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ON THE SEMIMETRIC ON A BOOLEAN ALGEBRA INDUCED BY A FINITELY ADDITIVE PROBABILITY MEASURE

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A finitely additive probability measure μ on a Boolean algebra $\mathscr{B}$ induces a semi-metric d_μ defined by $d_\mu(A, B) = \mu(A \Delta B)$. When $\mathscr{B}$ is a σ-algebra and μ countably additive $\mathscr{B}$ is complete as is well known. The converse is shown to be true. More precisely, if $\mathscr{B}_\mu$ is the quotient of $\mathscr{B}$ via μ-null sets then $\mathscr{B}_\mu$ is d_μ-complete iff μ is countably additive on $\mathscr{B}_\mu$ and $\mathscr{B}_\mu$ is complete as a Boolean algebra. Furthermore $\mathscr{B}_\mu$ is d_μ-complete iff every $\nu \ll \mu$ has a Hahn decomposition iff (when $\mathscr{B}$ is an algebra of sets) every $\nu \ll \mu$ has a $\mathscr{B}$-measurable Radon-Nikodym derivative. If $\mathscr{B}_\mu$ is not d_μ-complete it is either meager in itself or fails to have the property of Baire in it's completion. Examples are given of both situations with the density character of $\mathscr{B}_\mu$ an arbitrary infinite cardinal number.

If $\mathscr{B}$ is a Boolean algebra with supremum X and μ is a finitely additive probability measure on $\mathscr{B}$ (i.e., $\mu \in BA_1^+(\mathscr{B})$) there is a semi-metric d_μ on $\mathscr{B}$ given by $d_\mu(A, B) = \mu(A \Delta B)$ (where Δ denotes symmetric difference) for $\{A, B\} \subset \mathscr{B}$. Drewnowski [13] calls such semi-metrics Frechet-Nikodym semi-metrics. The metric space obtained by identifying A and B if $d_\mu(A, B) = 0$ is the quotient Boolean algebra $\mathscr{B}_\mu = \mathscr{B}/\mathscr{N}_\mu$ where $\mathscr{N}_\mu$ is the ideal of μ-negligible sets. We consider μ and d_μ to be defined on $\mathscr{B}_\mu$ in the usual manner so that $\mu(A \Delta B) = d_\mu(A, B)$ if $\{A, B\} \subset \mathscr{B}_\mu$. The operation of complementation is an isometry in $\mathscr{B}$ or $\mathscr{B}_\mu$ for d_μ.

When $\mathscr{B}_\mu$ is σ-complete and μ is countably additive on $\mathscr{B}_\mu$ then $\mathscr{B}_\mu$ is complete both as a Boolean algebra and as a metric space. This fact has been very useful for analysts in the special case where $\mathscr{B}$ is a σ-algebra of subsets of X and μ a countably additive measure on $\mathscr{B}$. In [12] it was asked to what extent this remains true if μ is only finitely additive. If μ is a $\{0, 1\}$-valued measure on the Boolean algebra $\mathscr{B}$ then $\mathscr{B}_\mu$ is a two point space $\{\phi, X\}$ with $d_\mu(\phi, X) = 1$. Thus, the theorem is true in this case. Of course, μ is then countably additive on $\mathscr{B}_\mu$. We may ask when $\mathscr{B}_\mu$ has an isolated point.

PROPOSITION 1. *$\mathscr{B}_\mu$ has an isolated point iff it is finite iff μ is a finite convex combination of $\{0, 1\}$-valued measures.*

Proof. If $\mathscr{B}_\mu$ is finite it has a finite number of μ-atoms and μ is a finite convex combination of $\{0, 1\}$-valued measures. If μ isn't a finite convex combination of $\{0, 1\}$-valued measures there is an infinite sequence $\{A_n\} \subset \mathscr{B}_\mu \backslash \{\phi\}$ with $\lim_{n\to\infty} \mu(A_n) = 0$. If $A \in \mathscr{B}_\mu$ then $A \neq A \Delta A_n$ for large n and $\lim_{n\to\infty} d_\mu(A, A \Delta A_n) = 0$. Thus, A isn't isolated. This suffices to establish the proposition. □

Thus, except in trivial cases, $\mathscr{B}_\mu$ is an infinite perfect metric space. It turns out that the only time $\mathscr{B}_\mu$ is complete under d_μ is when $\mathscr{B}_\mu$ is complete as a Boolean algebra and μ is countably additive on $\mathscr{B}_\mu$.

PROPOSITION 2. *In order that $\mathscr{B}_\mu$ be a complete metric space under d_μ it is necessary and sufficient that $\mathscr{B}_\mu$ be a complete Boolean algebra and that μ be countably additive on $\mathscr{B}_\mu$.*

Proof. First suppose that $\mathscr{B}_\mu$ isn't a complete Boolean algebra. Since $\mathscr{B}_\mu$ satisfies the countable chain condition it can't be a σ-complete Boolean algebra. Thus, there is an increasing sequence $\{A_n\} \subset \mathscr{B}_\mu$ without a supremum in $\mathscr{B}_\mu$. Let $\lambda = \lim_{n\to\infty} \mu(A_n)$. We have $d_\mu(A_n, A_{n+k}) = \mu(A_{n+k} \backslash A_n) \leqq \lambda - \mu(A_n) \to 0$ as $n \to \infty$. Thus, $\{A_n\}$ is d_μ-Cauchy. If $A \in \mathscr{B}_\mu$ were such that $\lim_{n\to\infty} d_\mu(A_n, A) = 0$ then $\lim_{n\to\infty} \mu(A_n \backslash A) = 0$ so $\mu(A_n \backslash A) = 0$ for all n hence $A_n \subset A$ for all A. From $\lim_{n\to\infty} \mu(A \backslash A_n) = 0$ it would follow that $A = \sup_n A_n$ which is impossible. Thus, if $\mathscr{B}_\mu$ is d_μ-incomplete it is incomplete as a Boolean algebra.

Now suppose that $\mathscr{B}_\mu$ is a complete Boolean algebra with μ not countably additive. There exists an increasing sequence $\{A_n\} \subset \mathscr{B}_\mu$ with union A so that $\lim_{n\to\infty} \mu(A_n) = \lambda < \mu(A)$. Once again, $\{A_n\}$ must be d_μ-Cauchy and if $C \in \mathscr{B}_\mu$ with $\lim_{n\to\infty} d_\mu(A_n, C) = 0$ then $C = A$. Since $\lim_{n\to\infty} d_\mu(A_n, A) = \mu(A) - \lambda \neq 0$, $\mathscr{B}_\mu$ is d_μ-incomplete. Thus, if $\mathscr{B}_\mu$ is d_μ-complete then μ is countably additive on $\mathscr{B}_\mu$. This suffices to establish the proposition. □

Plachky, [23] gives a characterization of extreme extensions ν of a finitely additive probability μ on $\mathscr{B}_1$ to $\mathscr{B}_2$. He denotes by $ba(\mathscr{B}_1, \mu, \mathscr{B}_2)$ all such extensions. We denote by $\xi ba(\mathscr{B}_1, \mu, \mathscr{B}_2)$ the extreme elements of the compact convex set $ba(\mathscr{B}_1, \mu, \mathscr{B}_2)$. In terms of the semi-metric d_ν elements ν of $\xi\, ba(\mathscr{B}_1, \mu, \mathscr{B}_2)$ are characterized by the condition that for all $A_2 \in \mathscr{B}_2$ and $\varepsilon > 0$ there is an $A_1 \in \mathscr{B}_1$ with $d_\nu(A_1, A_2) < \varepsilon$. That is, $\nu \in \xi\, ba(\mathscr{B}_1, \mu, \mathscr{B}_2)$ iff $\mathscr{B}_1$ is d_ν-dense in $\mathscr{B}_2$.

COROLLARY 2.1. *Let $\mathscr{B}_1 \subset \mathscr{B}_2$ be Boolean algebras and let μ be* *a probability measure with* $\mathscr{B}_1$ d_μ*-complete. For* $\nu \in \mathrm{ba}(\mathscr{B}_1, \mu, \mathscr{B}_2)$ *to be in* $\xi\, ba(\mathscr{B}_1, \mu, \mathscr{B}_2)$ *it is necessary and sufficient that for all* $A_2 \in \mathscr{B}_2$ *there be an* $A_1 \in \mathscr{B}_1$ *with* $d_\nu(A_1, A_2) = 0$*. If* $\nu \in ba(\mathscr{B}_1, \mu, \mathscr{B}_2)$ *then* $\mathscr{B}_2$ *is* d_ν*-complete.*

Proof. It is only necessary to show that if $\nu \in \xi\, ba(\mathscr{B}_1, \mu, \mathscr{B}_2)$ and $A_2 \in \mathscr{B}_2$ there is an $A_1 \in \mathscr{B}_1$ with $d_\nu(A_1, A_2) = 0$. By Plachky's condition we may construct a sequence $\{A^n\} \subset \mathscr{B}_1$ which d_ν-converges to A_2. Any d_μ-limit A_1 of this sequence will suffice. □

REMARKS. (1) Bogdan and Oberle in Proposition 1.1.1 of [9] obtain a result closely related to Proposition 2. M. Bhaskara Rao and K. P. S. Bhaskara Rao in [25] essentially obtain Propositions 1 and 2.

(2) Corollary 2.1 yields a method for obtaining noncountably additive μ with $\mathscr{B}_\mu$ d_μ-complete.

Recall that a finitely additive measure μ of bounded variation on a Boolean algebra $\mathscr{B}$ (i.e., $\mu \in BA(\mathscr{B})$) has a Hahn decomposition iff there is an $A \in \mathscr{B}$ so that $\mu(A) = \|\mu^+\|$ and $\mu(A^c) = \|u^-\|$. Thus, $\mu^+(E) = \mu(A \cap E)$ and $\mu^-(E) = \mu(E \cap A^c)$ if $E \in \mathscr{B}$. Here, μ^+ and μ^- are the positive and negative variations of μ. $|\mu| = \mu^+ + \mu^-$ is the total variation of μ.

PROPOSITION 3. *Let* μ *be a probability measure on the algebra* $\mathscr{B}$. $\mathscr{B}_\mu$ *is* d_μ*-complete iff every* $\nu \in BA(\mathscr{B})$ *with* $|\nu| = \mu$ *has a Hahn decomposition iff every* $\nu \in BA(\mathscr{B})$ *with* $\nu \ll \mu$ *has a Hahn decomposition.*

Proof. If μ is countably additive on the complete algebra $\mathscr{B}_\mu$ then every $\nu \in BA(\mathscr{B})$ with $\nu \ll \mu$ is countably additive on $\mathscr{B}_\mu$ hence has a Hahn-decomposition in $\mathscr{B}_\mu$ and in $\mathscr{B}$ (we are using the $\varepsilon - \delta$ definition of absolute continuity $\ll$ as in [8]). Only the converse needs to be established.

We must show that if every $\nu \in BA(\mathscr{B})$ with $|\nu| = \mu$ has a Hahn-decomposition then $\mathscr{B}_\mu$ is d_μ-complete. Suppose that μ isn't countably additive on $\mathscr{B}_\mu$. There exists $\{A_n\}$ an increasing sequence in $\mathscr{B}_\mu$ with supremum X such that $0 < \lim_{n\to\infty} \mu(A_n) = \lambda < 1$. Let $\mu'(A) = \lim_{n\to\infty} \mu(A \cap A_n)$ define $\mu'(A)$ for $A \in \mathscr{B}_\mu$ so that $\mu' \in BA^+(\mathscr{B}_\mu)$ hence $\mu' \in BA^+(\mathscr{B})$. Let $\mu'' = \mu - \mu' \in BA^+(\mathscr{B})$. Let $\nu = \mu' - \mu'' \in BA(\mathscr{B})$. Since μ' and μ'' may be verified to be singular, $\nu^+ = \mu'$, $\nu^- = \mu''$ and $|\nu| = \mu$. Let $A \in \mathscr{B}_\mu$ be such that $\nu(A) = \nu^+(A)$ and $-\nu(A^c) = \nu^-(A)$. We have $\nu^+(A) = \mu'(A) = \lim_{n\to\infty} \mu(A \cap A_n) = \|\mu'\| = \lambda$. Thus, $A_n \subset A$ for all n. Thus, $A = X$ and $\mu'' = 0$ which is impossible. Thus, μ must be countably additive on $\mathscr{B}_\mu$.

If $\mathscr{B}_\mu$ isn't σ-complete there is an increasing sequence $\{A_n\}$ without a supremum. Define $\mu'(A) = \lim_{n\to\infty} \mu'(A_n \cap A)$ so that $\mu' \in BA^+(\mathscr{B}_\mu)$ hence $\mu' \in BA^+(\mathscr{B})$ let $\mu'' = \mu - \mu'$ and let $\nu = \mu' - \mu''$. If $\mu'' = 0$ then $X = \sup_n A_n$ and if $\mu' = 0$ then $\phi = \sup_n A_n$ which are impossible. If $A \in \mathscr{B}_\mu$ is such that $\nu^+(E) = \nu(E \cap A)$ and $\nu^-(E)$: $-\nu(E \cap A^c)$. Once again, A would have to be $\sup_n A_n$ which is impossible. Since such an A is guaranteed to exist $\mathscr{B}_\mu$ must be σ-complete hence complete. □

$\mathscr{B}$ may be an algebra of subsets of X. This is the case if X is the Stone space $X_{\mathscr{B}}$ of $\mathscr{B}$ and $\mathscr{B}$ is regarded as the clopen algebra of $X_{\mathscr{B}}$. If $\mu \in BA(\mathscr{B})$ one may integrate simple step functions $f = \sum_{i=1}^n \lambda_i \chi_{A_i}$ with $\{A_1, \cdots, A_n\}$ in the usual manner. One may integrate any f which is the uniform limit of simple step functions as the limit of the integrals of the step functions. The totality of such f will be called bounded $\mathscr{B}$-measurable functions. More generally $f: X \to [-\infty, \infty]$ is called $\mathscr{B}$-measurable iff $f \wedge n \vee (-m)$ is a bounded $\mathscr{B}$-measurable function for all integers $n, m \geqq 0$. One defines $\int f d\mu$, for any $\mathscr{B}$-measurable f, to be $\lim_{(m,n)\to(\infty,\infty)} \int f \wedge n \vee (-m) d\mu$ provided this limit exists. For any $\mathscr{B}$-measurable f on X with $\int |f| d\mu < \infty$ one may define the measure $f\mu$ on $\mathscr{B}$ by the requirement that $(f\mu)(A) = \int f\chi_A d\mu$ for $A \in \mathscr{B}$. Then, $f\mu \in BA(\mathscr{B})$ and is absolutely continuous with respect to μ. If $\mu \in BA^+(\mathscr{B})$ one has $(f\mu)^+ = (f \vee 0)\mu$, $(f\mu)^- = -(f \wedge 0)\mu$ and $|f\mu| = |f|\mu$. If g is $\mathscr{B}$-measurable and $\int g d(f\mu)$ exists it is $\int gf d\mu$. If $\nu \ll \mu \in BA^+(\mathscr{B})$ one says that ν has a Radon-Nikodym derivative, $f = d\nu/d\mu$, iff f is $\mathscr{B}$-measurable with $\nu = f\mu$. When μ is a countably additive probability on the σ-complete $\mathscr{B}_\mu$ (i.e., when $\mathscr{B}_\mu$ is d_μ-complete) every $\nu \ll \mu$ has a Radon-Nikodym derivative on $\mathscr{B}_\mu$ with respect to μ and on $\mathscr{B}$ if μ is countably additive on $\mathscr{B}$.

PROPOSITION 4. *Let $\mathscr{B}$ be a Boolean set algebra and let $\mu \in BA_1^+(\mathscr{B})$. $\mathscr{B}_\mu$ is d_μ-complete iff every $\nu \ll \mu$ has a Radon-Nikodym derivative on $\mathscr{B}$ (hence on $\mathscr{B}_\mu$).*

Proof. There is a Banach lattice isomorphism between the M-space of bounded $\mathscr{B}$-measurable functions on X and the continuous functions on the Stone space $X_{\mathscr{B}}$. If the bounded $\mathscr{B}$-measurable f on X has corresponding to it $\tilde{f}$ and the finitely additive $p \in BA(\mathscr{B})$ has corresponding to it $\tilde{p} \in \mathscr{M}(X_{\mathscr{B}})$ under the Stone correspondence then $\int_X f dp = \int_{X_{\mathscr{B}}} \tilde{f} d\tilde{p}$. For $\nu \in BA(\mathscr{B})$, $\nu = f\mu$ with f bounded and $\mathscr{B}$-measurable iff $\tilde{\nu} = \tilde{f}\tilde{\mu}$ with $\tilde{f} \in C(X_{\mathscr{B}})$. If $|\nu| = \mu$ then $|\tilde{f}|\tilde{\mu} =$

$|\tilde{\nu}| = \tilde{\mu}$ so $|\tilde{f}| = 1$ on supp $(\tilde{\mu})$. There is a clopen set $[A] \subset X_{\mathscr{B}}$ corresponding to $A \in \mathscr{B}$ so that $\tilde{f} = \chi_{[A]} - \chi_{[A^c]}$ on supp $(\tilde{\mu})$ consequently $\tilde{\nu} = (\chi_{[A]} - \chi_{[A^c]})\tilde{\mu}$ and $\nu = (\chi_A - \chi_{A^c})\mu$. Thus, ν has a Hahn-decomposition. Since ν was arbitrary with $|\nu| = \mu$ $\mathscr{B}_\mu$ is d_μ-complete by Proposition 3.

Now suppose that $\mathscr{B}_\mu$ is d_μ-complete. If $\lambda^{-1}\mu \leqq \nu \leqq \lambda\mu$ for some $\lambda \in (0, \infty)$ then, on $X_{\mathscr{B}_\mu}$, there is a Radon-Nikodym derivative g for ν with respect to μ which is bounded and $\mathscr{B}_\mu$-measurable hence continuous ($\mathscr{B}_\mu$ is considered to be the clopen algebra of $X_{\mathscr{B}_\mu}$). If $\tilde{\nu}$ and $\tilde{\mu}$ are the Radon measures on $X_{\mathscr{B}_\mu}$ corresponding to ν and μ we have $\tilde{\nu} = g\tilde{\mu}$. Extend g continuously from $X_{\mathscr{B}_\mu}$, considered as a closed subspace of $X_{\mathscr{B}}$, to a continuous function f on $X_{\mathscr{B}}$. Then, $\tilde{\nu} = f\tilde{\mu}$ where $\{\tilde{\nu}, \tilde{\mu}\}$ are considered as Radon measures on $X_{\mathscr{B}}$. f is $\mathscr{B}$-measurable on $X_{\mathscr{B}}$ hence is the uniform limit of simple step functions $\{f_n\}$. If $X = X_{\mathscr{B}}$ then $\nu = f\mu$. Otherwise $\{f_n\}$ corresponds to a uniformly convergent sequence $\{f'_n\}$ of simple step functions on X (where $f'_n(x) = f_n(\hat{x})$ where $\hat{x} \in X_{\mathscr{B}}$ is the ultrafilter of supersets of x in $\mathscr{B}$). Once again $\nu = f'\mu$ where $f' = \lim_{n\to\infty} f'_n$.

If $\nu \ll \mu$ then ν is the limit in the variation norm of $\nu_n = \nu \wedge (n\mu) \vee (-n\mu)$ as $n \to \infty$. We have $\nu_n = \nu_{n+k} \wedge (n\mu) \vee (-n\mu)$ for all $k > 0$. Since $-n\mu \leqq \nu \leqq n\mu$ we have $\nu = f_n\mu$ and $f_n = f_{n+k} \wedge n \wedge -n$ for $k > 0$ where $\{f_n\}$ are $\mathscr{B}$-measurable on X. Define $f(x) = f_n(x)$ if $f_{n+k}(x) = f_n(x)$ for all $k > 0$. If $f(x)$ isn't defined either $f_n(x) = n$ for all n or $f_n(x) = -n$. In the first case set $f(x) = \infty$ and in the second set $f(x) = -\infty$. Since $f \wedge n \vee -n = f_n$ is $\mathscr{B}$-measurable it follows that f is $\mathscr{B}$-measurable. If $A \in \mathscr{B}$ then $\nu(A) = \lim_{n\to\infty} \nu_n(A) = \lim_{n\to\infty} \int_A f_n d\mu = \int_A f d\mu$. Thus, $f = d\nu/d\mu$. This establishes the proposition. □

REMARK. Since $\mathscr{B}_\mu$ remains unchanged if $\mathscr{B}$ is enlarged, and μ redefined, by only an enlargement of η_μ we may consider $\hat{\eta}_\mu$ the set of A with $A \subset X$ such that for all $\varepsilon > 0$ there is an $A^\varepsilon \in \mathscr{B}$ with $A \subset A^\varepsilon$ and $\mu(A^\varepsilon) \leqq \varepsilon$. Let $\mathscr{B} \Delta \hat{\eta}_\mu$ denote all sets A' in X differing from an $A \in \mathscr{B}$ by an $N \in \hat{\eta}_\mu$. For such A' set $\mu(A') = \mu(A)$ so that $\hat{\eta}_\mu$ is the ideal of μ-negligible sets in $\mathscr{B} \Delta \hat{\eta}_\mu$. Propositions 3 and 4 remain unchanged when $\mathscr{B}$ is replaced by $\mathscr{B} \Delta \hat{\eta}_\mu$.

In general $\mathscr{B}_\mu$ isn't complete under d_μ but its completion is easily identified.

PROPOSITION 5. Let $\mu \in BA_1^+(\mathscr{B})$, $X_{\mathscr{B}}(X_{\mathscr{B}_\mu})$ be the Stone space of $\mathscr{B}(\mathscr{B}_\mu)$ and let $\tilde{\mu}$ be the Radon probability measure on $X_{\mathscr{B}}(X_{\mathscr{B}_\mu})$ corresponding to μ. The d_μ-completion of $\mathscr{B}$ is the quotient of the Baire algebra, $\mathscr{B}^0$, of $X_{\mathscr{B}}(X_{\mathscr{B}_\mu})$ modulo $\tilde{\mu}$-negligible sets (i.e., $\mathscr{B}^0_{\tilde{\mu}}$)

under d_μ.

Proof. It is easiest to work with $\mathscr{B}_\mu$ considered as the clopen algebra of $X_{\mathscr{B}_\mu}$. Then $\mathscr{B}_\mu \subset \mathscr{B}^0$ and the metric d_μ on $\mathscr{B}_\mu$ is the same as is induced by the semi-metric $d_{\tilde{\mu}}$. As a result $\mathscr{B}_\mu$ is isometric to a subset of the $d_{\tilde{\mu}}$-complete $\mathscr{B}_\mu^0$. Since $\mathscr{B}^0$ is the monotone sequential closure of $\mathscr{B}_\mu$ it follows that $\mathscr{B}_\mu$ is d_μ-dense in $\mathscr{B}^0$ hence in $\mathscr{B}_{\tilde{\mu}}^0$. Thus, $\mathscr{B}_{\tilde{\mu}}^0$ must be the completion of $\mathscr{B}_\mu$. □

REMARK. Proposition 2 is an immediate corollary of Proposition 5.

One may extend μ defined on the algebra, $\mathscr{B}$ of subsets of X, not just to $\mathscr{B}\Delta\hat{\eta}_\mu$ but to an even larger algebra $\hat{\mathscr{B}}_\mu$ of subsets of X in a unique manner. $\hat{\mathscr{B}}_\mu$ is the *μ-completion* of $\mathscr{B}$ and consists of those sets $E \subset X$ so that $\mu^*(E) = \inf\{\mu(A): E \subset A \in \mathscr{B}\} = \mu_*(E) = \sup\{\mu(A): E \supset A \in \mathscr{B}\}$. One sets, for $E \in \hat{\mathscr{B}}^\mu$, $\mu(E) = \mu_*(E) = \mu^*(E)$. $\hat{\eta}_\mu$ is then the ideal of μ negligible sets in $\hat{\mathscr{B}}^\mu$ and $\hat{\mathscr{B}}\Delta\hat{\eta}_\mu \subset \hat{\mathscr{B}}^\mu$. One may ask whether $\hat{\mathscr{B}}^\mu$ is ever d_μ-complete. To answer this it is convenient to characterize $\hat{\mathscr{B}}^\mu$ in terms of the Stone space $X_{\mathscr{B}}$.

Let $j_{\mathscr{B}}(x) = \{A \in \mathscr{B}, x \in A\} \in X_{\mathscr{B}}$. The mapping $j_{\mathscr{B}}$ from X to $X_{\mathscr{B}}$ is such that if A is in $\mathscr{B}$ then $[A] = \overline{j_{\mathscr{B}}(A)}$ so that $A = j_{\mathscr{B}}^{-1}([A])$. The inverse image of the clopen algebra of $X_{\mathscr{B}}$ is the algebra $\mathscr{B}$. It is convenient to identify X with the dense subset $j_{\mathscr{B}}(X)$ of $X_{\mathscr{B}}$ even though this is only proper if $j_{\mathscr{B}}$ is injective iff $\mathscr{B}$ separates X.

PROPOSITION 6. *$E \subset X$ is in $\hat{\mathscr{B}}^\mu$ iff there is a closed G_δ, F and an open F_σ, G, in $X_{\mathscr{B}}$ with $G \subset F$, $\tilde{\mu}(F\backslash G) = 0$, and $j_{\mathscr{B}}^{-1}(G) \subset E \subset j_{\mathscr{B}}^{-1}(F)$. In particular $\tilde{\mu}(\partial\overline{j_{\mathscr{B}}(E)}) = 0$.*

Proof. Let $E \in \hat{\mathscr{B}}^\mu$. Let $\{A_n\}$ be an increasing sequence in $\mathscr{B}$ and $\{A^n\}$ be a decreasing sequence in $\mathscr{B}$ with $A_n \subset E \subset A^n$ so that $\mu(A^n\backslash A^n) \to 0$ as $n \to \infty$. Let $G = \bigcup_{n=1}^\infty [A_n]$ and $F = \bigcap_{n=1}^\infty [A^n]$. We have $G \subset F$ with $\tilde{\mu}(F\backslash G) = 0$ and we have $j_{\mathscr{B}}^{-1}(G) = \bigcup_{n=1}^\infty A_n \subset E \subset \bigcap_{n=1}^\infty A^n = j_{\mathscr{B}}^{-1}(F)$.

Conversely, if G is an open F_σ and F a closed G_δ in $X_{\mathscr{B}}$ with $j_{\mathscr{B}}^{-1}(G) \subset E \subset j_{\mathscr{B}}^{-1}(F)$ and with $\tilde{\mu}(F\backslash G) = 0$ then $G = \bigcup_{n=1}^\infty [A_n]$ and $F = \bigcap_{n=1}^\infty [A^n]$ with $\{A^n, A_n : n \in N\} \subset \mathscr{B}$ with $A_n \subset E \subset A^n$ for $n \in N$ and with $\mu(A^n\backslash A_n) \to 0$ as $n \to \infty$. Thus, $E \in \hat{\mathscr{B}}^\mu$. □

PROPOSITION 7. *$\hat{\mathscr{B}}^\mu$ is d_μ-complete iff* (i) *$\hat{\mathscr{B}}^\mu$ is a σ-algebra of subsets of X and* (ii) *μ is countably additive on $\hat{\mathscr{B}}^\mu$. In this case $\hat{\mathscr{B}}^\mu$ is μ-complete as a σ-algebra.*

Proof. From Proposition 2, d_μ-completeness of $\hat{\mathscr{B}}^\mu$ follows from (i) and (ii). Also, d_μ-completeness of $\hat{\mathscr{B}}^\mu$ implies (ii) and that $\hat{\mathscr{B}}^\mu$ is σ-complete as a Boolean algebra. If $\{E_n\}$ is an increasing sequence in $\hat{\mathscr{B}}^\mu$ we must show that $E = \bigcup_{n=1}^\infty E_n \in \hat{\mathscr{B}}^\mu$. Let E^∞ be the supremum of $\{E_n\}$ in $\hat{\mathscr{B}}^\mu$. Let $\{A_n\}$ be chosen increasing in $\mathscr{B}$ with $A_n \subset E_n$ and $\mu(E_n \backslash A_n) < 1/n$ for all n. Let $\{A^n\}$ be chosen decreasing in $\mathscr{B}$ with $E^\infty \subset A^n$ and $\mu(A^n \backslash E_\infty) < 1/n$ for all n. We have $A_n \subset E \subset A^n$ and $\mu(A^n \backslash A_n) \to 0$ as $n \to \infty$. Thus, $E \in \hat{\mathscr{B}}^\mu$. □

PROPOSITION 8. *$\hat{\mathscr{B}}^\mu$ is d_μ-complete iff $\tilde{\mu}$ is a category measure on $X_{\mathscr{B}_\mu}$.*

Proof. A residual Radon measure is a category measure on its support, [2].

Let $\hat{\mathscr{B}}^\mu$ be d_μ-complete. We must show that if Θ is an open set in $X_{\mathscr{B}_\mu}$ then $\tilde{\mu}(\partial\Theta) = 0$, [3]. There is an open F_σ $\Theta' \subset \Theta$ with $\tilde{\mu}(\Theta/\Theta') = 0$. Let $\tilde{\Theta}'$ be an open F_σ in $X_{\mathscr{B}}$ with $\tilde{\Theta}' \cap X_{\mathscr{B}_\mu} = \Theta'$ (where $X_{\mathscr{B}_\mu}$ is considered to be $\operatorname{supp}(\tilde{\mu}) \subset X_{\mathscr{B}}$). We have $\tilde{\mu}(\partial\tilde{\theta}') = 0$. Thus, considering closure in $X_{\mathscr{B}_\mu}$, $\tilde{\mu}(\partial\theta') = 0$. Since $X_{\mathscr{B}_\mu} = \operatorname{supp}(\tilde{\mu})$, $\bar{\theta}' = \bar{\theta}$. Since Θ differs from Θ' by a $\tilde{\mu}$-negligible set and Θ' differs from $\bar{\Theta}$ by a negligible set $\tilde{\mu}(\partial\Theta) = 0$ which shows that $\tilde{\mu}$ is residual on $X_{\mathscr{B}_\mu}$.

Let $\tilde{\mu}$ be residual on $X_{\mathscr{B}_\mu}$. From Oxtoby [20, Theorem 4] any Borel set A in $X_{\mathscr{B}_\mu}$ has the property that $\tilde{\mu}(A) = \tilde{\mu}(A^0) = \tilde{\mu}(\bar{A})$. Thus, if A is a Baire set in $X_{\mathscr{B}_\mu}$ there is an open F_σ $G \subset A$ and a closed G_δ $F \supset A$ with $\tilde{\mu}(F \backslash G) = 0$. Represent G as $\bigcup_{n=1}^\infty \{[A_n] \cap X_{\mathscr{B}_\mu}\}$ where $\{A_n\} \subset \mathscr{B}$ is increasing, and F as $\bigcap_{n=1}^\infty \{[A^n] \cap X_{\mathscr{B}}\}$ with $\{A^n\} \subset \mathscr{B}$ decreasing with $A_n \subset A^n$ for all n, and with $\mu(A^n \backslash A_n) = 0$. Let $E \subset X$ be $\bigcap_{n=1}^\infty A^n$. Since $A_n \subset E \subset A^n$ for all n we have $E \in \hat{\mathscr{B}}^\mu$. It is easily checked that E is the d_μ-limit of the Cauchy sequence $\{A_n\} \subset \mathscr{B}$ and that E corresponds to the element A in the d_μ-completion of $\mathscr{B}_\mu$ as given in Proposition 5. □

By Proposition 4, $\hat{\mathscr{B}}^\mu$ is d_μ-complete iff every ν with $|\nu| = \mu$ has a $\hat{\mathscr{B}}^\mu$-measurable Radon-Nikodym derivative. One may ask what is the case if one allows *Eudoxus integrable*, [14], Radon-Nikodym derivatives. A bounded function f is Eudoxus integrable iff there an increasing sequence $\{f_n\}$ of bounded $\mathscr{B}$-measurable functions and a decreasing sequence $\{f^n\}$ of bounded $\mathscr{B}$-measurable functions such that $f_n \leqq f \leqq f^n$ for all n and $\lim_{n\to\infty} \int f^n - f_n d_\mu = 0$. Since bounded $\hat{\mathscr{B}}^\mu$-measurable functions are Eudoxus integrable no more Endoxus integrable functions are obtained if one only requires $\hat{\mathscr{B}}^\mu$-measurability of $\{f_n\}$ and $\{f^n\}$. $\int f d_\mu$ is defined by $\lim_{n\to\infty} \int f_n d_\mu$ or $\lim_{n\to\infty} \int f^n d_\mu$.

COROLLARY 8.1. *$\hat{\mathscr{B}}^\mu$ is d_μ-complete iff every ν with $|\nu| = \mu$ has a Eudoxus integrable Radon-Nikodym derivative.*

Proof. One direction is clear. For the other suppose that all ν with $|\nu| = \mu$ have Eudoxus integrable derivatives. We shall consider X as identified with a subset of $X_{\mathscr{B}}$ via the map $j_{\mathscr{B}}$. Let ν have $|\nu| = \mu$ and let f be a Eudoxus integrable Radon-Nikodym derivative. Let $\{f_n\}$ and $\{f^n\}$ be the monotone sequences of bounded $\mathscr{B}$-measurable functions with $f_n \leqq f \leqq f^n$ for all n so that $\lim_{n\to\infty} \int f^n - f_n d_\mu = 0$. Let $\{\tilde{f}_n\}$ and $\{\tilde{f}^n\}$ be the corresponding sequences in $\mathscr{C}(X_{\mathscr{B}})$. Let $\check{f} = \inf_n \tilde{f}^n$ and $\hat{f} = \sup_n \tilde{f}_n$. $\check{f}$ is upper semicontinuous and $\check{f}$ is lower semi-continuous. The restrictions of $\check{f}$ and $\hat{f}$ to X are themselves Eudoxus integrable Radon-Nikodym derivatives of ν. Both $|\hat{f}|$ and $|\check{f}|$ are equal to 1 $\tilde{\mu}$ a.e. Let K be the compact $G_\delta \{\check{f} \geqq 1\}$. One has $\check{f} = \chi_K - \chi_{K^c}\tilde{\mu}$ a.e. Since ν was arbitrary $\tilde{\nu}$ could have been of the form $(\chi_\theta - \chi_{\theta^c})\tilde{\mu}$ for an open set θ in $X_{\mathscr{B}}$. Thus, for each open θ there is a compact $G_\delta K$ in $X_{\mathscr{B}}$ with $\tilde{\mu}(\theta \Delta K) = 0$. The closure of $\theta \cap X_{\mathscr{B}_\mu}$ must be contained in $K \cap X_{\mathscr{B}_\mu}$ since $\operatorname{supp}(\tilde{\mu}) = X_{\mathscr{B}_\mu}$. Thus, in $X_{\mathscr{B}_\mu}$, $\tilde{\mu}(\partial(\theta \cap X_{\mathscr{B}_\mu})) = 0$. Since $\theta \cap X_{\mathscr{B}_\mu}$ may be an arbitrary open set in $X_{\mathscr{B}_\mu} \tilde{\mu}$ is a category measure on $X_{\mathscr{B}_\mu}$. Proposition 8 shows that $\hat{\mathscr{B}}^\mu$ is d_μ-complete. □

REMARKS. Can Eudoxus integrability be replaced by μ-integrability? Recall that f is μ-integrable iff there is a sequence of simple $\mathscr{B}$-measurable functions which converges to f in μ-measure or in μ-probability.

The maximal ideal space $Z_{\tilde{\mu}}$ of $L^\infty(X_{\mathscr{B}_\mu}, \tilde{\mu})$ is the Gleason space or projective cover of $X_{\mathscr{B}_\mu}$ iff $\tilde{\mu}$ is a category measure on $X_{\mathscr{B}_\mu}$, [3]. This is true iff the projection dual to the injection $C(X_{\mathscr{B}_\mu}) \subset L^\infty(X_{\mathscr{B}_\mu}, \tilde{\mu})$ is irreducible. This yields a method for constructing $\hat{\mathscr{B}}^\mu$ which are d_μ-complete, yet such that $\mathscr{B} \Delta \hat{\eta}_\mu$ isn't d_μ-complete no matter how $\mathscr{B}$ is represented as an algebra of sets. One need only take an irreducible totally disconnected image Y of the maximal ideal space Z of $L^\infty(\Omega, \Sigma, P)$ where (Ω, Σ, P) is a probability measure space. Letting $\mathscr{B}$ be the clopen algebra of Y one has $Y = X_{\mathscr{B}}$. One may take $X(= j_{\mathscr{B}}(X))$ any dense subset of $X_{\mathscr{B}}$ regarding $\mathscr{B}$ now to be equal to it's trace on X. One way to obtain Y from Z is to identify two nonisolated points in Z (or even to identify a closed nowhere dense subset of Z).

COROLLARY 8.2. *There exists a set X, a Boolean algebra $\mathscr{B}$ of subsets of X and a strictly positive finitely additive probability μ on $\mathscr{B}$ so that $\mathscr{B}_\mu$ isn't d_μ-complete yet $(\hat{\mathscr{B}}^\mu)_\mu$ is d_μ-complete.*

The completion $\mathscr{B}_{\tilde{\mu}}^{0}$ of $\mathscr{B}_{\mu}$ under $d_{\tilde{\mu}}$ is a complete metrizable abelian topological group when the group operation is symmetric difference. Since $\mathscr{B}_{\mu}$ is a dense subgroup of $\mathscr{B}_{\tilde{\mu}}^{0}$ the regular open algebra of $\mathscr{B}_{\mu}$ is isomorphic to that of $\mathscr{B}_{\tilde{\mu}}^{0}$, [18], [20]. If F is a closed subset of $\mathscr{B}_{\mu}$ its interior is the intersection of $\bar{F}^{0}$ with $\mathscr{B}_{\mu}$ where closure and interior are taken in $\mathscr{B}_{\tilde{\mu}}^{0}$. Thus, F is nowhere dense in $\mathscr{B}_{\mu}$ iff F is nowhere dense in $\mathscr{B}_{\tilde{\mu}}^{0}$. Thus, $\mathscr{B}_{\mu}$ is meager in itself iff it is meager in $\mathscr{B}_{\tilde{\mu}}^{0}$. When $\mathscr{B}_{\mu}$ is incomplete yet nonmeager it must be badly behaved as a subset of $\mathscr{B}_{\tilde{\mu}}^{0}$. In Kelley [16, Problem 6P] it is shown that any nonmeager dense subgroup of a Baire topological group fails to have the property of Baire.

PROPOSITION 9. *If $\mathscr{B}_{\mu}$ is not complete then it either*
(a) *is meager in itself under d_{μ} or*
(b) *fails to have the property of Baire in its d_{μ}-completion.*

When $\mathscr{B}_{\mu}$ is d_{μ}-incomplete it may be meager. One instance is when $\mathscr{B}_{\mu}$ is countably infinite in particular when $\mathscr{B}$ is countable and $\mathscr{B}_{\mu}$ is infinite. In this case each point of $\mathscr{B}_{\mu}$ is nowhere dense hence $\mathscr{B}$ is meager. In quite a few instances $\mathscr{B}_{\mu}$ will be meager.

PROPOSITION 10. *Let $\mu \in BA_{1}^{+}(\mathscr{B})$. If $A \in \mathscr{B}_{\mu}$ (or $\mathscr{B}$) let $\mathscr{I}(A) = \{A' \in \mathscr{B}_{\mu}: A' \subset A\}$ and let $\mathscr{F}(A) = \{A' \in \mathscr{B}_{\mu}: A \subset A'\}$ be the principal ideal and filter in $\mathscr{B}_{\mu}$ generated by A.*
(a) *Both $\mathscr{F}(A)$ and $\mathscr{I}(A)$ are d_{μ}-closed.*
(b) *$\mathscr{I}(A)$ is nowhere dense iff A^{c} isn't a finite union of μ-atoms and is open if A^{c} is a finite union of μ-atoms.*
(c) *$\mathscr{F}(A)$ is nowhere dense iff A isn't a finite union of μ-atoms and is open if A is a finite union of μ-atoms.*

Proof. Only statements about $\mathscr{I}(A)$ need be proven for the statements about $\mathscr{F}(A)$ follow from those for $\mathscr{I}(A)$ upon applying the isometry $E \to E^{c}$.

(a) To show that $\mathscr{I}(A)$ is d_{μ}-closed consider a sequence $\{A_{n}\} \subset \mathscr{I}(A)$ converging to $C \in \mathscr{B}_{\mu}$. We have $\mu(C \backslash A_{n}) = \mu(C \backslash A) + \mu(C \cap (A \backslash A_{n})) \geqq \mu(C \backslash A)$. From $\lim_{n \to \infty} \mu(C \backslash A_{n}) = 0$ it follows that $\mu(C \backslash A) = 0$ so $C \in \mathscr{I}(A)$. This establishes (a).

(b) If A^{c} is a finite union of atoms then $\mathscr{B}_{\mu} = \cup \{\mathscr{I}(A) \Delta F: F \subset A^{c}\}$, where $\mathscr{I}(A) \Delta F = \{E \Delta F: E \in \mathscr{I}(A)\}$, is a finite disjoint union. The map $E \to E \Delta F$ is an isometry of $\mathscr{B}_{\mu}$ for d_{μ}. Thus, $\mathscr{I}(A) \Delta F$ is a closed set for each $F \subset A^{c}$. Since $\mathscr{B}_{\mu}$ is a finite union of disjoint closed sets each is a clopen set. Thus, $\mathscr{I}(A)$ is clopen.

Conversely, if A^{c} is not a finite union of atoms there are $F \subset \mathscr{B}_{\mu}$ $F \subset A^{c}$ with $\mu(F) > 0$ but arbitrarily small. If $A' \in \mathscr{I}(A)$ then

$d_\mu(A', A' \cup F)$ is arbitrarily small yet $A' \cup F \notin \mathscr{I}(A)$. Thus, no $A' \in \mathscr{I}(A)$ is an interior point of $\mathscr{I}(A)$. Thus, $\mathscr{I}(A)$ is nowhere dense. □

To show that $\mathscr{B}_\mu$ was meager it would suffice to show that there was a countable family $\{A_n\} \subset \mathscr{B}_\mu \backslash \{\phi\}$, with $\mathscr{F}(A_n)$ nowhere dense for all n, with $\mathscr{B}_\mu = \bigcup_{n=1}^\infty \mathscr{F}(A_n)$. That is, $\{A_n\}$ should be a family such that if $A \in \mathscr{B}_\mu$ there is an A_n with $A_n \subset A$ and so that no A_n is a finite union of atoms. A collection $\{A_\alpha\} \subset \mathscr{B}_\mu \backslash \{\phi\}$ such that any $A \in \mathscr{B}_\mu \backslash \{\phi\}$ contains an A_α is called a *pseudo base* of the algebra $\mathscr{B}_\mu$, [21]. Included in any pseudo base for $\mathscr{B}_\mu$ is the, at most countable, collection of atoms. If every $A \in \mathscr{B}_\mu$ contains an atom then the collection of atoms is a pseudo base and is minimal as a pseudo base. This is the case iff $X_{\mathscr{B}_\mu}$ is the closure of its countable set of isolated points iff $X_{\mathscr{B}_\mu}$ is between $N \cup \{\infty\}$ and βN as a compact Hausdorff space.

PROPOSITION 11. *Suppose that $\mathscr{B}_\mu$ is such that there exists an $A \in \mathscr{B}_\mu \backslash \{\phi\}$ not containing a μ-atom and such that the restriction of $\mathscr{B}_\mu$ to A has a countable pseudo base. $\mathscr{B}_\mu$ is meager.*

Proof. Let μ_A be the restriction of μ to A normalized to be a probability measure. $\mathscr{B}_{\mu_A}$ is the restriction of $\mathscr{B}_\mu$ to A. $\mathscr{B}_{\mu_A}$ is meager as the preceding remarks have shown. Let μ_{A^c} be the normalized restriction of μ to A^c. If μ_{A^c} doesn't exist then $\mathscr{B}_\mu = \mathscr{B}_{\mu_A}$ is meager. It is easily verified that $\mathscr{B}_\mu$ may be represented as the product $\mathscr{B}_{\mu_A} \times \mathscr{B}_{\mu_{A^c}}$. Furthermore the metric d_μ is given by $d_\mu((E_1, F_1), (E_2, F_2)) = \mu(A) d_{\mu_A}(E_1, E_2) + \mu(A^c) d_{\mu_{A^c}}(F_1, F_2)$ which yields a topology on $\mathscr{B}_{\mu_A} \times \mathscr{B}_{\mu_{A^c}}$ which is the product topology. Since $\mathscr{B}_{\mu_A}$ is meager so is $\mathscr{B}_{\mu_A} \times \mathscr{B}_{\mu_{A^c}} = \mathscr{B}_\mu$. □

REMARK. Every nonnegligible element of $\hat{\mathscr{B}}^\mu$ contains a nonnegligible element of $\mathscr{B}$ hence this proposition extends to the case of $\hat{\mathscr{B}}^\mu$. We may even extend this proposition to cover the case of the Boolean algebra completion of $\mathscr{B}$ or $\hat{\mathscr{B}}^\mu$.

PROPOSITION 12. *If $\mathscr{B}$ is an infinite Boolean algebra there is a probability measure μ on $\mathscr{B}$ such that $\mathscr{B}_\mu$ is meager, μ may be taken to be non-atomic if $\mathscr{B}$ admits a non-atomic measure and may always be chosen to be atomic otherwise.*

Proof. If $\mathscr{B}$ admits a non-atomic measure μ there is, [4], [24] a countable subalgebra $\mathscr{B}_0$ of $\mathscr{B}$ isomorphic to the clopen algebra of the Cantor set Δ. The algebra $\mathscr{B}_0$ has a countable base hence a countable pseudo base. Let $\Phi\colon X_{\mathscr{B}} \to X_{\mathscr{B}_0} \cong \Delta$ be the canonical surjection. Let $\tilde{\nu}$ be any non-atomic Radon probability measure on $X_{\mathscr{B}_0}$ with support equal to $X_{\mathscr{B}_0}$. Let X be a minimal closed subset of $X_{\mathscr{B}}$ such that $\Phi(Y) = X_{\mathscr{B}_0}$. The map Φ is irreducible on Y, [27], [4], hence Y has a countable pseudo base, [27]. Let $\tilde{\mu}$ be a Radon probability measure on Y (hence on $X_{\mathscr{B}}$) whose image under Φ is $\tilde{\nu}$. As in [4], $\tilde{\mu}$ is non-atomic on $X_{\mathscr{B}}$. Let μ be the measure on $\mathscr{B}$ corresponding to $\tilde{\mu}$ under the Stone correspondence. We have $Y = X_{\mathscr{B}_\mu}$. Since Y has a countable pseudo base and μ is non-atomic it follows from Proposition 11 that $\mathscr{B}_\mu$ is meager.

If $\mathscr{B}$ admits no nonzero non-atomic measure there is no nonzero non-atomic Radon measure on $X_{\mathscr{B}}$ hence $X_{\mathscr{B}}$ is scattered, [27], as is any closed subset. Since $X_{\mathscr{B}}$ is infinite there is a probability $\tilde{\mu} = \sum_{n=1}^{\infty} 2^{-n}\delta_{x_n}$ where $\{x_n\}$ is an infinite sequence in $X_{\mathscr{B}}$. The support Y of $\tilde{\mu}$ is a separable scattered space. If μ is the measure on $\mathscr{B}$ corresponding to $\tilde{\mu}$ under the Stone correspondence then $Y = X_{\mathscr{B}_\mu}$. The algebra $\mathscr{B}_\mu$ is the clopen algebra of Y. Every clopen set in Y contains one of the countable many isolated points. Thus, $\mathscr{B}_\mu$ has a countable pseudobase. □

REMARK. Again if $\mathscr{B}$ is an algebra of sets this proposition is valid for $\hat{\mathscr{B}}^\mu$.

We may improve Proposition 11 to some extent in the following proposition.

PROPOSITION 13. *Let $\mathscr{B}$ be an algebra and μ be a finitely additive probability on $\mathscr{B}$ so that $\mathscr{B}_\mu$ has a nonprincipal ultrafilter with a countable base. $\mathscr{B}_\mu$ is d_μ-meager.*

Proof. Let $\{A_n : n \in N\}$ be a countable base for an ultrafilter $\mathscr{F}$ in $\mathscr{B}_\mu$ so that $A_n \supset A_{n+1}$ for all n and so that $\mu(A_n \backslash A_{n+1}) > 0$ for all n. $\mathscr{F}$ is equal to $\bigcup_{n=1}^{\infty} \mathscr{F}(A_n)$. By Proposition 10 each $\mathscr{F}(A_n)$ is nowhere dense hence $\mathscr{F}$ is meager for d_μ. Consequently, the maximal ideal $\mathscr{I}$ dual to $\mathscr{F}$ is also meager. Since $\mathscr{B}_\mu = \mathscr{F} \cup \mathscr{I}$. $\mathscr{B}_\mu$ is meager. □

PROPOSITION 14. *For any infinite cardinal number m there is a Boolean algebra $\mathscr{B}$ and a finitely additive probability μ on $\mathscr{B}$ so that $\mathscr{B}_\mu$ is meager and has density character m.*

Proof. (The density character of a topological space is the minimum cardinal number of a dense subset.)

Let $\mathscr{B}'$ be the clopen algebra of the maximal ideal space $X_{\mathscr{B}'}$ of $L^\infty(\{0, 1\}^m, \hat{\mu})$ where $\hat{\mu}$ is the coin flip measure. Let $\tilde{\mu}$ be the probability Radon measure on $X_{\mathscr{B}'}$ corresponding to $\hat{\mu}$ under the Banach lattice isomorphism between $\mathscr{M}(X_{\mathscr{B}'})$ and $L^{\infty *}(\{0, 1\}^m, \hat{\mu})$ dual to that between $\mathscr{C}(X_{\mathscr{B}'})$ and $L^\infty(\{0, 1\}^m, \hat{\mu})$. Let μ be the countably additive probability on $\mathscr{B}'$ corresponding to $\tilde{\mu}$ under the Stone correspondence. Consider the cardinal m to be the first ordinal of cardinal m. Let $\hat{A}_\alpha$, for α an ordinal less than m, denote the clopen subset of $\{0, 1\}^m$ consisting of those elements whose αth coordinate is 0. Let A_α be the element of $\mathscr{B}'$ corresponding to $\hat{A}_\alpha$ for ordinals $\alpha < m$. The subalgebra of $\mathscr{B}'$ generated by $\{A_\alpha : \alpha < m\}$ is of cardinality m and is d_μ-dense in $\mathscr{B}'$. Thus, the d_μ density character of $\mathscr{B}'$ is at most m. It is easily verified that $d_\mu(A_\alpha, A_\beta) = 1/2$ for all $\alpha \neq \beta$. Thus, the density character of $\mathscr{B}'$ is at least m. This establishes the (well known) fact that $\mathscr{B}'$ has density character m. The same reasoning shows that $\mathscr{I}(A_\alpha^c)$, the principal ideal in $\mathscr{B}'$ generated by A_α^c has density character m as a closed subset of $\mathscr{B}'$. Choose a decreasing sequence $\{E_n : n \in N\} \subset \mathscr{B}'$ with $E_1 = A_1$ and $\mu(E_j \backslash E_{j+1}) > 0$. Let $\mathscr{F}$ be the filter $\bigcup_{n=1}^\infty \mathscr{F}(E_n)$ and let $\mathscr{I}$ be the ideal dual to $\mathscr{F}$. Let $\mathscr{B}$ be the algebra $\mathscr{F} \cup \mathscr{I}$. From Proposition 13, $\mathscr{B} = \mathscr{B}_\mu$ is d_μ-meager. Since $\mathscr{I}(A_1^c) \subset \mathscr{I}$ there is a closed subset of the metric space $\mathscr{B}$ of density character m. Thus, $\mathscr{B}$ has density character at least m and, since $\mathscr{B} \subset \mathscr{B}'$, the density character of $\mathscr{B}$ is equal to m. □

REMARK. Under this construction μ is never countably additive. Can μ be constructed to be countably additive?

If one wishes to find an algebra $\mathscr{B}$ and a finitely additive probability measure μ on $\mathscr{B}$ so that $\mathscr{B}_\mu$ is not meager for d_μ yet not complete one should choose $\mathscr{B}_\mu$ very large in its d_μ-completion $\mathscr{B}^0_{\tilde{\mu}}$. Considering $\mathscr{B}_\mu$ as a subalgebra of $\mathscr{B}^0_{\tilde{\mu}}$ one has the Stone space $X_{\mathscr{B}_\mu}$ a continuous image of the Stone space $X_{\mathscr{B}^0_{\tilde{\mu}}} \cdot X_{\mathscr{B}_\mu}$ is obtained by identifying points in $X_{\mathscr{B}^0_{\tilde{\mu}}}$. To make $\mathscr{B}_\mu$ large one should identify as few points as possible. For our construction we will start out with a given infinite hyperstonian space Z satisfying the countable chain condition so that Z is the maximal ideal space of $L^\infty(\Omega, \Sigma, P)$ for some probability measure space (Ω, Σ, P) not consisting of finitely many P atoms. We will consider $\tilde{\mu}$ to be the Radon probability measure on Z associated with P and will denote by $\mathscr{B}^0_{\tilde{\mu}}$ the clopen algebra of Z so that $Z = X_{\mathscr{B}^0_{\tilde{\mu}}}$. We will identify finitely many non-isolated points of Z to obtain a totally disconnected Z' whose clopen algebra will be denoted by $\mathscr{B}$. We will again denote by $\tilde{\mu}$ the Radon probability measure on Z' which is the image of $\tilde{\mu}$ under the canonical projection of Z onto Z'. By μ we will mean the finitely additive probability on $\mathscr{B}$ (or $\mathscr{B}^0_{\tilde{\mu}}$) corresponding to $\tilde{\mu}$. Since μ is strictly positive on $\mathscr{B}^0_{\tilde{\mu}}$ and on $\mathscr{B} = \mathscr{B}_\mu$ and $Z' = X_{\mathscr{B}_\mu}$. Consequently, we are in the desired setting for this proposition.

PROPOSITION 15. *Let (Ω, Σ, P) be a (countably additive) probability measure space not consisting of finitely many P-atoms. There is a subalgebra $\tilde{\Sigma}$ of Σ so that $\tilde{\Sigma}_P$ is incomplete, nonmeager for d_P with d_P-completion Σ_P.*

Proof. Assume the notation in the paragraph preceding this proposition. If we show that $\mathscr{B}_\mu$ is d_μ-incomplete we may obtain $\tilde{\Sigma}$ from $\mathscr{B}_\mu \subset \mathscr{B}^0_{\tilde{\mu}} = \Sigma_P$ by using a lifting λ for $L^\infty(\Omega, \Sigma, P)$ and taking $\tilde{\Sigma}$ to be the image of $\mathscr{B}_\mu$ under λ.

Let $\{x_1, \cdots, x_n\}$ be the points identified in Z to get $x \in Z'$. Each of $\{x_1, \cdots, x_n\}$ is an ultrafilter on $\mathscr{B}^0_{\tilde{\mu}}$ which contains elements of $\mathscr{B}^0_{\tilde{\mu}}$ of arbitrarily small μ measure (since each x_i is nonisolated). Let $\mathscr{F}$ be the filter $x_1 \cap \cdots \cap x_n$ which again contains elements of arbitrarily small $\tilde{\mu}$ measure. Let $\mathscr{I}$ be the ideal of $\mathscr{B}^0_{\tilde{\mu}}$ dual to $\mathscr{F}$ so $\mathscr{I} = \{A^c: A \in F\}$. $\mathscr{I}$ is a subgroup of $\mathscr{B}^0_{\tilde{\mu}}$ and is dense for d_μ since $\mathscr{F}$ contains sets of arbitrarily small measure. Thus, $\mathscr{I}$ is either meager or fails to have the property of Baire. $\mathscr{I}$ is a subgroup of $\mathscr{B}^0_{\tilde{\mu}}$ of finite index. This is because $\mathscr{I} = \mathscr{I}_1 \cap \cdots \cap \mathscr{I}_n$ where $\mathscr{I}_j$ is the maximal ideal of $\mathscr{B}^0_{\tilde{\mu}}$ dual to the ultrafilter x_j. No subgroup of $\mathscr{B}^0_{\tilde{\mu}}$ of finite index can be meager. Thus, $\mathscr{I}$ is non meager. The algebra $\mathscr{B}_\mu$ is easily seen to be $\mathscr{I} \cup \mathscr{F}$ hence is a nonmeager, dense, incomplete subgroup of $\mathscr{B}^0_{\tilde{\mu}}$. Thus, $\mathscr{B}_\mu$ fails to have the property of Baire. □

REMARKS. (1) It may be shown that as constructed, P is not countably additive on $\tilde{\Sigma}$ nor is $\tilde{\Sigma}$ complete as a Boolean algebra. (2) Is it true that if the projection of $X_{\tilde{\mathscr{B}}_\mu}$ onto $X_{\mathscr{B}_\mu}$ is irreducible that $\mathscr{B}_\mu$ is nonmeager? We conclude with a variation of Proposition 14 valid for complete Boolean algebras but with density characters restricted to cardinals between $\aleph_0$ and $2^{\aleph_0}$.

PROPOSITION 16. *Let $\mathscr{B}$ be an infinite complete Boolean algebra and m a cardinal number between $\aleph_0$ and $2^{\aleph_0}$. There is a finitely additive probability measure μ on $\mathscr{B}$ such that $\mathscr{B}_\mu$ is d_μ-meager and has density character m.*

Proof The first step of the proof is the construction of a probability measure μ_1 on 2^N so that 2^N has d_{μ_1} density character m. Let $\mathscr{A}_0$ be a free subalgebra of 2^N with m generators (since $m \leqq 2^{\aleph_0}$ $\mathscr{A}_0$ exists). On $\mathscr{A}_0$ let μ_1 be the usual coin toss measure so that each of the m generators of $\mathscr{A}_0$ receives measure 1/2 and so that the generators are μ_1-independent. The density character of $\mathscr{A}_0$ for d_{μ_1} is equal to m. Under any extension of μ_1 to 2^N, 2^N will have d_{μ_1}-density character at least m. If μ_1 is extended to 2^N so that $\mathscr{A}_0$ is d_{μ_1}-dense in 2^N then the density character of 2^N will be equal to m. To accomplish this we extend μ_1 by a transfinite inductive definition. Suppose, for ordinals $\beta < \alpha$, μ_1 has been extended from $\mathscr{A}_0$ to an algebra $\mathscr{A}_\beta$ so that $\mathscr{A}_\gamma \subset \mathscr{A}_\beta \subset 2^N$ if $\gamma < \beta$ and μ_1 when restricted to $\mathscr{A}_\gamma$ from $\mathscr{A}_\beta$ is the extension to $\mathscr{A}_\gamma$ of μ_1 from $\mathscr{A}_0$ and so that $\mathscr{A}_0$ is d_{μ_1}-dense in $\mathscr{A}_\beta$ for all $\beta < \alpha$. If α is a limit ordinal let $\mathscr{A}_\alpha = \bigcup_{\beta<\alpha} \mathscr{A}_\beta$ and let μ_1 be the unique extension to $\mathscr{A}_\alpha$ whose restrictions to $\mathscr{A}_\beta$ are the already given extension of μ_1 for $\beta < \alpha$. It is immediate that $\mathscr{A}_0$ is d_{μ_1}-dense in $\mathscr{A}_\alpha$ in this case. If α is not limit ordinal, β is its predecessor, and if $\mathscr{A}_\beta \neq 2^N$ select an $A \in 2^N \backslash \mathscr{A}_\beta$ and let $\mathscr{A}_\alpha$ be the algebra generated by $\mathscr{A}_\beta$ and A. It is well known that, if $(\mu_1)_*(A)$ and $(\mu_1)_*(A)$ are the outer and inner measures of A with respect to μ_1 on $\mathscr{A}_\beta$, there is an extension of μ_1 to $\mathscr{A}_\alpha$ with $\mu_1(A) = \lambda$ whenever $(\mu_1)_*(A) \leqq \lambda \leqq (\mu_1)_*(A)$. Select an extension μ_1 so that $\mu_1(A) = (\mu_1)_*(A)$. It is easily deduced that A is in the d_{μ_1}-closure of $\mathscr{A}_\beta$ so there is a sequence $\{A_n\} \subset \mathscr{A}_\beta$ with $d_{\mu_1}(A_n, A) \to 0$. From this it follows that $d_{\mu_1}(A_n \cap B, A \cap B) \to 0$ and $d_{\mu_1}(A_n^c \cap B, A^c \cap B) \to 0$ for all $B \in \mathscr{A}_\beta$. Thus, $\mathscr{A}_\beta$ is d_{μ_1}-dense in $\mathscr{A}_\alpha$. Thus, $\mathscr{A}_0$ is d_{μ_1}-dense in $\mathscr{A}_\alpha$. For all ordinals α we have $\mathscr{A}_0 d_{\mu_1}$-dense in $\mathscr{A}_\alpha$. For some ordinal α, $\mathscr{A}_\alpha = 2^N$. At this stage the desired extension has been accomplished.

The second step of the proof is to construct a probability measure μ on 2^N such that 2^N is d_μ-meager with density character m. Let μ_0 be the countably additive measure on N with $\mu_0(\{n\}) = 2^{-n}$ for $n \in N$. Let $\mu = 1/2(\mu_0 + \mu_1)$ where μ_1 is constructed in the preceding paragraph. Since μ is strictly positive on N, Proposition 11 shows that 2^N is d_μ-meager. From the construction of μ_1 it follows that there is a set $\{A_\alpha : \alpha < m\}$ (where m is considered the first ordinal of cardinality m) with $\mu_1(A_\alpha \varDelta A_\beta) = 1/2$ for $\alpha \neq \beta$. Thus, $d_\mu(A_\alpha, A_\beta) = \mu(A_\alpha \varDelta A_\beta) \geqq (1/2)\mu_1(A_\alpha \varDelta A_\beta) = 1/4$. Thus, the density character of 2^N is at least m. Let $\{E_\alpha : \alpha < m\}$ be a d_{μ_1}-dense set in 2^N. Let N_f be the d_{μ_0}-dense set of finite subsets of 2^N. All sets which differ from an E_α by an element of N_f form a d_μ-dense set of cardinality m. Thus, the density character of 2^N under d_μ is at most m, hence is equal to m. This establishes the proposition for the case $\mathscr{B} = 2^N$.

The third step of the proof consists of extending from the case $\mathscr{B} = 2^N$ to the case where $\mathscr{B}$ is an arbitrary complete Boolean algebra. This is done imitating arguments given in [4]. An infinite complete algebra contains an infinite disjoint sequence $\{A_n : n \in N\}$

hence contains a subalgebra isomorphic to the clopen algebra of the Alexandroff compactification, $N \cup \{\infty\}$, of N. There is a continuous surjection from the Stone space, $X_{\mathscr{B}}$, of $\mathscr{B}$ onto $N \cup \{\infty\}$. Thus, by results on projective covers on Gleason spaces, [3], there is a continuous surjection of $X_{\mathscr{B}}$ onto βN the Gleason space of $N \cup \{\infty\}$. Consequently, by results in [4], there is a closed subspace Y of $X_{\mathscr{B}}$ on which the surjection from $X_{\mathscr{B}}$ to βN is a homeomorphism. The closed set Y is the Stone space of the algebra $\mathscr{B}/\mathscr{I}$ where $\mathscr{I}$ is some ideal of $\mathscr{B}$. Thus, there is a Boolean isomorphism $j: \mathscr{B}/\mathscr{I} \rightarrow 2^N$. Let μ denote both the measure constructed in the previous paragraph on 2^N and its pull back under j to $\mathscr{B}/\mathscr{I}$. Let μ also denote the measure on $\mathscr{B}$ obtained by defining $\mathscr{I}$ to consist of μ-negligible sets. $\mathscr{B}/\mathscr{I} = \mathscr{B}_\mu$ is d_μ-meager and has density character m. This complete the proof of the proposition. □

REMARKS. (1) This result is best possible in that on 2^N any measure μ yields density character at most the cardinality, $2^{\aleph_0}$, for 2^N.

(2) Can higher cardinals be obtained for d_μ-density character of sufficiently large complete Boolean algebras $\mathscr{B}$ with $\mathscr{B}_\mu$ d_μ-meager?

(3) There is no hope, by Proposition 2, that μ can be constructed in a countably additive fashion. This is because $\mathscr{B}_\mu$ as the quotient of a complete algebra by an ideal is an F-algebra, [4], which satisfies the countable chain condition hence is complete.

(4) The measure μ constructed in Proposition 16 is non-atomic. Candeloro and Sacchetti, [10] in the proof of Theorem 2.4 show that if $\mathscr{B}$ is 2^X and μ is non-atomic there is a σ-algebra $\mathscr{A}$ of subsets of X such that $\mathscr{A}$ under d_μ is homeomorphic to $\{0, 1\}^N$. Thus, $\mathscr{B}_\mu$ while d_μ-meager is fairly large.

(5) Seever in [26] shows that the Vitali-Hahn-Saks theorem is valid for finitely additive measures on $\mathscr{B}_\mu$ if $\mathscr{B}_\mu$ is σ-complete. Labuda, [17], shows that the Vitali-Hahn-Saks theorem is true when $\mathscr{B}_\mu$ isn't d_μ-meager. Propositions 15 and 16 demonstrates the independence of their results.

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