PACIFIC JOURNAL OF MATHEMATICS
Vol. 100, No. 2, 1982

MAXIMAL FUNCTIONS FOR A SEMIFLOW IN AN INFINITE MEASURE SPACE

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Let $(X, \mathscr{F}, \mu)$ be a σ-finite measure space and $\Gamma = (\theta_t: t \geqq 0)$ a measurable semiflow of measure preserving transformations on $(X, \mathscr{F}, \mu)$. The *maximal function* f^* of a function $f \in L_1(\mu) + L_\infty(\mu)$ is defined by

$$f^*(x) = \sup_{b>0} \frac{1}{b}\int_0^b |f(\theta_t x)|\, dt .$$

The purpose of this paper is to prove that if $\mu(X) = \infty$ and the semiflow Γ is conservative and ergodic then for every constant $\alpha > 0$

$$\alpha\mu\{f^* > \alpha\} = \int_{\{f^*>\alpha\}} |f|\, d\mu .$$

As a corollary we also prove that if $w \geqq 0$ is a constant then

$$\int_{\{|f|>t\}} |f| \left(\log \frac{|f|}{t} \right)^{w+1} d\mu < \infty \text{ for every } t > 0$$

if and only if

$$\int_{\{f^*>t\}} f^* \left(\log \frac{f^*}{t} \right)^{w} d\mu < \infty \text{ for every } t > 0 .$$

It is well known in ergodic theory (see e.g. [2], Chapter VIII) that if $1 < p \leqq \infty$ and $f \in L_p(\mu)$ then $f^* \in L_p(\mu)$; while if $\mu(X) < \infty$ and $|f| \log^+ |f| \in L_1(\mu)$ then $f^* \in L_1(\mu)$. In 1979 Petersen [8] proved, under the hypothesis $\mu(X) < \infty$, that if $(\theta_t: -\infty < t < \infty)$ is an ergodic flow of measure preserving transformations on $(X, \mathscr{F}, \mu)$ then $f^* \in L_1(\mu)$ implies $|f| \log^+ |f| \in L_1(\mu)$. This is the continuous parameter version of Ornstein's result [7] for an invertible, ergodic and measure preserving transformation on the finite measure space. Petersen proved his result by using a reverse maximal ergodic theorem. Later, Marcus and Petersen [5] improved Petersen's maximal ergodic theorem as follows:

If $\mu(X)=1$, $0 \leqq f \in L_1(\mu)$ and $\alpha \geqq \int f d\mu$, and if $(\theta_t: -\infty < t < \infty)$ is ergodic, then

$$\alpha\mu\{f^* > \alpha\} = \int_{\{f^*>\alpha\}} f d\mu .$$

(A slightly different form in which f need not be nonnegative, may be seen in [5].) In the present paper we shall show that a similar maximal ergodic theorem holds for a conservative and ergodic semiflow $(\theta_t: t \geqq 0)$ on an infinite measure space, and that a ratio version of the Marcus-Petersen maximal ergodic theorem holds.

Our proof uses Derriennic's reverse maximal ergodic theorem [1], together with familiar approximation techniques. The referee has remarked that the proof given in [5] yields also our result for a conservative and ergodic flow on an infinite measure space, and then the semiflow result follows from standard extension techniques. The author thinks that the approach given below remains to be of some interest since it is more direct.

2. Results.

THEOREM 1. *Let $\Gamma = (\theta_t: t \geqq 0)$ be a measurable semiflow of measure preserving transformations on a σ-finite measure space $(X, \mathscr{F}, \mu)$. Assume that Γ is conservative and ergodic. Then the following hold:*

(I) *If $\mu(X) = \infty$ then for any $0 \leqq f \in L_1(\mu) + L_\infty(\mu)$ and any $\alpha > 0$*

$$\alpha\mu\{f^* > \alpha\} = \int_{\{f^*>\alpha\}} f d\mu \; ; \tag{1}$$

(II) *Given $f, g \in L_1(\mu)$ with $f \geqq 0$ a.e. and $g > 0$ a.e., define*

$$f_g^*(x) = \sup_{b>0} \left(\int_0^b f(\theta_t x)dt\right) \Big/ \left(\int_0^b g(\theta_t x)dt\right) .$$

Then for any $\alpha > \int f d\mu \Big/ \int g d\mu$

$$\alpha\int_{\{f_g^*>\alpha\}} g d\mu = \int_{\{f_g^*>\alpha\}} f d\mu \; . \tag{2}$$

For the proof of the above theorem we need two lemmas. Before mentioning these, we note that the function f_g^* may be defined for any $f, g \in L_1(\mu) + L_\infty(\mu)$ with $f \geqq 0$ a.e. and $g > 0$ a.e. With this understanding we have the

LEMMA 1. *Given $f, g \in L_1(\mu) + L_\infty(\mu)$, with $f \geqq 0$ a.e. and $g > 0$ a.e., for any $\alpha > 0$ we have*

$$\alpha\int_{\{f_g^*>\alpha\}} g d\mu \leqq \int_{\{f_g^*>\alpha\}} f d\mu \; .$$

Proof. First suppose $0 \leqq f \in L_1(\mu)$. For $n \geqq 1$, let

$$f_n(x) = 2^n\int_0^{2^{-n}} f(\theta_t x)dt \; , \quad g_n(x) = 2^n\int_0^{2^{-n}} g(\theta_t x)dt$$

and

$$f'_n(x) = \sup_{k\geq 0}\Big(\sum_{i=0}^{k} T_n^i f_n(x)\Big)\Big/\Big(\sum_{i=0}^{k} T_n^i g_n(x)\Big)$$

where $T_n h(x) = h(\theta_{2^{-n}} x)$ for $h \in L_1(\mu) + L_\infty(\mu)$. Then, directly,

$$f'_n \uparrow f_g^* \text{ and } \{f'_n > \alpha\} = \Big\{\sup_{k\geq 0}\sum_{i=0}^{k} T_n^i(f_n - \alpha g_n) > 0\Big\} .$$

Since T_n is a positive contraction on $L_1(\mu)$ and $(f_n - \alpha g_n)^+ \in L_1(\mu)$ because $(f_n - \alpha g_n)^+ \leqq f_n \in L_1(\mu)$, it follows from Garsia's proof (see e.g. [3], p. 23) of the Hopf maximal ergodic lemma that

$$\int_{\{f'_n > \alpha\}}(f_n - \alpha g_n)d\mu \geqq 0 .$$

On the other hand, since $g \in L_1(\mu) + L_\infty(\mu)$, it follows from [6] that $\lim_n g_n(x) = g(x)$ a.e. Furthermore we see that $\lim_n \|f_n - f\|_1 = 0$. Hence, by Fatou's lemma,

$$\begin{aligned}\alpha\int_{\{f_g^* > \alpha\}} g d\mu &\leqq \liminf_n \alpha \int_{\{f'_n > \alpha\}} g_n d\mu \\ &\leqq \lim_n \int_{\{f'_n > \alpha\}} f_n d\mu = \int_{\{f_g^* > \alpha\}} f d\mu .\end{aligned}$$

Next let us consider the general case $0 \leqq f \in L_1(\mu) + L_\infty(\mu)$. Choose ${}_nf \in L_1(\mu)$, $n = 1, 2, \cdots$, so that $0 \leqq {}_nf \uparrow f$. Then ${}_nf_g^* \uparrow f_g^*$, and thus

$$\begin{aligned}\alpha\int_{\{f_g^* > \alpha\}} g d\mu &= \lim_n \alpha\int_{\{{}_nf_g^* > \alpha\}} g d\mu \\ &\leqq \lim_n \int_{\{{}_nf_g^* > \alpha\}} {}_nf d\mu = \int_{\{f_g^* > \alpha\}} f d\mu ,\end{aligned}$$

which completes the proof.

LEMMA 2. *Let θ be a measure preserving transformation on a σ-finite measure space $(X, \mathscr{F}, \mu)$. Given f, $g \in L_1(\mu) + L_\infty(\mu)$ with $f \geqq 0$ a.e. and $g > 0$ a.e., define*

$$f'_g(x) = \sup_{k\geq 0}\Big(\sum_{i=0}^{k} f(\theta^i x)\Big)\Big/\Big(\sum_{i=0}^{k} g(\theta^i x)\Big) .$$

If θ is conservative then for any $\alpha > 0$ we have

$$\int_{[f'_g \leqq \alpha]\cap\{f'_g > \alpha\}} f d\mu \leqq \alpha\int_{[f'_g \leqq \alpha]\cap(\{f'_g > \alpha\}\cup\theta^{-1}\{f'_g > \alpha\})} g d\mu$$

where $[f'_g \leqq \alpha]$ denotes the smallest θ-invariant subset in $\mathscr{F}$ which contains the set $\{f'_g \leqq \alpha\}$.

Proof. See Derriennic [1]. (A minor change is sufficient for the proof.)

Proof of Theorem 1. Since the proof of (I) is similar to that of (II), we only prove (II). For $n \geqq 1$, let f_n, g_n and f'_n be defined as in the proof of Lemma 1. It follows that

$$\lim_n \|f_n - f\|_1 = 0 = \lim_n \|g_n - g\|_1$$

and

$$f'_n \uparrow f^*_g .$$

Denoting by $_n[f'_n \leqq \alpha]$ the smallest subset in $\mathscr{F}$ which is invariant with respect to $\theta_{2^{-n}}$ and contains the set $\{f'_n \leqq \alpha\}$, we have, by Lemma 2, that

$$\int_{_n[f'_n \leqq \alpha] \cap \{f'_n > \alpha\}} f_n d\mu \leqq \alpha \int_{_n[f'_n \leqq \alpha] \cap (\{f'_n > \alpha\} \cup \theta_{2^{-n}}^{-1}\{f'_n > \alpha\})} g_n d\mu .$$

(Recall that the conservativity of the semiflow implies the conservativity of each θ_t.) On the other hand, from an easy continuous version of Hopf's maximal ergodic lemma, we observe that

$$\mu\{f^*_g \leqq \alpha\} > 0 \text{ for every } \alpha > \int f d\mu \Big/ \int g d\mu .$$

Then the ergodicity of the semiflow implies that

$$_n[f^*_g \leqq \alpha] \uparrow X .$$

and therefore

$$\liminf_n {}_n[f'_n \leqq \alpha] = \lim_n {}_n[f^*_g \leqq \alpha] = X .$$

Consequently

$$\int_{\{f^*_g > \alpha\}} f d\mu = \lim_n \int_{_n[f'_n \leqq \alpha] \cap \{f'_n > \alpha\}} f_n d\mu .$$

Similarly

$$\lim_n \int_{_n[f'_n \leqq \alpha] \cap (\{f'_n > \alpha\} \cup \theta_{2^{-n}}^{-1}\{f'_n > \alpha\})} g_n d\mu = \lim_n \int_{\{f'_n > \alpha\} \cup \theta_{2^{-n}}^{-1}\{f'_n > \alpha\}} g d\mu ,$$

and let us prove that

$$\lim_n \int_{\{f'_n > \alpha\} \cup \theta_{2^{-n}}^{-1}\{f'_n > \alpha\}} g d\mu = \int_{\{f^*_g > \alpha\}} g d\mu .$$

To do this, given an $\varepsilon > 0$, choose $E \in \mathscr{F}$ so that

$$\mu(E) < \infty \quad \text{and} \quad \int_{X-E} g d\mu < \varepsilon .$$

Next put $E' = \{f_g^* > \alpha\} \cap E$, $E'_n = \{f'_n > \alpha\} \cap E$ and $E''_n = \{f'_n > \alpha\} \cap (X - E)$. It is easily seen that

$$\lim_n \mu(E \Delta \theta_{2^{-n}}^{-1} E) = \lim_n \| 1_E - 1_E \circ \theta_{2^{-n}} \|_1 = 0 ,$$
$$\lim_n \mu(E' \Delta \theta_{2^{-n}}^{-1} E') = 0 ,$$
$$\lim_n \mu(E' - E'_n) = 0 ,$$

and

$$\theta_{2^{-n}}^{-1} E''_n \subset \theta_{2^{-n}}^{-1}(X - E) = X - \theta_{2^{-n}}^{-1} E$$
$$\subset (X - E) \cup (E \Delta \theta_{2^{-n}}^{-1} E) .$$

Hence

$$\limsup_n \int_{(\theta_{2^{-n}}^{-1}\{f'_n > \alpha\}) - \{f'_n > \alpha\}} g d\mu \leqq \int_{X-E} g d\mu < \varepsilon ,$$

and since $\varepsilon > 0$ was arbitrary,

$$\lim_n \int_{\{f'_n > \alpha\} \cup \theta_{2^{-n}}^{-1}\{f'_n > \alpha\}} g d\mu = \lim_n \int_{\{f'_n > \alpha\}} g d\mu = \int_{\{f_g^* > \alpha\}} g d\mu ,$$

which shows that

$$\int_{\{f_g^* > \alpha\}} f d\mu \leqq \alpha \int_{\{f_g^* > \alpha\}} g d\mu .$$

This, together with Lemma 1, completes the proof of (II).

As a corollary of Theorem 1 we obtain the following dominated ergodic theorem.

THEOREM 2. *Let $\Gamma = (\theta_t \colon t \geqq 0)$ be conservative and ergodic, and let $w \geqq 0$ be a constant. Then the following hold*:

(I) *If $\mu(X) = \infty$ and $0 \leqq f \in L_1(\mu) + L_\infty(\mu)$, then*

$$\int_{\{f > t\}} f \left(\log \frac{f}{t}\right)^{w+1} d\mu < \infty \text{ for every } t > 0$$

if and only if

$$\int_{\{f^* > t\}} f^* \left(\log \frac{f^*}{t}\right)^{w} d\mu < \infty \text{ for every } t > 0 ;$$

(II) *Given $f, g \in L_1(\mu)$, with $f \geqq 0$ a.e. and $g > 0$ a.e.,*

$$\int_{\{f > g\}} f \left(\log \frac{f}{g}\right)^{w+1} d\mu < \infty$$

if and only if

$$\int_{\{f_g^*>1\}} f_g^*(\log f_g^*)^w g d\mu < \infty \ .$$

Proof. For the proof of (I) it suffices to consider the case $t=1$. Then, by Fubini's theorem and the above theorem, we have

$$\begin{aligned}
\int_{\{f^*>1\}} f^*[\log f^*]^w d\mu &= \int_{\{f^*>1\}} \int_1^{f^*(x)} [(\log t)^w + w(\log t)^{w-1}] dt d\mu(x) \\
&= \int_1^\infty [(\log t)^w + w(\log t)^{w+1}] \mu\{f^* > t\} dt \\
&= \int_1^\infty \frac{1}{t} [(\log t)^w + w(\log t)^{w-1}] \int_{\{f^*>t\}} f d\mu dt \\
&\geqq \int_1^\infty \frac{1}{t} (\log t)^w \int_{\{f>t\}} f d\mu dt \\
&= \int_{\{f>1\}} f(x) \int_1^{f(x)} \frac{1}{t} (\log t)^w dt d\mu(x) \\
&= \frac{1}{w+1} \int_{\{f>1\}} f(\log f)^{w+1} d\mu
\end{aligned}$$

On the other hand, Lemma 1 together with a standard argument gives

$$\mu\{f^* > t\} \leqq \frac{2}{t} \int_{\{f>t/2\}} f d\mu \ \text{ for every } t > 0 \ ,$$

thus

$$\begin{aligned}
&\int_{\{f^*>1\}} f^*[\log f^*]^w d\mu \\
&\qquad \leqq \int_1^\infty \frac{2}{t} [(\log t)^w + w(\log t)^{w-1}] \int_{\{f>t/2\}} f d\mu dt \\
&\qquad = \int_{\{2f>1\}} 2f[(w+1)^{-1}(\log 2f)^{w+1} + (\log 2f)^w] d\mu \ ,
\end{aligned}$$

and the last integral is finite if the first condition of (I) is satisfied. So (I) is proved.

The proof of (II) may be done similarly and omitted.

REMARK. The linear subclasses $R_w(\mu)$, $w \geqq 0$, of $L_1(\mu) + L_\infty(\mu)$ defined by

$$R_w(\mu) = \left\{ f \colon \int_{\{|f|>t\}} |f| \left(\log \frac{|f|}{t}\right)^w d\mu < \infty \ \text{ for all } \ t > 0 \right\}$$

were originally introduced by Fava [3] in order to obtain a weak type inequality for a product of maximal operators. In view of [3], [6] and [10], these subclasses are important in pointwise ergodic theory.

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Received August 20, 1980.
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