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PROJECTIVE COLORINGS

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All colorings of the points of (Desarguesian) projective planes in three colors so that no straight line contains points of all three colors are characterized in terms of the valuations of the field of coordinates. Generalizations to higher dimensions and applications to the Fundamental Theorem of Projective Geometry and the division of polygons into disjoint triangles of equal areas are given. We restrict our discussion to the Pappian (commutative coordinate field) case. For the general division ring case see [3].

1. Introduction. M. Bognár (Budapest University) recently posed the following question: can the points of the real projective plane be colored in three colors (nontrivially) so that no line contains all three colors? Here nontrivial means that no color is confined to one line. For the affine plane this question had been answered in the affirmative in [6]. In this paper we give a complete characterization of these colorings and extend our results in various directions (other fields, higher dimensions, and curves of higher degree). The results involve a nontrivial blending of combinatorics and valuation theory. In § 5, we give a generalization of the fundamental theorem of projective geometry suggested by our results. In § 6 we extend results of [6] to the division of polygons into disjoint triangles of equal areas.

After this paper was written we learned of the work of D. S. Carter and A. Vogt [3]. We wish to acknowledge their priority, especially in the discovery of Theorems 1 and 5 of this paper. In view of our somewhat different approach we give the comparatively short proofs of these two theorems.

2. Colorings of projective planes. Let $P_2(F)$ denote the projective plane over the field F. (For convenience we assume F to be commutative, although it does not appear necessary for our results. It is not clear, however, whether our results extend to the non-Desarguesian case.) We wish to color the points of $P_2(F)$ in three colors, say red, white, and blue, so that no line contains points of all three colors. To avoid trivialities we assume, of course, that each color is used at least once. One type of coloring can be obtained by coloring a point p red, say, and coloring each line through p (with p deleted) either solid white or solid blue, randomly. Another type, essentially the dual of the above, is obtained by coloring the points on a line l either white or blue, randomly, and coloring all points not on l red. We shall call colorings of either of the above types *trivial*. Notice the following easily verified facts:

(a) the trivial colorings are precisely the ones in which some color is confined to a line.

(b) a coloring in which some line contains only one point of a given color must be trivial.

(c) a three-coloring of the points of $\boldsymbol{P}_2(F)$ leads to a dual three-coloring of the lines of $\boldsymbol{P}_2(F)$ by the pairs of colors RW, RB, or WB of the points of the line. This coloring has the property that the lines of a pencil involve exactly two of those colors. Note that for the corresponding coloring of the affine plane not every line may be two-colored. Thus the dual coloring of the lines of the affine plane exists for the trivial coloring of the first kind in which no two colors are restricted to a single line, but not for the trivial coloring of the second kind.

(d) the two types of trivial colorings have in common colorings in which one point p is red, say, all points on a line l through p are white (except p), and all points not on l are blue. We call such a coloring a *flag coloring*.

We show that there is an intimate, and rather surprising, connection between colorings of $\boldsymbol{P}_2(F)$ and valuations on F (see also [3, Theorem 3.7]).

THEOREM 1. *The projective plane $\boldsymbol{P}_2(F)$ has a nontrivial 3-coloring with no 3-colored lines if and only if the field F has a nontrivial non-Archimedean valuation.*

Before proceeding to the proof we mention several consequences.

COROLLARY. *$\boldsymbol{P}_2(F)$ has no nontrivial coloring if and only if F is an algebraic extension of a finite field.*

For a discussion of valuations see e.g., [5, XII, 4].

Hence $\boldsymbol{P}_2(\boldsymbol{R})$ has a nontrivial coloring whereas, for instance, finite projective planes do not. Our proof of Theorem 1 will in fact show that nontrivial colorings of $\boldsymbol{P}_2(F)$ are induced from trivial colorings of $\boldsymbol{P}_2(\bar{F})$ where $\bar{F}$ is the residue class field of F with respect to some valuation. This, together with the nonconstructive nature of valuations on $\boldsymbol{R}$ yields the following.

COROLLARY. *All measurable colorings of $\boldsymbol{P}_2(\boldsymbol{R})$ are trivial, and* *the existence of nontrivial colorings cannot be proved without the Axiom of Choice.*

No nontrivial valuation ring of $\boldsymbol{R}$ can be measurable, since $\boldsymbol{R}$ has no proper subgroup of positive measure. For the relation to the Axiom of Choice see e.g., [2].

We now proceed to the proof of Theorem 1.

Proof. First, suppose that there is a nontrivial non-Archimedean valuation $x \to |x|$ on F. Let $\mathscr{O} = \{x \mid |x| \leqq 1\}$ be the associated valuation ring, $\mathscr{M} = \{x \mid |x| < 1\}$ its maximal ideal, and $\bar{F} = \mathscr{O}/\mathscr{M}$ the residue class field. We color $\boldsymbol{P}_2(F)$ as follows:

Choose a homogeneous coordinate system $(x, y, z) \in F^3$ for $\boldsymbol{P}_2(F)$. Then color a point $p = (x, y, z)$ as follows:

(1) red if $|x| < |z|$, $|y| < |z|$

(2) white if $|x| \geqq |z|$, $|y| < |x|$

(3) blue if $|y| \geqq |z|$, $|y| \geqq |x|$.

It is easy to see that this gives a nontrivial 3-coloring of $\boldsymbol{P}_2(F)$. To see that no line is 3-colored suppose that $p_i = (x_i, y_i, z_i)$, $i = 1, 2, 3$, are points with p_1 red, p_2 white, and p_3 blue. Consider the determinant

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} .$$

If p_1, p_2, p_3 were collinear then this determinant would be zero. However, from the colors of the p_i, it is clear that the term $z_1x_2y_3$ strictly dominates the other terms in the determinant expansion (with respect to the valuation), so the determinant cannot vanish[1].

Notice that this coloring can be viewed as follows: choose a representative (x, y, z) of p with $x, y, z \in \mathscr{O}$ but not all of x, y, z in $\mathscr{M}$. Then $(\bar{x}, \bar{y}, \bar{z}) \in \bar{F}^3$ corresponds to a point of $\boldsymbol{P}_2(\bar{F})$. The flag coloring of $\boldsymbol{P}_2(\bar{F})$ which colors the origin red, the rest of the x-axis white, and the rest of the plane blue, pulls back to the above-described coloring of $\boldsymbol{P}_2(F)$. Similarly we can pull back any trivial coloring of $\boldsymbol{P}_2(\bar{F})$ to get a nontrivial coloring of $\boldsymbol{P}_2(F)$.

Now suppose that we are given a nontrivial coloring of $\boldsymbol{P}_2(F)$. We may choose coordinates so that the origin is red, the point at infinity on the x-axis is white, and the point at infinity on the y-axis is blue. Let $R_x = \{x \mid (x, 0, 1) \text{ is colored red}\}$ and $R_y = \{y \mid (0, y, 1)$

[1] We recently became aware that this coloring was previously introduced in **[6]** for the affine plane.

is colored red}. Then if $(x, y, 1)$ is red the horizontal line through $(x, y, 1)$ is red-white so $(0, y, 1)$ is also red (since the y-axis is red-blue). Hence $y \in R_y$. Similarly $x \in R_x$. Conversely, if $x \in R_x$ and $y \in R_y$ it is easy to see that $(x, y, 1)$ is red. Hence the red points are precisely $R_x \times R_y$.

Suppose now that the point $(b, 1, 0)$ on the line at infinity is blue. If this point had originally been chosen to lie at infinity on the y-axis, the effect would have been that of a coordinate change $x \mapsto x - by$, $y \mapsto y$, $z \mapsto z$. Hence, if $(x, y, 1)$ was originally red, $(x - by, y, 1)$ is red in the new coordinates, so $(x - by, 0, 1)$ was originally red. Hence $x - by \in R_x$. Conversely, if $x - by \in R_x$ and $y \in R_y$ then $x \in R_x$. Hence R_x is closed under translation by the subgroup $\langle bR_y \rangle$ generated by bR_y. Similarly, if $(1, w, 0)$ is white, R_y is closed under translation by $\langle wR_x \rangle$. In particular, we have $\langle bR_y \rangle \subseteq R_x$ and $\langle wR_x \rangle \subseteq R_y$.

Now let $B_z = \{b \mid (b, 1, 0) \text{ is blue}\}$ and $W_z = \{w \mid (1, w, 0) \text{ is white}\}$. Then from the above we have that the subring $\langle\langle B_z W_z \rangle\rangle$ generated by $B_z W_z = \{bw \mid b \in B_z,\ w \in W_z\}$ maps R_x into R_x and R_y into R_y under multiplication. Let $\mathscr{O}_z = \{t \mid t\langle\langle B_z W_z \rangle\rangle \subseteq \langle\langle B_z W_z \rangle\rangle\}$. Then $\mathscr{O}_z$ is a subring of F. Furthermore $\mathscr{O}_z \neq F$, since $\mathscr{O}_z = F$ would imply $\langle\langle B_z W_z \rangle\rangle = F$. (If B_z or W_z is $\{0\}$ then the coloring is trivial, by comment (b) early in this section.) But this, in conjunction with $\langle\langle B_z W_z \rangle\rangle R_x \subseteq R_x$, implies that either $R_x = \{0\}$ or $R_x = F$, again forcing the coloring to be trivial. We now show that $\mathscr{O}_z$ is a valuation ring, giving the desired valuation on F.

Suppose $t \in F$. If $tB_z \subseteq B_z$ then $t\langle\langle B_z W_z \rangle\rangle \subseteq \langle\langle B_z W_z \rangle\rangle$, so $t \in \mathscr{O}_z$. Otherwise, $tb \notin B_z$ for some $b \in B_z$, so $(tb, 1, 0)$ is white, so $t^{-1}b^{-1} \in W_z$. Hence $t^{-1} \in B_z W_z$, so $t^{-1} \in \mathscr{O}_z$. This concludes the proof of Theorem 1.

If we define $W_x = \{w \mid (1, 0, w) \text{ is white}\}$ and $B_y = \{b \mid (0, 1, b) \text{ is blue}\}$ then in an exactly analogous way we get the nontrivial rings $\langle\langle R_x W_x \rangle\rangle$ and $\langle\langle R_y B_y \rangle\rangle$ and the corresponding valuation rings $\mathscr{O}_x$ and $\mathscr{O}_y$ of F.

In order to prove that $\mathscr{O}_x = \mathscr{O}_y = \mathscr{O}_z$ we analyze the ring $\langle\langle B_z W_z \rangle\rangle$ in more detail. We know that for each $b \in B_z$, $w \in W_z$, $b \neq 0$ we have $|w| \leqq |b^{-1}|$. If equality can occur then $\langle\langle B_z W_z \rangle\rangle$, being an $\mathscr{O}_z$-module and containing a unit of $\mathscr{O}_z$, must be $\mathscr{O}_z$. If equality does not occur then $\langle\langle B_z W_z \rangle\rangle = \mathfrak{M}_z$, the maximal ideal of $\mathscr{O}_z$. In either case R_x and R_y are closed under multiplication by $\mathfrak{M}_z$ and hence $\mathscr{O}_x$ and $\mathscr{O}_y$ are $\mathfrak{M}_z$-modules, but this implies that $\mathscr{O}_x = \mathscr{O}_y = \mathscr{O}_z = \mathscr{O}$.

Assume first that one of the rings $\langle\langle B_z W_z \rangle\rangle$, $\langle\langle R_x W_x \rangle\rangle$, $\langle\langle R_y W_y \rangle\rangle$ is $\mathscr{O}$; say $\langle\langle B_z W_z \rangle\rangle = \mathscr{O}$. Then there exist $b_0 \in B_z$, $w_0 \in W_z$ so that $|b_0 w_0| = 1$ so, by a change of scale $(x, y, 0) = (w_0 x', w_0 y', 0)$ we get $B_{z'}, W_{z'} \subseteq \mathscr{O}$.

Now R_x, R_y are $\mathscr{O}$-modules and hence are given by a "Dedekind cut" in the valuation group Γ corresponding to $\mathcal{O}$. That is, there exist α, β in the completion of Γ so that

$$R_x = \{x \mid x \in F,\ |x| < \alpha\} \text{ or } R_x = \{x \mid x \in F,\ |x| \leqq \alpha\}$$

and

$$R_y = \{y \mid y \in F,\ |y| < \beta\} \text{ or } R_y = \{y \mid y \in F,\ |y| \leqq \beta\}\ .$$

If $\alpha, \beta \in \Gamma$ pick $u, v \in F$ so that $|u| = \alpha$, $|v| = \beta$ and make a change of scale $x = ux'$, $y = vy'$. Then $R_{x'} = \mathfrak{M}$ or $\mathcal{O}$ and $R_{y'} = \mathfrak{M}$ or $\mathcal{O}$. Correspondingly $W_{x'} = \mathcal{O}$ or $\mathfrak{M}$ and $B_{y'} = \mathcal{O}$ or $\mathfrak{M}$. Hence $\langle\!\langle R_{x'} W_{x'} \rangle\!\rangle = \langle\!\langle R_{y'} B_{y'} \rangle\!\rangle = \mathfrak{M}$. Thus $R_{x'}$, $R_{y'}$, $W_{x'}$, $B_{y'}$ are unions of residue classes (mod $\mathfrak{M}$).

Now $W_{z'}$ and $B_{z'}$ are $(1 + \mathfrak{M})$-closed. That is, $W_{z'} + \mathfrak{M} W_{z'} = W_{z'}$, $B_{z'} + \mathfrak{M} B_{z'} = B_{z'}$. Since, by our choice of scale we have $1 \in W_{z'}$ and $b_0 w_0 \in B_{z'}$ it follows that $B_{z'}$ and $W_{z'}$ are unions of residue classes (mod $\mathfrak{M}$).

Thus our coloring is a pull-back of a coloring of $\boldsymbol{P}_2(\bar{F})$. It is easy to see that the coloring of $\boldsymbol{P}_2(\bar{F})$ is trivial of the first type described when $R_{x'} = R_{y'} = \mathfrak{M}$ and of the second type when $R_{x'} = R_{y'} = \mathcal{O}$.

In case $\langle\!\langle B_z W_z \rangle\!\rangle = \langle\!\langle R_x W_x \rangle\!\rangle = \langle\!\langle R_y B_y \rangle\!\rangle = \mathfrak{M}$ all the color classes are given by Dedekind cuts and in case these cuts are in Γ we can change scales to make all the sets $R_x, R_y, W_x, W_z, B_y, B_z$ either $\mathfrak{M}$ or $\mathcal{O}$ so that the coloring is again a pull-back of a trivial coloring of $\boldsymbol{P}_2(\bar{F})$.

We still need to consider the case in which the Dedekind cuts are not in Γ. In the first case we extend F to a field E by adjoining transcendentals u or v or both and extending the valuation by setting $|u| = \alpha$, $|v| = \beta$. The valuation ring $\mathcal{O}$ and its maximal ideal $\mathfrak{M}$ are then extended to $\mathcal{O}_E$ and $\mathfrak{M}_E$ and the coloring of $\boldsymbol{P}_2(F)$ is extended to that of $\boldsymbol{P}_2(E)$ by making the sets R_x, R_y, W_x, B_y equal to $\mathfrak{M}_E$ or $\mathcal{O}_E$. The extension of B_z and W_z may not be unique, but we can set

$$W_{z,E} = (W_z \cup \{w \mid |w| = 1,\ w \notin F\})\ (1 + \mathfrak{M}_E)$$
$$B_{z,E} = B_z + B_z \mathfrak{M}_E\ .$$

This coloring is a pull-back of a trivial coloring of $\boldsymbol{P}_2(\bar{E}) = \boldsymbol{P}_2(\bar{F})$ and it induces a coloring on the subspace isomorphic to $\boldsymbol{P}_2(F)$.

The extension, if necessary, in case $\langle\!\langle B_z W_z \rangle\!\rangle = \langle\!\langle R_x W_x \rangle\!\rangle = \langle\!\langle R_y B_y \rangle\!\rangle = \mathfrak{M}$ is entirely analogous.

To sum up:

THEOREM 2. *Every 3-coloring of* $\boldsymbol{P}_2(F)$ *defines a unique non-*

Archimedean valuation v on F and the coloring of $\boldsymbol{P}_2(F)$ is a pull-back of a trivial coloring (corresponding to the trivial valuation) on $\boldsymbol{P}_2(\bar{F})$ where $\bar{F}$ is the residue class field determined by v. This pull-back may be obtained via an embedding of $\boldsymbol{P}_2(F)$ in $\boldsymbol{P}_2(E)$ where E is a (finitely generated) extension field of F.

REMARK. In every 4-coloring of $\boldsymbol{P}_2(F)$ there must occur 3-colored lines, regardless of the nature of F. To see this, let p, q, r, s be red, white, blue and yellow respectively. Since the lines pq and rs intersect at least one of them is three-colored.

On the other hand there are $2^{\aleph_0}$-colorings of $\boldsymbol{P}_2(\boldsymbol{R})$ in which no line has more than three colors. For example color the points of a strictly convex curve with distinct colors other than red and all other points red.

3. Higher dimensions. We now consider colorings of projective n-space $\boldsymbol{P}_n(F)$ in $n+1$ colors so that no hyperplane contains all $n+1$ colors. As before we consider colorings to be trivial if any color is confined to a hyperplane. One such trivial coloring can be obtained by generalizing our previous flag colorings in an obvious way: color a point red, the rest of a line through that point white, the rest of a plane through that line blue, *etc.* Then we have the following extension of our previous result.

THEOREM 3. *The projective space $\boldsymbol{P}_n(F)$ has a nontrivial $n+1$-coloring with no $n+1$-colored hyperplane if and only if the field F has a nontrivial valuation.*

Proof. Suppose $x \rightarrow |x|$ is a valuation with associated $\mathscr{O}$, $\mathscr{M}$, and $\bar{F}$ as before. We obtain a nontrivial coloring of $\boldsymbol{P}_n(F)$ by pulling back a generalized flag coloring of $\boldsymbol{P}_n(\bar{F})$. More specifically, we color $(x_1, x_2, \cdots, x_{n+1})$ red if $|x_i| < |x_{n+1}|$ for $i < n+1$, white if $|x_1| \geqq |x_{n+1}|$ and $|x_i| < |x_1|$ for $2 \leqq i \leqq n$, blue if $|x_2| \geqq |x_{n+1}|$, $|x_2| \geqq |x_1|$, and $|x_i| < |x_2|$ for $3 \leqq i \leqq n$, *etc.* Then a determinant argument as in § 2 shows that no hyperplane is $n+1$-colored.

Now suppose that a nontrivial coloring exists. Choose n points involving n colors and pass a hyperplane through them. This gives an n-coloring of $\boldsymbol{P}_{n-1}(F)$. Furthermore no hyperplane of this $\boldsymbol{P}_{n-1}(F)$ contains all n colors, indeed, no k-plane can be $(k+2)$-colored $(k = 1, \cdots, n-1)$, since then it could be enlarged to a hyperplane of $\boldsymbol{P}_n(F)$ containing $n+1$ colors. Hence, by induction on n, either this coloring of $\boldsymbol{P}_{n-1}(F)$ is trivial or F has a nontrivial valuation. It only remains to show that, if the n-coloring of $\boldsymbol{P}_{n-1}(F)$ is trivial, then the $n+1$-coloring of $\boldsymbol{P}_n(F)$ can't be nontrivial. (This is really an extension of comment (b) of § 2.) Suppose, for instance, that the color red is confined to an $n-2$ dimensional subspace of $\boldsymbol{P}_{n-1}(F)$. Pick a red point p on this $n-2$ dimensional subspace and a line l through p in $\boldsymbol{P}_{n-1}(F)$ but not in the $n-2$ dimensional subspace. Then this line l is solid white, say, except for p. Choose now $n-1$ points involving all colors except red and white. Then any hyperplane through these points and a red point must cut l in p but not contain l (or it would involve all $n+1$ colors), so these $n-1$ points and p determine a hyperplane containing all red points. This contradicts the nontriviality of the original coloring.

This completes the proof of Theorem 3. A more detailed analysis, which we omit, shows that any nontrivial coloring is "essentially" a pull back of a trivial coloring of $\boldsymbol{P}_n(\bar{F})$, as in the planar case.

4. Linear systems of curves. Suppose one wishes to color the projective plane $\boldsymbol{P}_2(F)$ in six colors so that no conic contains more than five colors. As before we insist that no color is confined to a conic to avoid trivialities. Our previous results enable us to do this, and in fact to do something much more general.

Consider $n+1$ algebraic curves in $\boldsymbol{P}_2(F)$ which are the loci of $n+1$ homogeneous algebraic equations

$$f_i(x, y, z) = 0;\ i = 0, 1, 2, \cdots, n$$

where $f_i \in F[x, y, z]$ are polynomials of the same degree and linearly independent over F. These curves define an n-dimensional linear system of curves, the loci of

$$c_0 f_0 + c_1 f_1 + \cdots + c_n f_n = 0\ ,$$

where $(c_0, c_1, \cdots, c_n) \in \boldsymbol{P}_n(F)$.

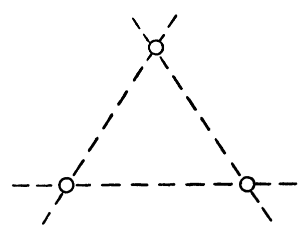
THEOREM 4. *Let C be an n-dimensional linear system of algebraic curves in $\boldsymbol{P}_2(F)$, such that no point in $\boldsymbol{P}_2(F)$ lies on all curves in C. If F admits a nontrivial valuation then there is a coloring of $\boldsymbol{P}_2(F)$ in $n+1$ colors so that no curve in C contains all $n+1$ colors and no color is confined to a curve in C.*

Proof. By duality points of $\boldsymbol{P}_2(F)$ can be regarded as (one dimensional subspaces of) linear functions on C. In other words there is a map ϕ of $\boldsymbol{P}_2(F)$ into the n-dimensional projective space $\boldsymbol{P}(C^*)$ determined by the dual space $C^* = \mathrm{Hom}_F(C, F)$ of C. But we can color $\boldsymbol{P}(C^*)$ in $n+1$ colors using a valuation as in Theorem 3. This will give an $(n+1)$-coloring of $\boldsymbol{P}(C^*)$ which we restrict to $\phi(\boldsymbol{P}_2(F))$, giving a coloring of $\boldsymbol{P}_2(F)$. Since curves in C correspond to hyperplanes in $\boldsymbol{P}(C^*)$, no curve in C will contain all $n+1$ colors.

All that remains is to show that the coloring can be chosen nontrivial on $\phi(P_2(F))$. In fact, we must first arrange that all $n+1$ colors appear on $\phi(P_2(F))$, and then nontriviality will follow. To do this choose points $p_0, p_1, p_2, \cdots, p_n$ in $P_2(F)$ whose images under ϕ do not lie in a hyperplane of $P(C^*)$; this can be done since no curve in C contains all of $P_2(F)$—remember F cannot be finite! Then coordinatize $P(C^*)$ with respect to the $n+1$ points $\phi(p_i)$ and use as the coloring of $P(C^*)$ a pull back of a generalized flag coloring with respect to these coordinates. This guarantees that $p_0, p_1, \cdots, p_n$ are all colored differently, so the proof is complete.

Theorem 4 can clearly be generalized to linear systems of hypersurfaces in higher dimensions. In view of Theorems 1 and 3, it is also tempting to conjecture that the converse of Theorem 4 holds, i.e., that the existence of a nontrivial coloring with respect to a linear system implies the existence of a nontrivial valuation on F. If a coloring of $P_2(F)$ is given we can assign to each curve in C one of the colors not on it, obtaining a "dual" coloring of $P(C)$. We cannot apply Theorem 3 to this coloring of $P(C)$ to get a valuation unless we know that all hyperplanes of $P(C)$ omit at least one color, and our hypotheses only guarantee this for hyperplanes of $P(C)$ corresponding to points of $P(C^*)$ that lie in $\phi(P_2(F))$. Without an 'ad hoc' hypothesis to cover the other points we do not see how to proceed.

5. The fundamental theorem of projective geometry. Colorings of $P_2(F)$ in three colors can be considered as mappings from $P_2(F)$ to the 3-point projective plane



which preserve collinearity. (One can consider this as the projective plane over the "one-element field", since $n^2+n+1=3$ when $n=1$). Furthermore the colorings we use in Theorem 1 are obtained via mappings from $P_2(F)$ to $P_2(\bar{F})$ which preserve collinearity. These observations suggest the validity of an extension of the Fundamental Theorem of Projective Geometry (FTPG)[2]. There are many versions

[2] In fact it is a question of this kind which appears to have motivated Bognár's original question (private written communication).

of the FTPG in the literature. The one we refer to is essentially that of Artin's [1]. We state it here for projective planes over fields only, and in non-coordinate-free notation:

(*FTPG*): *Let* ϕ *be a* $1-1$ *mapping from* $\boldsymbol{P}_2(F_1)$ *onto* $\boldsymbol{P}_2(F_2)$ *which preserves collinearity. Then there is an isomorphism* μ *from* F_1 *onto* F_2 *and a* 3×3 *invertible matrix* M *over* F_2 *so that* ϕ *is induced by the mapping from* F_1^3 *to* F_2^3 *given by* $(x, y, z) \mapsto (x^\mu, y^\mu, z^\mu)M$. *Furthermore* ϕ *determines* μ *uniquely, and determines* M *up to multiplication by a scalar matrix.*

The extension of this theorem we give allows ϕ to be neither $1-1$ nor onto. In a sense we classify all morphisms in the appropriate category whereas the FTPG classifies only isomorphisms.

THEOREM 5. *Let* ϕ *be a mapping from* $\boldsymbol{P}_2(F_1)$ *to* $\boldsymbol{P}_2(F_2)$ *which preserves collinearity, and whose image contains four points, no three collinear. Then there is a place* μ *from* F_1 *into* F_2 *and invertible* 3×3 *matrices* M_1, M_2 *over* F_1, F_2 *respectively, so that* ϕ *is induced by the map from* F_1^3 *to* F_2^3 *given by* $\vec{v} = (x, y, z) \mapsto (\vec{v}M_1)^\mu M_2$.

(Here $(x, y, z)^\mu$ means the vector obtained by scaling (x, y, z) to have integral coordinates, not all in the maximal ideal, and then applying μ to each coordinate.)

Proof. By a change of coordinates in $\boldsymbol{P}_2(F_2)$ we may assume that the four points, no three collinear, in the image of ϕ are $(0, 0, 1)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(1, 1, 1)$. (In affine coordinates the origin $(0, 0)$, the points at ∞ on the x and y axes, and $(1, 1)$). This change of coordinates gives the matrix M_2. Notice that one of the standard methods for introducing coordinates in a projective plane [4] applied to $\phi(\boldsymbol{P}_2(F_1))$ now shows that $\{t \mid (t, t, 1) \in \phi(\boldsymbol{P}_2(F_1))\}$ is a subfield F_2' of F_2 and $\phi(\boldsymbol{P}_2(F_1))$ consists precisely of points whose coordinates lie in F_2', i.e., a copy of $\boldsymbol{P}_2(F_2')$.

Now, by a change of coordinates in $\boldsymbol{P}_2(F_1)$, we may assume that ϕ maps $(0, 0, 1)$ to $(0, 0, 1)$, $(1, 0, 0)$ to $(1, 0, 0)$, $(0, 1, 0)$ to $(0, 1, 0)$, and $(1, 1, 1)$ to $(1, 1, 1)$. This change of coordinates gives the matrix M_1. It only remains to be shown that, with respect to the new coordinates, ϕ is induced by $(x, y, z) \mapsto (x, y, z)^\mu$ for some place μ.

Let $\mathcal{O} = \{t \mid \phi(t, t, 1) \neq (1, 1, 0)\}$, i.e., the coordinates of points on the line $y = x$ in $\boldsymbol{P}_2(F_1)$ that are not mapped by ϕ to a point at infinity. Define $\mu: F_1 \rightarrow F_2$ by $\mu(t) = \infty$ if $t \notin \mathcal{O}$, and $\mu(t) = t'$ if $t \in \mathcal{O}$ and $\phi(t, t, 1) = (t', t', 1)$. We claim that μ is a place and that μ induces ϕ.

The argument here is essentially the one used in introducing coordinates in a plane, namely that addition and multiplication are determined by geometric configurations. First, observe that by considering horizontal and vertical lines and their intersections we obtain that the image of a "finite" point (x_0, y_0) under ϕ is finite if and only if x_0 and y_0 lie in $\mathcal{O}$. Furthermore the point $(1, 1, 0)$ is mapped to $(1, 1, 0)$ by ϕ. If t_1, t_2 lie in $\mathcal{O}$ then the line $y = x + t_1$ determined by $(0, t_1, 1)$ and $(1, 1, 0)$ is mapped into a line not at ∞ by ϕ, so the intersection of this line with $y = t_2$ is mapped to a finite point. In other mords $(t_2 - t_1, t_2, 1)$ is mapped to a finite point, so $\mathcal{O}$ is closed under subtraction. Furthermore the configuration used in this argument is preserved under ϕ, so we must have $(t_2 - t_1)^\mu = t_2^\mu - t_1^\mu$ also.

The line $y = t_1 x$ determined by $(0, 0, 1)$ and $(1, t_1, 1)$ is mapped under ϕ into the line determined by $(0, 0, 1)$ and $(1, t_1^\mu, 1)$. Hence, the intersection of this line with $x = t_2$ is mapped to a finite point, i.e., $(t_2, t_1 t_2, 1)$ is mapped to a finite point. Hence $\mathcal{O}$ is closed under multiplication, and as before we also obtain $(t_1 t_2)^\mu = t_1^\mu t_2^\mu$.

We now have $\mu: \mathcal{O} \to F_2$ being a ring homomorphism with kernel $\mathcal{M} = \{t \mid \phi(t, t, 1) = (0, 0, 1)\}$ and image the subfield F_2' of F_2 alluded to earlier. Suppose that $t \in F_1$, $t \neq 0$. Then the points $(0, 0, 1)$, $(1, t, 1)$, and $(t^{-1}, 1, 1)$ are collinear, lying on $y = tx$. Then their images must be collinear, so t and t^{-1} cannot both lie outside $\mathcal{O}$ (or their images would be $(0, 0, 1)$, $(0, 1, 0)$, $(1, 0, 0)$). Hence $\mathcal{O}$ is a valuation ring, and μ is the associated place. Furthermore it is now clear that μ determines the mapping ϕ, so the proof is complete.

The extension of Theorem 5 to higher dimensions is completely straightforward. We omit the details.

A coordinate-free restatement of Theorem 5 would look something like this: Let V_1 and V_2 be 3-dimensional vector spaces over F_1 and F_2, and let $\phi: \boldsymbol{P}(V_1) \to \boldsymbol{P}(V_2)$ preserve collinearity and contain 4 points, no 3 collinear, in its image. Then there is a valuation of F_1, a free rank 3 $\mathcal{O}$-submodule W of V_1, an embedding $\mu: \mathcal{O}/\mathcal{M} \to F_2$, and an F_2-isomorphism from $(W/\mathcal{M}W) \otimes_{\mathcal{O}/\mathcal{M}} F_2$ onto V_2 which induces ϕ. It is unfortunate that W need be specified, but this appears unavoidable.

Finally, a word about the uniqueness of the ingredients in Theorem 5. It is not difficult to show that ϕ uniquely determines the place μ. However, M_1 and M_2 are not uniquely determined up to multiplication by scalar matrices. We must also allow for replacing M_1 and M_2 by $(M_1 M^{-1})$ and $(M^\mu M_2)$ where M is an invertible (with respect to $\mathcal{O}$) 3×3 matrix with entries in $\mathcal{O}$.

6. Divisibility of quadrilaterals into disjoint triangles of equal areas. Monsky [6] applied the coloring described in § 2 to the affine plane $A_2(\boldsymbol{R})$. That is, the point (x, y) is

(1) red if $|x| < 1$, $|y| < 1$
(2) white if $|x| \geqq 1$, $|y| < |x|$
(3) blue if $|y| \geqq 1$, $|y| \geqq |x|$.

The argument for the noncollinearity of three points of different color in the proof of Theorem 1 then yields that the area, A, of a triangle with vertices of different colors (3-colored triangle for short) satisfies

$$|A| = \left| \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right| \geqq \left| \frac{1}{2} \right| .$$

Using the fact that the vertices (0, 0), (1, 0), (1, 1), (0, 1) of the unit square are colored $RWBB$ so that the square has, say, a single RW edge, a simple combinatorial argument shows that every triangulation of the unit square must contain a 3-colored triangle. A 2-adic valuation of $\boldsymbol{R}$ shows that if the area of such a triangle is the reciprocal of an integer, that integer must be even. In other words the unit square cannot be divided into an odd number of disjoint triangles (that is, triangles with disjoint interiors) of equal areas.

We now can ask the question: which other quadrilaterals have the property that they cannot be divided into an odd number of disjoint triangles of equal areas? Since this property is an affine invariant we may assume that three of the vertices are (0, 0), (1, 0), (0, 1) and let the fourth vertex be (x, y), $x \geqq y$.

If there exists a 2-adic valuation of $\boldsymbol{R}$ for which $\max\{|x|, |y|\} \geqq 1$ then the vertex (x, y) is not colored red. Thus any triangulation must involve a 3-colored triangle and if the area of the quadrilateral satisfies

$$|(x + y)/2| < 2 \leqq |\text{area of 3-colored triangle}|$$

then the number of triangles must be even. All of these conditions are satisfied if x, y are odd rational numbers. We have thus proved the following:

THEOREM 6. *The set of quadrilaterals which cannot be divided into an odd number of disjoint triangles of equal areas is everywhere dense.*

Using any p-adic valuation, $p>2$, and (x, y) with $\max\{|x|, |y|\} \geqq$

1, $|(x+y)/2| < 1 \leqq |\text{area of 3-colored triangle}|$, which is satisfied by p-adically integral rationals x, y with $x + y \equiv 0 (\bmod p)$, we get a more general result.

THEOREM 7. *For any prime p there exists an everywhere dense set of quadrilaterals which cannot be divided into k disjoint triangles of equal areas unless k is divisible by p.*

We now observe that the set of quadrilaterals which can be divided into k disjoint triangles of equal areas is clearly a closed set. To see this let the fourth vertex (x_n, y_n) approach (x_0, y_0). Then for a subsequence the vertices of the dividing triangles $(x_{n,i}, y_{n,i})$ converge to vertices $(x_{0,i}, y_{0,i})$ and for a subsequence of these the corresponding triangulations converge. Since Theorem 7 shows that this closed set is nowhere dense, we get a surprising corollary.

THEOREM 8. *The set of quadrilaterals which can be divided into a number of disjoint triangles of equal areas is of the first category.*

The restriction to quadrilaterals was only for the convenience of exposition and Theorem 8 holds for n-gons with $n \geqq 4$. It seems that the vertices (x, y) for which the quadrilateral can be divided into k triangles of equal areas lie on the union of a finite number of algebraic curves and thus form a set of measure 0. Thus the set in Theorem 8 would also have measure 0.

There are no analogous results for the division of polygons into disjoint quadrilaterals of equal areas. In fact it is quite easy to see that any polygon can be divided into a finite number of quadrilaterals of equal areas.

Generalizations to the division of polyhedra into disjoint simplices of equal volumes are possible, but in addition to the colorings discussed in § 3 we need generalizations of Monsky's combinatorial lemma on triangulations. We therefore do not go into the details.

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