PACIFIC JOURNAL OF MATHEMATICS
Vol. 101, No. 1, 1982

THE POINTS OF LOCAL NONCONVEXITY OF STARSHAPED SETS

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The notion of point of local nonconvexity has been an important tool in the study of the geometry of nonconvex sets, since Tietze characterized, more than fifty years ago, the convex subsets of E^n as those connected sets without points of local nonconvexity. It is proved here that for each convex component K of a closed connected set S in a locally convex space there exist points of local nonconvexity of S arbitrarily close to K, unless S itself be convex. Klee's generalization of the just quoted Tietze's theorem follows immediately. The notion of "higher visibility" is introduced in the last section, and three Krasnosselsky-type theorems involving the points of local nonconvexity are proved.

1. Notations and basic definitions.

The interior, closure, boundary and convex hull of a set S are denoted by int S, cl S, bdry S and conv S, respectively. The closed segment joining x and y is denoted $[xy]$. If $x \in S$ and $y \in S$, we say that x sees y *via* S if $[xy] \subset S$. The *star* of x with respect to S is the set $st(x; S)$ of all points of S that see x via S. A *star-center* of S is a point $x \in S$ such that $st(x; S) = S$, that is a point of S that sees the whole S. The *kernel* of S is the set ker S of all the star-centers of S. S is *starshaped* if $\ker S \neq \emptyset$. A *convex component* of S is a maximal convex subset of S. The point $x \in \operatorname{bdry} S$ is a *point of local nonconvexity* of S if for every neighborhood U of x, the set $U' = U \cap S$ is not convex. The set of all points of local nonconvexity of S is denoted lnc S. The origin (null-vector) of a linear topological space is denoted by θ, and the family of its neighborhoods by $\mathscr{N}(\theta)$.

2. Points of local nonconvexity and convex components.

THEOREM 2.1. *Let S be a closed connected nonconvex set in a locally convex linear topological space, and K be a convex component of S. Then $(K + V) \cap \operatorname{lnc} S \neq \emptyset$ for each $V \in \mathscr{N}(\theta)$.*

Proof. It is clear that K is closed and, a fortiori, closed in the relative topology of S. Assume there exists a $V \in \mathscr{N}(\theta)$ such that $(K + V) \cap \operatorname{lnc} S = \emptyset$. If $x \in K$ there must exist a $U \in \mathscr{N}(\theta)$ such that $U \subset V$ and $U' = (U + x) \cap S$ be convex. We intend to prove that $U' \subset K$. On the contrary, suppose there exist $y \in U'$ and $z \in K$ such that y does not see z via S. Let y_0 be the last point of $[xz]$

(going from x to z) that is visible from y. By Lemma 1 of [7], it is easy to verify that $[yy_0] \cap \operatorname{bdry} S$ would contain a point $p \in \operatorname{lnc} S$. But then $p \in K + U \subset K + V$, in contradiction with our basic assumption. Hence there are no such points y and z. That is $\forall y \in U'$ and $\forall z \in K$, $[yz] \subset S$. This implies that $\operatorname{conv}(U' \cup K) \subset S$, and by the maximality of K, $\operatorname{conv}(U' \cup K) = K$ and $U' \subset K$. Since this is true for every $x \in K$, K is open in the relative topology of S. The connectedness of S implies that $K = S$, a fact that contradicts the nonconvexity of S. Hence no such V can exist.

We are tempted to substitute the thesis of 2.1 by the stronger statement "$K \cap \operatorname{lnc} S \neq \varnothing$". Unfortunately this is false, as the following counterexample shows. If we define $S = \{(x; y) \in R^2 \mid y \leqq |x|^{-1}\}$ then $K = \{(x; y) \mid y \leqq 0\}$ is a convex component of S but $K \cap \operatorname{lnc} S = \varnothing$. The next corollary considers a situation where this stronger statement holds.

COROLLARY 2.2. *Let S be a closed connected set in a locally convex linear topological space such that* $\operatorname{lnc} S$ *be compact or empty, and let K be a convex component of S. Then the following statements are equivalent*: (i) $K = S$ (ii) $K \cap \operatorname{lnc} S = \varnothing$.

Proof. Clearly (i) implies (ii). On the other hand, assume that $K \cap \operatorname{lnc} S = \varnothing$. We intend to prove the existence of a neighborhood V_0 of θ such that $(K + V_0) \cap \operatorname{lnc} S = \varnothing$. The inexistence of a neighborhood would allow us to pick a net $\{t_V, V \in \mathscr{N}(\theta)\}$ in $\operatorname{lnc} S$ such that for every $V \in \mathscr{N}(\theta)$ $t_V \in (K + V) \cap \operatorname{lnc} S$. The compactness of $\operatorname{lnc} S$ would imply the existence of a converging subnet, which in turn would contradict (ii). Hence the existence of V_0 is proved, in contradiction with the thesis of the previous theorem. Then S must be convex and (i) holds.

We conclude this section with a new proof for the classical Tietze-Klee theorem, originally stated in [3].

THEOREM 2.3 (*Tietze-Klee*). *Let S be a closed connected set in a locally convex linear topological space. Then the following statements are equivalent*: (i) $\operatorname{lnc} S = \varnothing$. (ii) *$S$ is convex.*

Proof. It is clear that (ii) implies (i). On the other hand, (i) contradicts the conclusion of Theorem 2.1. Hence S must be convex.

3. Three Krasnosselsky-type theorems. The point p *has higher visibility* via S *than* the point q if $\operatorname{st}(p; S) \supset \operatorname{st}(q; S)$. The relation "has higher visibility via S than" is a partial ordering in S, and the star-centers of S (if there exist such points) should be the maximal elements for this ordering. The *visibility cell* of p is the set $\mathrm{vis}(p)$ of all the points of S having higher visibility via S than p. Of course, $p \in \mathrm{vis}(p)$ always.

LEMMA 3.1. *The visibility cell of p is the intersection of all the convex components of S that include p.*

Proof. Let $x \in \mathrm{vis}(p)$ and K be a convex component of S that includes p. Then $K \subset \mathrm{st}(p; S) \subset st(x; S)$. This inclusion implies that $K' = \mathrm{conv}(\{x\} \cup K) \subset S$, and the maximality of K yields $K = K'$. Hence $x \in K$. Conversely, let x belong to the intersection of all the convex components of S that include p, and let $z \in \mathrm{st}(p; S)$. There is a convex component K_0 of S such that $[zp] \subset K_0$. But $x \in K_0$ by construction. Hence x sees z via S. Since the argument holds for each $z \in \mathrm{st}(p; S)$, $x \in \mathrm{vis}(p)$.

It is important to observe that the preceding characterization of $\mathrm{vis}(p)$ uses no topological structure whatsoever.

THEOREM 3.2. *Let S be a closed connected nonconvex set in a locally convex linear topological space, such that* $\mathrm{lnc}\, S$ *be compact. The kernel of S is the intersection of the visibility cells of all its points of local nonconvexity.*

Proof. Let $A = \cap \{\mathrm{vis}(p) \mid p \in \mathrm{lnc}\, S\}$. Corollary 2.2 and Lemma 3.1 imply that A is the intersection of all the convex components of S. Whence, by the lemma that precedes Theorem 2 of [6], $A = \ker S$.

Three well-known theorems concerning intersections of families of convex sets are quoted here for later reference.

THEOREM 3.3 (*Helly* [2]). *Let $\mathscr{K}$ be a collection of compact convex sets in E^n, containing at least $n+1$ members, and such that each subfamily of $n+1$ members have nonempty intersection. Then, the intersection of all the members of $\mathscr{K}$ is not empty.*

THEOREM 3.4 (*Klee* [4]). *Let $\mathscr{K}$ be a collection of compact a convex sets in E^n, containing at least $n+1$ members, and let C be compact convex set in E^n such that for each subfamily of $n+1$ members of $\mathscr{K}$ there exists a translate of C included in the intersection of the subfamily. Then, there exists a translate of C included* *in the intersection of all the collection* $\mathscr{K}$.

THEOREM 3.5 (*Grünbaum* [1]). *Let n and k be integers such that $n \geqq k > 0$, and let $h(n; k)$ be defined by*: (i) $h(n; n) = n + 1$ (ii) $h(n; 1) = 2n$ (iii) $h(n; k) = 2n - k$ *for* $n > k > 1$. *Let* $\mathscr{K}$ *be a finite collection of convex sets in* E^n *containing at least* $h(n; k)$ *members, and such that each subfamily of* $h(h; k)$ *members has intersection of dimension at least* k. *Then the intersection of all the collection* $\mathscr{K}$ *is of dimension at least* k.

THEOREM 3.6. *Let S be a compact connected nonconvex set in* E^n *such that for every k-pointed subset* $\{t_1; \cdots; t_k\}$ *of* $\operatorname{lnc} S$, *with* $k \leqq n + 1$ *there exists a point having higher visibility via S than each* t_i. *Then S is starshaped.*

Proof. Consider the family $\mathscr{K} = \{\operatorname{vis}(p) \mid p \in \operatorname{lnc} S\}$. By Lemma 3.1 each member is convex and compact, and by hypothesis the intersection of every $n + 1$ members is not empty. Furthermore, $\operatorname{lnc} S$ is closed, hence compact. Then, Theorems 3.2 and 3.3 imply that $\ker S \neq \varnothing$.

THEOREM 3.7. *Let S be a compact connected nonconvex set in* E^n *and assume that there exists* $\delta > 0$ *such that for every k-pointed subset* $A \subset \operatorname{lnc} S$ *with* $k \leqq n + 1$, *there is a ball B of radius δ such that all the points of B have higher visibility via S than each of the points of A. Then the kernel of S contains a ball of radius δ.*

Proof. Let B be a ball of radius δ and $\mathscr{K}$ be the same family as in the previous theorem. Theorems 3.2 and 3.4 imply that $\ker S$ includes a translate of B.

THEOREM 3.8. *Let k and n be positive integers,* $k \leqq n$, *and let* $h(n; k)$ *be defined by*: (i) $h(n; n) = n + 1$; (ii) $h(n; 1) = 2n$; (iii) $h(n; k) = 2n - k$ *for* $n > k > 1$. *Let S be a closed connected nonconvex set in* E^n, *and assume that* $\operatorname{lnc} S$ *is finite and such that for each m-pointed subset A of* $\operatorname{lnc} S$, *with* $m \leqq h(n; k)$ *there are* $k + 1$ *affinely independent points having higher visibility than each of the points of A. Then the kernel of S is of dimension at least k.*

Proof. Consider once more the family of visibility cells of the points of local nonconvexity of S. The hypothesis of Theorem 3.5 holds and Theorem 3.2 implies that $\ker S$ has dimension at least k.

REMARK. Since $\ker S \subset \operatorname{vis}(p)$ for each $p \in S$, the hypothesis of

Theorem 3.6 is not only sufficient but also necessary for the validity of the thesis. The same statement can be made about Theorems 3.7 and 3.8.

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Received September 3, 1980 and in revised form June 10, 1981.

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