obtained from X by adjoining a new point ∞, Σ' is the σ-algebra on X' generated by Σ and $\{\infty\}$ and μ' is the measure on X' with $\mu'\{\infty\} = 1$ and whose restriction to Σ is μ.

PROPOSITION 4.1. (a) $\mathscr{S}_{\diamondsuit^+} = \{\text{conv}\,(0, S) : S \in \mathscr{S}^{f}\}$ *is a K-simplicial subdivision of* $\diamondsuit^+$ *whose associated simplicial complex is* $\mathscr{S}^f \cup \{\text{conv}\,(0, S) : R \in \mathscr{S}^f\}$.

(b) *If* $C \in C\text{-Chain}_f(\Sigma_\mu)$ *then* $\text{conv}\,(0, S(C))$ *is affinely homeomorphic with the weak topology where* ∞ *is adjoined as an isolated point to* C.

Proof. It is easily verified that $\mathscr{S}_{\diamondsuit^+}$ is a covering of $\diamondsuit^+$ by K-simplexes. If $\{S_1, S_2\} \subset \mathscr{S}^f$ then $\text{conv}\,(0, S_1) \cap \text{conv}\,(0, S_2) = \text{conv}\,(0, S_1 \cap S_2)$ is a norm closed face both of $\text{conv}\,(0, S_1)$ and $\text{conv}\,(0, S_2)$. Furthermore any norm closed face F of $\text{conv}\,(0, S)$ with $S \in \mathscr{S}^f$ is either a norm closed face of S or is of the form $\text{conv}\,(0, \tilde{F})$ for some norm closed face $\tilde{F}$ of S. These remarks suffice to establish (a).

(b) is immediate since 0 is not in the closure of $C(1)$ for any

$C \in C\text{-Chain}_f\,(\Sigma_\mu)$. □

REMARK. If (X, Σ, μ) is σ-finite and $g \in \Delta$ with $F_g = \Delta$ then in Proposition 4.1, $\mathscr{S}^f$ may be replaced by $\mathscr{S}_g$ to obtain a K-simplicial subdivision of $\diamondsuit^+$.

An isometry T of $L^1(X, \Sigma, \mu)$ carries the barycentric subdivision $\mathscr{S}_{\diamondsuit^+}$ into a barycentric subdivision $T(\mathscr{S}_{\diamondsuit^+}) = \{T(S): S \in \mathscr{S}_{\diamondsuit^+}\}$ of the K-simplex $T(\diamondsuit^+)$. By suitable choice of isometries T a barycentric K-simplicial subdivision of $\diamondsuit$ will be constructed as a union of the subdivisions $T(\mathscr{S}_{\diamondsuit^+})$. One isometry of $L^1(X, \Sigma, \mu)$ is that induced by a μ-measure preserving automorphism of Σ. For such an isometry T one has $T(\Delta) = \Delta$, $T(\diamondsuit^+) = \diamondsuit^+$. In fact, $T(S) \in \mathscr{S}_{\diamondsuit^+}$ if $S \in \mathscr{S}_{\diamondsuit^+}$ for T must be an order isomorphism of Σ hence preserve chains, complete chains or maximal chains. Such isometries T can be ignored for the purpose of constructing a simplicial subdivision of $\diamondsuit$. Any isometry of $L^1(X, \Sigma, \mu)$ is the composition (on either side) of an isometry arising from a measure preserving Σ-automorphism and an isometry of the form R_E where $E \in \Sigma_\mu$ and $R_E(f)$ is defined to be $(\chi_E - \chi_{E^c})f$ for any $f \in L^1(X, \Sigma, \mu)$. This may be established in several different ways, one being an appeal to the Banach-Stone Theorem. We have E defined for the isometry T by the requirement that the image of $1 \in L^\infty(X, \Sigma, \mu)$ under the adjoint isometry T^* be $\chi_E - \chi_{E^c}$.

The image $T(\Delta)$ under an isometry T is a maximal proper face of $\diamondsuit$, a one co-dimensional face in fact. $T(\Delta)$ is equal to $\{f \in \diamondsuit: \|f\|_1 = 1,\ (\chi_E - \chi_{E^c})f \geqq 0\}$ where $E \in \Sigma_\mu$ is associated with T. The image of $\diamondsuit^+$ under T has a similar characterization. The 1-codimensional skelton of $\diamondsuit$ consisting of all 1-codimensional faces of $\diamondsuit$ is precisely the set of maximal proper faces of $\diamondsuit$ by Lau in [5]. Lau also shows that any maximal proper face of $\diamondsuit$ is $R_E(\Delta)$ for a unique E in Σ_μ.

PROPOSITION 4.2. (a) *$\{R_E(\diamondsuit^+): E \in \Sigma_\mu\}$ is a K-simplicial subdivision of $\diamondsuit$.*

(b) *If $\{E_1, E_2\} \subset \Sigma_\mu$ then $R_{E_1}(\diamondsuit^+) \cap R_{E_2}(\diamondsuit^+) = R_{E_1}(\diamondsuit^+ \cap R_F(\diamondsuit^+)) = R_{E_1}(\{f \in \diamondsuit^+: f\chi_{F^c} = 0\})$ where F is $(E_1 \cap E_2) \cup (E_1^c \cap E_2^c)$.*

Proof. The proof of (b) is straight forward. To establish (a) it is enough to show that if E_1 and E_2 are in Σ_μ then $R_{E_1}(\diamondsuit^+) \cap R_{E_2}(\diamondsuit^+)$ is a face of $R_{E_1}(\diamondsuit^+)$. This is an isomorphic image of $\diamondsuit^+ \cap R_F(\diamondsuit^+) = \{f \in \diamondsuit^+: f\chi_{F^c} = 0\}$ where $F = (E_1 \cap E_2) \cup (E_1^c \cap E_2^c)$. Since this is a face of $\diamondsuit^+$, $R_{E_1}(\diamondsuit^+) \cap R_{E_2}(\diamondsuit^+)$ is a face of $R_{E_1}(\diamondsuit^+)$. □

PROPOSITION 4.3. *$\mathscr{S}_\diamondsuit = \{R_E(S): S \in \mathscr{S}_{\diamondsuit^+},\ E \in \Sigma_\mu\}$ forms a K-simplicial subdivision of $\diamondsuit$.*

Proof. It is only necessary to show that if $\{E_1, E_2\} \subset \Sigma_\mu$ and $S_1 \cap S_2 \in \mathscr{S}_{\diamond +}$ then $R_{E_1}(S_1) \cap R_{E_2}(S_2)$ is a face of $R_{E_1}(S_1)$. By (b) of Proposition 4.2, it may be assumed that $E_1 = X$ so that $R_{E_1}(S_1) = S_1$. In this case $S_1 \cap R_{E_2}(S_2) = S_1 \cap \{f \in S_2: f\chi_{E_2^c} = 0\}$. Since $\{f \in S_2: f\chi_{E_2^c} = 0\}$ is a norm closed face of S_2, Proposition 3.3 guarantees that $S_1 \cap R_{E_2}(S_2)$ is a face of S_1. □

REMARK. In the σ-finite case one may obtain a barycentric simplicial subdivision of $\diamondsuit$ as in Proposition 4.3 starting with the subdivision $\mathscr{S}_g$ of Δ rather than $\mathscr{S}^f$ for $g \in \Delta$ with $F_g = \Delta$.

5. *K*-simplicial subdivisions of barycentric type. In this section it is shown that the K-simplicial subdivision $\mathscr{S}^f$ of a K-simplex Δ is the only type of barycentric subdivision possible satisfying certain coherence and regularity properties.

PROPOSITION 5.1. *Let $\mathscr{S}$ be a K-simplicial subdivision of Δ (the positive face of the unit ball of $L^1(X, \Sigma, \mu)$) and $\mathscr{C}$ its associated K-simplicial complex.* (i) *Assume that if $S \in \mathscr{S}$,*

(a) $S = \text{cl conv}\,(\xi(S))$

(b) *$\xi(S)$ is linearly ordered by absolute continuity ($g_1 \ll g_2$ iff $S_{g_1} \subset S_{g_2}$ iff $F_{g_1} \subset F_{g_2}$)*

(ii) *If g_1, g_2 are in the zero skeleton, ${}^0\mathscr{S}$, of $\mathscr{S}$ then $g_1^A = g_2^A$ if $A = S_{g_1} \cap S_{g_2}$.*

Then, $\{\varnothing\} \cup \{S_g: g \in {}^0\mathscr{S}\}$ is an ideal in Σ_μ. If $g \in {}^0\mathscr{S}$ then the trace $\mathscr{C} \cap F_g$ is $\mathscr{C}_g$.

Proof. We first note that (ii) implies that for an $A \in \Sigma_\mu$ there is at most one $g \in {}^0\mathscr{S}$ with $S_g = A$. The assumptions (i) and (ii) assure that $\xi(S)$ is a norm closed set in Δ which is locally compact and, in fact, for which every bounded order interval is compact. If S_0 is the closed convex hull of a compact order interval in $\xi(S)$ then S_0 is a compact face of S which is a Bauer simplex. The σ-convex hull of the union all such S_0 is a norm closed face of the K-simplex S, [3], [4], which contains $\xi(S)$ hence equal S. If F is any closed face of S then F is the σ-convex hull of $F \cap S_0$ for such S_0 hence F is the closed convex hull of $F \cap \xi(S) = \xi(F)$. That is, $\mathscr{C}$ is the ensemble $\{\text{cl conv}\,(K): K \text{ closed in } \xi(S), S \in \mathscr{S}\}$. If $g_0 \in \Delta$ is such that $g_0^{S_g} = g$ for all $g \in \mathscr{S}$ with $S_g \subset S_{g_0}$ the trace $\mathscr{C} \cap F_{g_0} = \{S \cap F_{g_0}, S \in \mathscr{C}\}$ is $\{S \in \mathscr{C}, S \subset F_{g_0}\}$ and is a subset of $\mathscr{C}_{g_0}$ (the simplicial complex in Proposition 2.6 with F_{g_0} replacing Δ). Such g_0 include all elements of ${}^0\mathscr{S}$.

If $g \in {}^0\mathscr{S}$ and $A \in \Sigma_\mu \backslash \{\varnothing\}$ is in S_g then $g^A \in S$ for some $S \in \mathscr{S}$ hence $g^A \in S \cap F_g \in \mathscr{C}_g$. This is only possible if $g^A \in \xi(S \cap F_g) \subset \xi(S)$.

Thus, if $g \in {}^0\mathscr{S}$ and $\varnothing \neq A \subset S_g$ then $A = S_{g'}$, for some $g' \in {}^0\mathscr{S}$. If we are given $C \in C\text{-Chain}(\Sigma_\mu)$ with supremum in S_g one may construct an $h \in \Delta$ such that $C = C(h, g)$. For h to be in S for some $S \in \mathscr{S}$ it is necessary and sufficient that $C(h, g)$ be a closed subset of $\xi(S)$. Since $\mathscr{S}$ covers F_g it is easy to deduce that $\mathscr{C} \cap F_{g_0} = \mathscr{C}_g$.

To establish that $\{\varnothing\} \cup \{S_g : g \in {}^0\mathscr{S}\}$ is an ideal in Σ_μ we need to show that if $\{g_1, g_2\} \subset {}^0\mathscr{S}$ there is a $g_3 \in {}^0\mathscr{S}$ with $S_{g_3} = S_{g_1} \cup S_{g_2}$. It may be assumed, without loss of generality, that $S_{g_1} \cap S_{g_2} = \varnothing$. Hence we may assume that, in $F = F_{g_0}$ for $g_0 = (g_1 + g_2)/2$, F_{g_1} and F_{g_2} are complementary split faces. If $g \in {}^0\mathscr{S} \cap F$ then g is uniquely expressed as a convex combination $\lambda_g \tilde{g}_1 + (1 - \lambda_g)\tilde{g}_2$ where $\tilde{g}_1 \in F_{g_1}$ and $\tilde{g}_2 \in F_{g_2}$ are given by $\tilde{g}_j = g^{S_g \cap S_{g_j}} = g_j^{S \cap_g S_{g_j}}$ for $j = 1, 2$. If $S \in \mathscr{S}$ then $F \cap S$ is $\operatorname{cl\,conv}(F \cap \xi(S))$ where $\xi(S) \cap F$ is a closed initial interval of the linearly ordered $\xi(S)$. As g increases in $\xi(S) \cap F$, S_g increases in $S_{g_1} \cup S_{g_2}$ to S_0. An increasing cofinal sequence $\{g_n\}$ may be found in $\xi(S) \cap F$ so that $\{\lambda_{g_n}\}$ is convergent to λ_0, say. Then $\{g_n\}$ converges in norm to $g_\infty = \lambda_0 g_1^{S_{g_1} \cap S_0} + (1 - \lambda_0) g_2^{S_{g_2} \cap S_0}$. Since $\xi(S) \cap F$ is closed $g_\infty \in \xi(S) \cap F$. Thus, g_∞ is the maximum of $\xi(S) \cap F$. There is an $S \in \mathscr{S}$ so that $g_0 \in S \cap F$. If there is an $A \subsetneqq S_{g_1} \cup S_{g_2}$ such that $S_g \subset A$ for all $g \in \xi(S) \cap F$ then $S_h \subset A$ for all $h \in S \cap F$. Considering $h = g_0$ this is seen to be impossible so such an A doesn't exist. Thus, $S_{g_1} \cup S_{g_2} = S_{g_\infty}$. Thus, we may set $g_3 = g_\infty$. This establishes the proposition. $\square$

Any K-simplicial subdivision $\mathscr{S}$ of a K-simplex Δ which satisfies (i) and (ii) of Propotision 5.1 will be said to be of *barycentric type.*

PROPOSITION 5.2. *Let $\mathscr{S}$ be a K-simplicial subdivision of Δ of barycentric type. There is a measure ν on (X, Σ) so that Δ is affinely isometric with the positive face of the unit ball of $L^1(X, \Sigma, \nu)$ under an isometry Φ and so that $\Phi({}^0\mathscr{S})$ consists of elements of the form $\chi_A[\nu(A)]^{-1}$ for $0 < \nu(A) < \infty$.*

Proof. Select a maximal collection $\{g_\alpha\} \subset {}^0\mathscr{S}$ with disjoint $\{S_{g_\alpha}\}$. Select g_{α_0}. If $\alpha \neq \alpha_0$ there is a unique $g^\alpha \in {}^0\mathscr{S}$ with $S_{g^\alpha} = S_{g_\alpha} \cup S_{g_{\alpha_0}}$ and $g^\alpha = \lambda_\alpha(g_{\alpha_0} + \gamma_\alpha g_\alpha)$ where $\lambda_\alpha(1 + \gamma_\alpha) = 1$ with $\lambda_\alpha > 0$ and $\gamma_\alpha > 0$. Set $h = g_{\alpha_0} + \sum_{\alpha \neq \alpha_0}(1/\gamma_\alpha)g_\alpha$ and $\nu = h\mu$ so that $\int_X f d\nu = \int_X f h d\mu$ for all f. The map $\Phi: f \to f/h$ is a bipositive isometry from $L^1(X, \Sigma, \mu)$ onto $L^1(X, \Sigma, \nu)$ with $\Phi(g_{\alpha_0}) = \chi_{S_{g_{\alpha_0}}}$ and $\Phi(g_\alpha) = (1/\gamma_\alpha)\chi_{S_{g_\alpha}}$. For all α, $0 < \nu(S_{g_{\alpha_0}}) < \infty$. If $\varnothing \neq A \subset S_{g_\alpha}$ then $\Phi(g_\alpha^A) = \chi_A[\gamma_\alpha \mu(g_\alpha, A)]^{-1}$.

Suppose that $\alpha_0, \alpha_1, \alpha_2$ are distinct and that A_j is a non-empty subset of $S_{g_{\alpha_j}}$ for $j = 1, 2$. The unique $h \in {}^0\mathscr{S}$ with $S_h = A_0 \cup A_1 \cup A_2$

is a convex combination $\eta_0 g_{\alpha_0}^{A_0} + \eta_1 g_{\alpha_1}^{A_1} + \eta_2 g_{\alpha_2}^{A_2}$. We have $h^{A_0 \cap A_j} = [g^{\alpha_j}]^{A_0 \cap A_j} = [g_{\alpha_0} + \gamma_{\alpha_j} g_{\alpha_j}]^{A_0 \cup A_j} = (\mu(g_{\alpha_0}, A_0) g_{\alpha_0}^{A_0} + \gamma_{\alpha_j}\mu(g_{\alpha_j}, A_j) g_{\alpha_j}^{A_j})[\mu(g_{\alpha_0}, A_0) + \gamma_{\alpha_j}\mu(g_{\alpha_j}, A_j)]^{-1}$ for $j = 1, 2$. Also, $h^{A_0 \cup A_j} = (\eta_0 g_{\alpha_0}^{A_0} + \eta_j g_{\alpha_j}^{A_j})(\eta_0 + \eta_j)^{-1}$ for $j = 1, 2$. Thus, the vector (η_0, η_j) is proportional to the vector $(\mu(g_{\alpha_0}, A_0), \gamma_{\alpha_j}\mu(g_{\alpha_j}, A_{\alpha_j}))$ for $j = 1, 2$. Thus, (η_0, η_1, η_2) is proportional to $(\mu(g_\alpha, A_0), \gamma_{\alpha_1}\mu(g_{\alpha_0}, A_1), \gamma_{\alpha_2}\mu(g_{\alpha_2}, A_2))$. We have $h^{A_1 \cup A_2} = \tilde{h}$ as the unique element of ${}^0\mathscr{S}$ with $S_{\tilde{h}} = A_1 \cup A_2$. $h^{A_1 \cup A_2}$ is a multiple of $\gamma_{\alpha_1}\mu(g_{\alpha_1}, A_1) g_{\alpha_1}^{A_1} + \gamma_{\alpha_2}\mu(g_{\alpha_2}, A_2) g_{\alpha_2}^{A_2}$. Setting $\gamma_{\alpha_0} = 1$ we may deduce that if h is any element of ${}^0\mathscr{S}$ it may be represented as a countable convex combination $\sum_\alpha \eta_\alpha g_{g_\alpha}^{S_h \cap S_{g_\alpha}}$ when η_α is $\gamma_\alpha \mu(g_\alpha, S_{g_\alpha} \cap S_h)[\sum_\beta \mu(g_\beta, S_{g_\beta} \cap S_h)]^{-1}$. We have that $\Phi(h) = \sum_\alpha \eta_\alpha \Phi(g_\alpha^{S_h \cap S_{g_\alpha}}) = [\sum_\alpha \gamma_\alpha \mu(g_\alpha, S_{g_\alpha} \cap S_h)[\gamma_\alpha \mu(g_\alpha, S_{g_\alpha} \cap S_h)]^{-1} \chi_{S_{g_\alpha} \cap S_h}][\sum_\beta \gamma_\beta \mu(g_\beta, S_{g_\beta} \cap S_h)]^{-1} = \chi_{S_h}[\sum_\beta \mu(g_\beta, S_{g_\beta} \cap S_h)]^{-1}$. Since $\Phi(h)$ has norm 1 in $L^1(X, \Sigma, \nu)$ we have $\nu(S_h) = \sum_\beta \gamma_\beta \mu(g_\beta, S_{g_\beta} \cap S_h) \in (0, \infty)$. That is, if $h \in {}^0\mathscr{S}$ then $\Phi(h) = \chi_{S_h}[\nu(S_h)]^{-1}$ and $0 < \nu(S_h) < \infty$.

The mapping Φ sends $\mathscr{S}$ onto a simplicial subdivision $\Phi(\mathscr{S})$ of the positive face $\Delta(\nu)$ of the unit ball of $L^1(X, \Sigma, \nu)$ whose zero skeleton $\Phi({}^0\mathscr{S})$ is a subset of the zero skeleton of the simplicial subdivision $\mathscr{S}^f(\nu)$ of $\Delta(\nu)$ given by Proposition 3.3. Conditions (i) and (ii) asssure that $\Phi(\mathscr{S}) \subset \mathscr{C}^f(\nu)$ (the simplical complex associated with $\mathscr{S}^f(\nu)$). If $\nu(A) < \infty$ then $\chi_A[\nu(A)]^{-1}$ belongs to some simplex S in $\Phi(\mathscr{S})$. Since S is a face of some simplex $\tilde{S}$ in $\mathscr{S}^f(\nu)$ and $\chi_A[\nu(A)]^{-1} \in \xi(\tilde{S})$, $\chi_A[\nu(A)]^{-1} \in \xi(S)$ hence is in $\Phi({}^0\mathscr{S})$. That is, $\Phi({}^0\mathscr{S}) = {}^0\mathscr{S}^f$. If $S \in \mathscr{S}$ then $\Phi(\xi(S))$ is in $\text{Chain}_f(\Sigma_\nu)$. If $A_1 \subset A_2$ and $\chi_{A_2}[\nu(A_2)]^{-1} \in \Phi(\xi(S))$ then $\chi_{A_1}[\nu(A_1)]^{-1} \in \Phi(\xi(S))$. If there is an A with $0 < \nu(A) < \infty$ with $\chi_{A_1}[\nu(A_1)]^{-1} \ll \chi_A[\nu(A)]^{-1}$ for all $\chi_{A_1}[\nu(A_1)]^{-1} \in \Phi(S))$ and $\chi_A[\nu(A)]^{-1}$ isn't in $\Phi(\xi(S))$ we find that there is a simplex $\tilde{S}$ in $\mathscr{S}$ with $\Phi^{-1}(\chi_A[\nu(A)]^{-1} \in \tilde{S}$ with S a proper face of $\tilde{S}$ which is impossible since $\mathscr{S}$ is a simplicial subdivision. Thus, $\Phi(\xi(S))$ must belong to $M\text{-Chain}_f(\Sigma_\nu)$. That is, $\Phi(S) \in \mathscr{S}^f(\nu)$ for any $S \in \mathscr{S}$. For any $S \in \mathscr{S}^f(\nu)$ there is an $h \in L^1(X, \Sigma, \nu)$ so that the chain $\{\{h > t\}: 0 < t < \|h\|\}$ has closure $\xi(S)$. The simplex S is the smallest in $\mathscr{C}^f(\nu)$ containing h. Since $\Phi(\mathscr{S})$ covers $\Delta(\nu)$, $h \in \Phi(\tilde{S})$ for some $\tilde{S} \in \mathscr{S}$. Thus, $S = \Phi(\tilde{S})$. It follows that $\mathscr{S}^f(\nu) = \Phi(\mathscr{S})$. This completes the proof of the proposition. □

COROLLARY 5.2.1. *Let $\{F_\alpha\}$ be a disjoint collection of norm closed faces of Δ. Let $\mathscr{S}_\alpha$ be a simplicial subdivision of F_α of barycentric type. There is a simplicial subdivision $\mathscr{S}$ of Δ such that each $\mathscr{S}_\alpha$ is in the K-simplicial complex $\mathscr{C}$ associated with $\mathscr{S}$.*

Proof. Let Δ be represented as the positive face of the unit ball of $L^1(X, \Sigma, \mu)$. For each α let $A_\alpha = \bigcup \{S_g: g \in F_\alpha\}$ so F_α is representable as the positive face of the unit ball of $L^1(A_\alpha, \Sigma, \mu)$. Let ν_α be a measure on A_α so that Δ_α is $\mathscr{S}^f(\nu_\alpha)$ as in Proposition

5.2. Let F_∞ be the face of Δ complementary to $\bigcup_\alpha F_\alpha$ and $A_\infty = X\backslash\bigcup_\alpha A_\alpha$ (in Σ_μ) so that $F_\infty = \bigcup\{S_g: g \in A_\infty\}$. Let ν_∞ be the restriction of μ to A_∞. Let $\nu = \sum_\alpha \nu_\alpha \chi_{A_\alpha} + \nu_\infty \chi_{A_\infty}$. $L^1(X, \Sigma, \nu)$ is isometric with $L^1(X, \Sigma, \mu)$ under a positive isometry Φ. The image of $\mathscr{S}^f(\nu)$ under Φ is a K-simplicial subdivision $\mathscr{S}$ of Δ whose associated K-simplicial complex contains $\bigcup_\alpha \mathscr{S}_\alpha$. □

We may improve Corollary 2.5.1 in Proposition 5.3 and provide a basis for giving barycentric subdivisions of arbitrary K-simplicial complexes in Proposition 5.4.

PROPOSITION 5.3. *Let $\{F_\alpha\}$ be a collection of norm closed faces of the K-simplex Δ. Let $\mathscr{S}_\alpha$ be a K-simplicial subdivision of F_α for each α which is barycentric type. If the trace of $\mathscr{S}_\alpha$ and $\mathscr{S}_\beta$ agree on $F_\alpha \cap F_\beta$ for all α, β, then there is a K-simplicial subdivision $\mathscr{S}$ of Δ with associated simplicial complex $\mathscr{C}$ such that $\mathscr{S}_\alpha \subset \mathscr{C}$ for all α.*

Proof. For any α there is a minimal collection $\{F_\beta: \beta \in \Lambda_\alpha\}$ so that $\alpha \in \Lambda_\alpha$ and so that if $\gamma \notin \Lambda_\alpha$ then F_γ is disjoint from the norm closed face Δ_α generated by $\{\bigcup F_\beta: \beta \in \Lambda_\alpha\}$. The relation $\alpha_1 \sim \alpha_2$ iff $\alpha_2 \in \Lambda_{\alpha_1}$ is an equivalence relation on the index set of $\{F_\alpha\}$. Δ_α depends only on the equivalence class of α. If $\alpha_1 \not\sim \alpha_2$ then $\Delta_{\alpha_1} \cap \Delta_{\alpha_2} = \emptyset$. If we show how to give a simplicial subdivision $\mathscr{S}$ of a Δ_α so that $\mathscr{S}_\beta \subset \mathscr{C}$ if $\beta \in \Lambda_\alpha$ we will be done upon appeal to Corollary 5.2.1. Thus, without loss of generality it may be assumed that $\Delta = \Delta_{\alpha_0}$ for some α_0. We may enumerate $\{F_\alpha\}$ by ordinals α so that $F_{\alpha_0} = F_0$ and so that if β is an ordinal then $F_\beta \cap F^{\beta-} \neq \emptyset$ where $F^{\beta-}$ is the norm closed face of Δ generated by $\{F_\alpha: \alpha < \beta\}$. Let F'_β be the norm closed face of F_β complementary in F_β to $F_\beta \cap F^{\beta-}$. Let $S_\beta = \bigcup\{S_g: g \in F_\beta\}$, $S^{\beta-} = \bigcup\{S_\alpha: \alpha < \beta\}$, $S^\beta = S^{\beta-} \cup S_\beta$ and $S'_\beta = S_\beta \backslash S^{\beta-}$. We wish to construct a measure ν on (X, Σ) and a Banach lattice isomorphism $\Phi: L^1(X, \Sigma, \mu) \to L^1(X, \Sigma, \nu)$ so that $\Phi(\mathscr{S}_\alpha) \subset \mathscr{C}^f(\nu)$. The measure ν and isomorphism Φ will be constructed by transfinite induction by constructing ν_α, the restriction of ν to S^α, for each ordinal α. Each ν_α will be of the form $h\chi_{S_\alpha}\mu$ for some positive measurable h and Φ will be the map $g \to g/h$. Suppose ν_β has been constructed for all $\beta < \alpha$ so that $\nu_\beta = h\chi_{S^\beta}\mu$ for some measurable h and so that $\Phi(\mathscr{S}_\beta) \subset \mathscr{C}^f(\nu_\beta)$ when $\mathscr{C}^f(\nu_\beta)$ is the simplicial complex on the positive face $F^\beta = F^{\beta-} \cup F_\beta$ of the unit ball of $L^1(S^\beta, \Sigma, \nu_\beta)$. Now ν_α, which is to be defined, must equal, on $S^{\alpha-}$, the measure $h\chi_{S^{\alpha-}}\mu$. ν_α must be defined on S'_α, if this is non-empty. By Proposition 5.2, there is a measure ω_α on S_α so that $\omega_\alpha = h_\alpha \chi_{S_\alpha}\mu$ for some measurable h_α, and so that if $\Phi_\alpha: g \to g/h_\alpha$ is the isomorphism of $L^1(S_\alpha, \Sigma, \mu)$ to $L^1(S_\alpha, \Sigma, \omega_\alpha)$ then $\Phi(\mathscr{S}^\alpha) = \mathscr{S}^f(\omega_\alpha)$. In the construction of ω_α one may actually set $\omega_\alpha = h\mu$ on $S^{\alpha'} \cap S_\alpha$ to begin with. The function h is defined only on $S^{\alpha-}$. Extend h to S^α so that $h = h_\alpha$ on S'_α. Let $\nu_\alpha = h\chi_{S^\alpha}\mu$. It must be verified that $\Phi(\mathscr{S}_\alpha) \subset \mathscr{C}^f(\nu_\alpha)$. It suffices to verify that $\Phi({}^0\mathscr{S}_\alpha) \subset {}^0\mathscr{S}^f(\nu_\alpha)$. If $g \in {}^0\mathscr{S}_\alpha$, write it uniquely as the convex combination $\lambda_\alpha g^{S^\alpha} + (1 - \lambda_\alpha) g^{S'_\alpha}$. We have $\Phi(g) = \Phi_\alpha(g) = \chi_{S_g}[\omega_\alpha(S_g)]^{-1} = \chi_{S_g}[\nu_\alpha(S_g)]^{-1} \in {}^0\mathscr{S}(\nu_\alpha)$. Thus, $\Phi(\mathscr{S}_\alpha) \subset \mathscr{C}^f(\nu_\alpha)$. At the termination of transfinite induction h is defined on all of X, ν is defined on Σ, and Φ is defined on $L^1(X, \Sigma, \mu)$ to $L^1(X, \Sigma, \nu)$ so that $\Phi(\mathscr{S}_\alpha) \subset \mathscr{C}^f(\nu)$ for all α. This establishes the proposition. □

PROPOSITION 5.4. *Let $\mathscr{C}$ be a K-simplicial complex. There is a K-simplicial complex $\mathscr{C}'$ such that*

(i) *If $S' \in \mathscr{C}'$ and $S \in \mathscr{C}$ then $S' \cap S$ is a face of S'.*

(ii) *If $S \in \mathscr{C}$ the trace, $S \cap \mathscr{C}'$, is the K-simplicial complex associated with a K-simplicial subdivision of S of barycentric type.*

Proof. Enumerate $\mathscr{C}$ as $\{S_\alpha\}$ where α ranges over an initial set of ordinals. Suppose that for all ordinals $\gamma < \alpha$, S_γ has been provided with a K-simplicial subdivision $\mathscr{S}_\gamma$ of barycentric type with associated K-simplicial complex $\mathscr{C}_\gamma$. Suppose further that the traces of $\mathscr{C}_{\gamma_1}$ and $\mathscr{C}_{\gamma_2}$ on $S_{\gamma_1} \cap S_{\gamma_2}$ are the same. On S_α we have a collection of norm closed faces $\{S_\alpha \cap S_\gamma : \gamma < \alpha\}$ each of which has the simplicial complex $\mathscr{C}_\gamma \cap S_\alpha$ corresponding to a K-simplicial subdivision S_γ of $S_\alpha \cap S_\gamma$ of barycentric type. Furthermore, if $\gamma_1 \neq \gamma_2$ then $\mathscr{C}_{\gamma_1} \cap (S_\alpha \cap S_{\gamma_1} \cap S_{\gamma_2}) = \mathscr{C}_{\gamma_2} \cap (S_\alpha \cap S_{\gamma_1} \cap S_{\gamma_2})$. By Proposition 5.3, there is a K-simplicial subdivision $\mathscr{S}_\alpha$ of S_α of barycetric type so that the associated K-simplicial complex $\mathscr{C}_\alpha$ has trace equal to $\mathscr{C}_\gamma \cap (S_\alpha \cap S_\gamma)$ for all $\gamma < \alpha$. By transfinite induction each simplex S_α in $\mathscr{C}$ is simplicially subdivided in a barycentric fashion by $\mathscr{S}_\alpha$. Let $\mathscr{C}' = \bigcup_\alpha \mathscr{C}_\alpha$. $\mathscr{C}'$ is easily verified to be a K-simplicial complex which satisfies (i) and (ii) of this proposition. □

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