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ZERO DISTRIBUTION OF FUNCTIONS WITH SLOW OR MODERATE GROWTH IN THE UNIT DISC

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We consider functions which are analytic or meromorphic in the unit disc D. Our concern is to obtain information about values a function may assume when we know the function satisfies certain growth conditions. For analytic functions whose growth measured by two related growth indicators is distinct, we explore conditions under which this distinctness leads to the presence of zeros for the functions.

Let f be a meromorphic function in D. We shall assume knowledge of the usual functions of Nevanlinna theory as in [1]. In addition we define $m_2(r, f)$ for $0 < r < 1$ by

$$m_2(r, f) = \left\{\frac{1}{2\pi}\int_0^{2\pi}(\log|f(re^{i\theta})|)^2 d\theta\right\}^{1/2} .$$

A number of properties of $m_2(r, f)$ and some value distribution theorems involving $m_2(r, f)$ appear in [5]. We further define α and $\tilde{\alpha}$ by

$$\alpha = \limsup_{r\to 1} \frac{\log m_2(r, f)}{-\log(1-r)} ,$$

and

$$\tilde{\alpha} = \limsup_{r\to 1} \frac{\log T(r, f)}{-\log(1-r)} ,$$

where $T(r, f)$ is the value of the Nevanlinna characteristic function at r. (Of course α and $\tilde{\alpha}$ depend on f, but as we use them no confusion should result from suppressing that dependence, since we are ordinarily dealing with one function at a time.) In this paper we assume $\alpha < +\infty$. We prove the following theorem.

THEOREM 1. *Let f be an analytic function in D. Then $\tilde{\alpha} \leqq \alpha \leqq \tilde{\alpha} + 1/2$.*

If we consider f defined in D by $\exp((1+z)/(1-z))$, it can be shown that $\tilde{\alpha} = 0$ and $\alpha = 1/2$. It is also possible to construct Blaschke products in D for which $0 < \alpha < 1/2$ [cf. 2].

For an arbitrary meromorphic function f in D it will be convenient to define $I_1(r)$ and $I_2(r)$ for $0 < r < 1$ by

$$I_1(r) = \int_0^{2\pi} \left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right| d\theta ,$$

and

$$I_2(r) = \int_0^{2\pi} \left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right|^2 d\theta .$$

We shall show:

THEOREM 2. *Let f be a meromorphic function in D. Assume f has at most a finite number of zeros and poles in D. Also assume $\alpha > 0$.*

(i) *If $I_1(r) = 0((1-r)^{-\beta})$, $(r \to 1)$, then $\beta \geqq \alpha$.*

(ii) *If $I_2(r) = 0((1-r)^{-\beta})$, $(r \to 1)$, then $\beta \geqq 2\alpha$.*

We now turn to results about value distribution. Our main concern is to obtain conclusions about the values assumed by a function for which $\tilde{\alpha} < \alpha$. Our theorems allow $\tilde{\alpha} = 0$, so they apply to functions which are bounded in D for which $\alpha > 0$ or to functions which have bounded characteristic in D for which $\alpha > 0$. We have:

THEOREM 3. *Let f be an analytic function in D for which $0 \leqq \tilde{\alpha} < \alpha$. Suppose f has at most a finite number of zeros. Assume further that*

$$I_1(r) = 0((1-r)^{-\beta}) , \quad (r \longrightarrow 1) , \tag{1.1}$$

where $\beta + \tilde{\alpha} < 2\alpha$. Then for each θ, $0 \leqq \theta < 2\pi$,

$$|\log|f(re^{i\theta})|| \neq 0((1-r)^{-\gamma}) , \quad (r \longrightarrow 1) ,$$

where $\gamma < 2\alpha - \tilde{\alpha}$.

Corollaries of special interest are:

COROLLARY 1. *Let f be an analytic function in D for which $\tilde{\alpha} < \alpha$. Assume further that (1.1) holds where $\beta + \tilde{\alpha} < 2\alpha$. If f has a finite nonzero radial asymptotic path in D, then f has infinitely many zeros.*

COROLLARY 2. *Let f be an analytic function in D for which $\tilde{\alpha} < \alpha$. Assume f has at most a finite number of zeros in D and $I_1(r) = 0((1-r)^{-\beta})$, $(r \to 1)$. If f has a finite nonzero radial asymptotic path in D, then $\beta \geqq 2\alpha - \tilde{\alpha}$.*

Corollary 2 should be compared with (i) of Therem 2.

COROLLARY 3. *Let f be an analytic function in D which has bounded characteristic. Suppose $\alpha > 0$. Assume f has at most a finite number of zeros. If $I_1(r) = 0((1-r)^{-\beta})$, $(r \to 1)$, then $\beta \geqq 2\alpha$.*

A second major result concerning value distribution is:

THEOREM 4. *Let f be an analytic function in D for which $0 \leqq \tilde{\alpha} < \alpha$. Assume further that*

$$I_2(r) = 0((1-r)^{-\beta}) , \quad (r \longrightarrow 1) , \tag{1.2}$$

where $\beta < 2 + 2\alpha$. Then f has an infinite number of zeros in D.

If we expect to conclude the presence of zeros for analytic functions in D for which $\tilde{\alpha} < \alpha$, it is clear that a condition such as (1.1) or (1.2) is necessary when one considers the function f defined in D by $f(z) = \exp((1+z)/(1-z))$. For this f it can be shown that $\tilde{\alpha} = 0$; $\alpha = 1/2$; $I_1(r) \sim C_1(1-r)^{-1}$; and $I_2(r) \sim C_2(1-r)^{-3}$ where C_1 and C_2 are constants. The corollaries below are immediate consequences of Theorem 4. It is interesting to compare them with (ii) of Theorem 2.

COROLLARY 4. *Let f be an analytic function in D. Assume $\tilde{\alpha} < \alpha$. Assume further that f has at most a finite number of zeros in D. If $I_2(r) = 0((1-r)^{-\beta})$, $(r \to 1)$, then $\beta \geqq 2\alpha + 2$.*

COROLLARY 5. *Let f be an analytic function in D. Assume $\alpha > 0$, and suppose f has no zeros in D. If $I_2(r) = 0((1-r)^{-\beta})$, $(r \to 1)$, then $\beta \geqq 2\alpha + 2$.*

One might suspect that for an analytic function in D for which the orders satisfy $\tilde{\alpha} < \alpha$ that it might be possible to conclude that the zeros of such a function could not be uniformly distributed about the circumference of the unit circle. However the function g defined in D by

$$g(z) = \left(\exp\left(\frac{1+z}{1-z}\right)\right)B(z)$$

where $B(z)$ is the Blaschke product constructed in [3, p. 599] by G. MacLane and L. Rubel has $\tilde{\alpha} = 0$ and $\alpha = 1/2$ and zeros rather uniformly distributed, so any theorem in the direction suggested would have to be rather refined.

The remaining sections contain the proofs of the theorems in order.

2. Proof of Theorem 1.

Without loss of generality we assume

$f(o) = 1$. In [4] it was shown that

$$m_2(r, f) \leqq \{1 + (8\sqrt{\log 2})/\sqrt{\log (R/r)}\}T(R, f) , \quad (0 < r < R) .$$

Thus, if we let $C = 8\sqrt{\log 2}$, $R = (1/2)(1 + r)$, and observe that

$$\log\left(1 + \frac{1 - r}{2r}\right) \geqq \frac{1 - r}{4r} , \quad (r_0 < r < 1)$$

we see

$$m_2(r, f) \leqq \left\{1 + 2C\left(\frac{1}{1 - r}\right)^{1/2}\right\} T\left(\frac{1}{2}(1 + r), f\right) , \quad (r_0 < r < 1) .$$

Hence $\alpha \leqq \tilde{\alpha} + 1/2$.

To see that $\tilde{\alpha} \leqq \alpha$, we use Schwarz's inequality to get

$$\frac{1}{2\pi} \int_0^{2\pi} |\log | f(re^{i\theta}) || d\theta \leqq m_2(r, f) .$$

But the First Fundamental Theorem of Nevanlinna theory then shows

$$2T(r, f) - N(r, 1/f) = \frac{1}{2\pi} \int_0^{2\pi} |\log | f(re^{i\theta}) || d\theta .$$

Therefore,

$$T(r, f) \leqq m_2(r, f) ,$$

and the desired inequality follows.

3. Proof of Theorem 2. We assume $f(o) = 1$. We first observe [cf. 5] that for $0 < r < 1$

$$m_2(r, f) = \left\{\sum_{k=-\infty}^{\infty} | c_k(r) |^2\right\}^{1/2}$$

where

$$c_k(r) = \frac{1}{2\pi} \int_0^{2} (\log | f(re^{i\theta}) |) e^{-ik\theta} d\theta .$$

We note that $c_k(r) = \overline{c_{-k}(r)}$ and $c_0(r) = N(r, 1/f) - N(r, f)$. Thus, for $0 < r < 1$,

$$m_2(r, f) = \left\{(N(r, 1/f) - N(r, f))^2 + 2 \sum_{k=1}^{\infty} | c_k(r) |^2\right\}^{1/2} . \tag{3.1}$$

Now for integral k define $H_k(r)$ by

$$H_k(r) = \frac{1}{2\pi} \int_0^{2\pi} (\log f(re^{i\theta})) e^{-ik\theta} d\theta , \quad (0 < r < 1) .$$

Then for $k \geqq 1$,

$$2c_k(r) = H_k(r) + \overline{H_{-k}(r)}\,, \quad (0 < r < 1)\,. \tag{3.2}$$

Using integration by parts, we can write for $k \geqq 1$ and $0 < r < 1$ that

$$H_k(r) = -\frac{1}{k}(n(r, 1/f) - n(r, f)) + \frac{r^k}{k}\frac{1}{2\pi i}\int_{|z|=r}\frac{f'(z)}{f(z)}\frac{dz}{z^k}\,, \tag{3.3}$$

and

$$H_{-k}(r) = \frac{1}{k}(n(r, 1/f) - n(r, f)) - \frac{1}{kr^k}\cdot\frac{1}{2\pi i}\int_{|z|=r}\frac{f'(z)}{f(z)}z^k dz\,. \tag{3.4}$$

From (3.2), (3, 3), and (3.4) we see for $k \geqq 1$ and $0 < r < 1$ that

$$2c_k(r) = \frac{1}{k}r^k\frac{1}{2\pi i}\int_{|z|=r}\frac{f'(z)}{f(z)}\frac{dz}{z^k} - \frac{1}{k}\frac{1}{r^k}\overline{\frac{1}{2\pi i}\int_{|z|=r}\frac{f'(z)}{f(z)}z^k dz}\,.$$

And so taking absolute values and making standard estimates, we have for $0 < r < 1$

$$2|c_k(r)| \leqq \frac{1}{k}\frac{2}{2\pi}\int_0^{2\pi}\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right|d\theta\,, \quad (k \geqq 1)\,. \tag{3.5}$$

Therefore, if $I_1(r) = 0((1 - r)^{-\beta})$, $(r \to 1)$, then there is a constant A_1 such that (3.5) implies

$$|c_k(r)| \leqq \frac{1}{k}A_1(1 - r)^{-\beta}\,, \quad (k \geqq 1)\,, \quad (0 < r < 1)\,. \tag{3.6}$$

Combining (3.1) and (3.6), we find

$$m_2(r, f) \leqq \left\{(N(r, 1/f) - N(r, f))^2 + \sum_{k=1}^{\infty}\left\{\frac{1}{k^2}(A_1)^2(1 - r)^{-2\beta}\right\}\right\}^{1/2}\,, \quad (0 < r < 1)\,. \tag{3.7}$$

Hence there is a constant A_2 such that (3.7) becomes

$$m_2(r, f)^2 \leqq (N(r, 1/f) - N(r, f))^2 + A_2(1 - r)^{-2\beta}\,, \quad (0 < r < 1)\,. \tag{3.8}$$

On the other hand we know for $0 < r < 1$ that

$$n(r, 1/f) - n(r, f) = \frac{1}{2\pi i}\int_{|z|=r}\frac{f'(z)}{f(z)}dz\,,$$

and

$$|n(r, 1/f) - n(r, f)| \leqq I_1(r)\,, \quad (0 < r < 1)\,. \tag{3.9}$$

We also have

$$
(3.10)\qquad
\begin{aligned}
|N(r, 1/f) - N(r, f)| &= \left|\int_0^r \frac{n(t, 1/f) - n(t, f)}{t}dt\right| ,\\
&\leqq \int_0^r \frac{|n(t, 1/f) - n(t, f)|}{t}dt , \quad (0 < r < 1) .
\end{aligned}
$$

Combining (3.9) with (3.10) and the assumption $f(o) = 1$, we see there are constants K_1 and K such that

$$
(3.11)\quad |N(r, 1/f) - N(r, f)| \leqq K_1 \int_{1/2}^r I_1(t)dt + K , \quad \left(\frac{1}{2} < r < 1\right) .
$$

Thus, if $I_1(r) = 0((1 - r)^{-\beta})$, $(r \to 1)$, there are constants K_2, K_3, and K_4 such that (3.11) yields for $1/2 < r < 1$

$$
(3.12)\qquad |N(r, 1/f) - N(r, f)| \leqq \begin{cases} K_2 \quad \text{if} \quad \beta < 1 \\ -K_3 \log(1 - r) \quad \text{if} \quad \beta = 1 \\ K_4(1 - r)^{-(\beta-1)} \quad \text{if} \quad \beta > 1 . \end{cases}
$$

Now if $\beta < \alpha$ we find (3.8) and (3.12) taken together lead to a contradiction. Hence $\beta \geqq \alpha$, and (i) is proved.

To see (ii) is true, we first observe that there is a constant K_5 such that

$$
I_1(r) \leqq K_5(I_2(r))^{1/2} , \quad (0 < r < 1) .
$$

So if $I_2(r) = 0((1 - r)^{-\beta})$, $(r \to 1)$, then

$$
(3.13)\qquad I_1(r) = 0((1 - r)^{-\beta/2}) , \quad (r \longrightarrow 1) .
$$

Part (i) and (3.13) imply $\beta \geqq 2\alpha$.

4. Proof of Theorem 3. Assume first that f has no zeros in D, $f(o) = 1$, and let $0 < r < R < 1$. If $z = re^{i\theta}$, then the Poisson-Jensen formula shows

$$
\log|f(z)| = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\log|f(Re^{i\phi})|}{R^2 - 2Rr\cos(\phi - \theta) + r^2}d\phi .
$$

Multiplying both sides by $(1/(2\pi))\log|f(z)|$ and integrating with respect to θ over the interval $[0, 2\pi]$, we get

$$
m_2(r, f)^2 = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\log|f(Re^{i\phi})|}{R^2 - 2Rr\cos(\phi - \theta) + r^2}d\phi\right)(\log|f(re^{i\theta})|)d\theta .
$$

An integration by parts on the right-hand side gives

$$
m_2(r, f)^2 = \left(\frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\log|f(Re^{i\phi})|}{R^2 - 2Rr\cos\phi + r^2}d\phi\right)\left(\frac{1}{2\pi}\int_0^{2\pi}(\log|f(re^{i\theta})|)d\theta\right)
$$

$$+\int_0^{2\pi}\left(\int_0^{\theta}\frac{1}{2\pi}(\log|f(re^{i\theta})|)d\theta\right)$$
$$\times\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{(R^2-r^2)(\log|f(Re^{i\phi})|((2Rr\sin(\phi-\theta))}{(R^2-2Rr\cos(\phi-\theta)+r^2)^2}d\phi\right)d\theta\ .$$

Using the Poisson-Jensen formula and the differentiated formula, the latter may be rewritten as

$$m_2(r,f)^2=(\log|f(r)|)\left(\frac{1}{2\pi}\int_0^{2\pi}(\log|f(re^{i\theta})|)d\theta\right)$$
$$+\int_0^{2\pi}\left(\int_0^{\theta}\frac{1}{2\pi}(\log|f(re^{i\theta})|)d\theta\right)\left(\operatorname{Im} re^{i\theta}\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right)d\theta\ .$$

Standard inequalities then imply

$$m_2(r,f)^2\leqq 2T(r,f)\left\{|\log|f(r)||+\int_0^{2\pi}\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right|d\theta\right\}\ .$$

For each $\varepsilon>0$, (1.1) and the definition of $\tilde{\alpha}$ enable us to obtain

$$m_2(r,f)^2=0((1-r)^{-(\tilde{\alpha}+\varepsilon)})\{|\log|f(r)||+(1-r)^{-\beta}\}\ ,\quad (r\longrightarrow 1)\ . \tag{4.1}$$

Since $\beta+\tilde{\alpha}<2\alpha$, we conclude from (4.1) that the conclusion is true for $\theta=0$. A simple rotation shows it to be valid for each θ. (Actually the bound in (4.1) is uniform with respect to θ.)

If f has zeros in D, we let $\{a_n\}$ be the zeros and r_0 such that $|a_n|<r_0<1$ for all n. Let $0<r_0<r<R<1$. If $z=re^{i\theta}$, we proceed in a similar manner to that above using the Poisson-Jensen formula to obtain $m_2(r,f)^2$. Further similar steps to the above along with the choice of $R=(1/2)(1+r)$ lead to the desired conclusion.

5. **Proof of Theorem 4.** We may assume without loss of generality that $f(o)=1$. It can be shown using Green's theorem as in [6] that if $f(z)=e^{g(z)}$ in $\{z||z|\leqq r\}$, where $0<r<1$, then

$$\frac{d}{dt}\left(\frac{1}{2\pi}\int_0^{2\pi}(\log|f(te^{i\theta})|)^2d\theta\right)=\frac{1}{\pi t}\int_0^t\int_0^{2\pi}|g'(\rho e^{i\theta})|^2\rho d\theta d\rho\ , \tag{5.1}$$

for $0<t<r$. Integrating (5.1) from 0 to r, we obtain

$$m_2(r,f)^2=\int_0^r\frac{1}{\pi t}\left(\int_0^t\int_0^{2\pi}\left|\frac{f'(\rho e^{i\theta})}{f(\rho e^{i\theta})}\right|^2\rho d\theta d\rho\right)dt\ . \tag{5.2}$$

Using (1.2) in the right-hand side of (5.2), we find

$$m_2(r,f)^2=\begin{cases}0(1),\ (r\longrightarrow 1)\ , & \text{if}\quad \beta\leqq 2;\\ 0((1-r)^{-\beta+2}),\ (r\longrightarrow 1)\ , & \text{if}\quad \beta>2\ .\end{cases} \tag{5.3}$$

Since $\alpha > 0$ and $\beta < 2 + 2\alpha$, we see that whatever the value of β the statements in (5.3) lead to contradictions. Hence if f satisfies the hypothesis of Theorem 4, it must have at least one zero in D.

So suppose f has zeros but that the number of zeros is finite. Let $\{a_n\}$ be the zeros, and assume r_0 is such that $|a_n| < r_0 < 1$, for all n. Define the function B in D by

$$B(z) = \prod_{|a_n|<r_0} \left(\frac{z - a_n}{1 - \bar{a}_n z} \right) .$$

We consider $r_0 < r_1 < r < 1$ and apply (5.1) to the function defined by the quotient f/B in D. Integrating from 0 to r, we obtain

$$\begin{aligned}(5.4) \qquad m_2(r, f)^2 = \frac{1}{\pi}\int_0^{2\pi} (\log|f(re^{i\theta})|)(\log|B(re^{i\theta})|)d\theta \\ + \frac{1}{2\pi}\int_0^{2\pi} (\log|B(re^{i\theta})|)^2 d\theta + \left(\log\left|\frac{f(o)}{B(o)}\right|\right)^2 \\ + O(1) + \int_{r_1}^{r} \frac{1}{\pi t} \int_{r_1}^{t} \int_0^{2\pi} \left| \frac{f'(\rho e^{i\phi})}{f(\rho e^{i\phi})} - \frac{B'(\rho e^{i\phi})}{B(\rho e^{i\phi})} \right|^2 \rho d\phi d\rho dt , \\ (r \longrightarrow 1) .\end{aligned}$$

Since the number of zeros of f is finite and $r > r_1 > r_0 > |a_n|$, for each n, we see the first term on the right-hand side of (5.4) is bounded by

$$O(T(r, f)) , \quad (r \longrightarrow 1) ,$$

and the second and third terms are bounded by a constant. Thus (5.4) becomes

$$\begin{aligned}m_2(r, f)^2 \leqq O(T(r, f)) \\ (5.5) \qquad + \int_{r_1}^{r} \frac{1}{\pi t} \int_{r_1}^{t} \int_0^{2\pi} \left| \frac{f'(\rho e^{i\phi})}{f(\rho e^{i\phi})} - \frac{B'(\rho e^{i\phi})}{B(\rho e^{i\phi})} \right|^2 \rho d\phi d\rho dt , \\ (r \longrightarrow 1) .\end{aligned}$$

Using standard inequalities, we then get

$$\begin{aligned}(5.6) \qquad m_2(r, f)^2 \leqq O(T(r, f)) + \int_{r_1}^{r} \frac{1}{\pi t} \int_{r_1}^{t} \rho \Big\{ I_2(\rho) + \int_0^{2\pi} \left| \frac{B'(\rho e^{i\phi})}{B(\rho e^{i\phi})} \right|^2 d\phi \\ + 2(I_2(\rho))^{1/2} \left(\int_0^{2\pi} \left| \frac{B'(\rho e^{i\phi})}{B(\rho e^{i\phi})} \right|^2 d\phi \right)^{1/2} \Big\} d\rho dt , \quad (r \longrightarrow 1) .\end{aligned}$$

We know that

$$\frac{B'(z)}{B(z)} = \sum_n \left(\frac{1}{z - a_n} + \frac{\bar{a}_n}{1 - \bar{a}_n z} \right) .$$

Hence the fact that

$$\int_0^{2\pi} \frac{d\theta}{|1 - se^{i\theta}|^2} = 0\left(\frac{1}{1-s}\right), \quad (s \longrightarrow 1),$$

enables one to obtain the estimate

$$\int_0^{2\pi} \left|\frac{B'(\rho e^{i\phi})}{B(\rho e^{i\phi})}\right|^2 d\phi = 0\left(\frac{1}{1-\rho}\right), \quad (\rho \longrightarrow 1). \tag{5.7}$$

We then observe

$$\int_{r_1}^{r} \frac{1}{\pi t} \int_{r_1}^{t} \rho\left(\frac{1}{1-\rho}\right) d\rho dt \leqq \int_{r_1}^{r} \left(\log \frac{1}{1-t}\right) dt \leqq 1. \tag{5.8}$$

Taking (5.7) and (5.8) together with (5.6) we get

$$\begin{aligned} m_2(r, f)^2 &\leqq 0(T(r, f)) + 0\left(\log \frac{1}{1-r}\right) \\ &\quad + K \int_{r_1}^{r} \frac{1}{\pi t} \int_{r_1}^{t} \rho \left\{ I_2(\rho) + (I_2(\rho))^{1/2} \left(\frac{1}{1-\rho}\right)^{1/2} \right\} d\rho dt, \\ &\qquad (r \longrightarrow 1). \end{aligned} \tag{5.9}$$

Recalling that $\tilde{\alpha} < \alpha$ and that (1.2) holds, we see as in the earlier case (5.9) leads to a contradiction whatever the value of β subject to $\beta < 2 + 2\alpha$ with $\alpha > 0$. Therefore, f has an infinite number of zeros.

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