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AUTOMORPHISMS OF QUOTIENTS OF $\prod GL(n_i)$

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Quotients of $GL(n)$ by finite subgroups can have radial algebraic automorphisms. More generally, quotients of $\Pi_1^s GL(n_i)$ by $(s-1)$-dimensional central subgroups can have automorphisms not induced by automorphisms of $\Pi GL(n_i)$. This paper works out an explicit description of all their algebraic group automorphisms. As a sample application, the normalizer of the $GL(n)$-action on $\Lambda^r(k^n)$ is computed.

The automorphisms of the general linear groups $GL(n, k)$ over a field k are quite well known [2, 4]. There are first of all the algebraic automorphisms, which (for $n > 2$) are just the inner automorphisms and transpose inverse. There are also the automorphisms induced by automorphisms of k. Finally, there may in some cases be radial automorphisms sending g to $\lambda(g)g$ for scalar $\lambda(g)$. Such radial automorphisms exist only when k has special properties; they cannot be defined systematically over rings containing k—that is to say, they are not algebraic automorphisms. Consequently, I was rather surprised when I observed that certain naturally occurring images of $GL(n)$ (quotients by finite central subgroups) do have algebraic radial automorphisms. The existence of such automorphisms seems not to have been pointed out before. It turns out to be implicit in one familiar context, but there the group is in disguise (see § 3).

In this paper we will work out precisely when such radial algebraic automorphisms exist and what they can be. More generally we will treat quotients $(\prod GL(n_i))/A$ that have one-dimensional center, and we will go on to compute the whole group of algebraic automorphisms. This will be interesting because a number of outer automorphisms here require appropriate scalar factors in their definition and are not simply induced by automorphisms of $\prod GL(n_i)$. The exact result also is useful when one wants to find the normalizers of these groups in larger ones, and we will conclude with a detailed example of such an application.

For brevity "group" will mean an algebraic group over a field k, and "homomorphism" will mean an algebraic homomorphism. More precisely, we will treat our objects as affine group schemes [5]. The groups $\prod GL(n_i)/A$ that we really care about will have the same automorphisms in any version of algebraic group theory, since they are smooth (and indeed are determined by their k-rational points when k is infinite); but the use of group schemes offers certain technical advantages. Most notably, it provides us with kernels even for inseparable homomorphisms, so that for instance a quotient map with trivial group scheme kernel is an isomorphism [5, § 15.4]. Any reader unfamiliar with group schemes may simply assume $\operatorname{char}(k) = 0$; this will involve no serious loss, because the whole point of using group schemes is that they allow the same arguments to work in all characteristics.

1. Radial automorphisms. We begin with the algebraic group $GL(n_1) \times \cdots \times GL(n_s)$. Its center C is $G_m \times \cdots \times G_m$. Its commutator subgroup is $SL(n_1) \times \cdots \times SL(n_s)$, so its abelianization H is again $\cong G_m \times \cdots \times G_m$. The map $\prod GL(n_i) \to H = \prod G_m$ is given by the determinant maps on each factor, and thus the induced map $C \to H$ raises scalars in the ith factor to the n_i power. The groups we study are those of the form $G = (\prod GL(n_i))/A$, where A is a subgroup of the center C. We write elements in $\prod GL(n_i)$ as g or $\langle g_i \rangle$, with $[g]$ or $[\langle g_i \rangle]$ for the typical image element in G. The center of G is C/A; that is, its elements come from scalars in the factors of $\prod GL(n_i)$.

Our concern in this section is to find the radial automorphisms of the algebraic group G; by this we mean those of the form $[g] \mapsto \lambda([g])[g]$, where each $\lambda([g])$ is an element of the center C/A. It is trivial to compute that a function of this form preserves multiplication in G iff $\lambda\colon G \to C/A$ is a homomorphism. Thus we must begin by computing $\operatorname{Hom}(G, C/A)$, which we do using character groups.

The character group $X = \operatorname{Hom}(C, G_m)$ of C is a free abelian group with basis $e_1, \cdots, e_s$ given by the projections of C onto its factors. As $C \to H$ is an epimorphism, we may identify $Y = \operatorname{Hom}(H, G_m)$ with a subgroup of X; it is the subgroup generated by the $n_i e_i$. The character group of C/A is similarly identified with a subgroup V of X, the subgroup of those characters vanishing on A. Our group G, which is determined by specifying A, is equally well determined by specifying the subgroup V of X.

Now a homomorphism $\lambda\colon G \to C/A$ is the same as a homomorphism $\prod GL(n_i) \to C/A$ vanishing on A. Any such homomorphism must factor through the abelianization H, and thus it corresponds to a map $Y \leftarrow V$ of character groups. For it to vanish on A, the image of the character map must again be contained in V. Thus radial endomorphisms of G correspond to abelian group maps $\varphi\colon V \to V \cap Y$. The condition that $\varphi(V)$ be in Y is just a divisibility condition which is equivalent to saying that $\phi = \operatorname{diag}(n_1, \cdots, n_s)\psi$

for some $\psi\colon V \to X$.

We now must determine when $[g] \mapsto \lambda([g])[g]$ is an automorphism. If its kernel is trivial, its image will have the same dimension as G, and this will force the image to equal G, since G is smooth and connected. Thus we only need to worry about the kernel. Clearly $\lambda([g])[g] = [e]$ forces $[g]$ to be central, so the kernel is contained in C/A. What we need then is that the map $C \to C/A$ given by $\lambda(g)[g]$ have kernel precisely A. When dualized to the character groups, this says that $v \mapsto \varphi(v) + v$ should have image precisely V. That condition automatically implies $\varphi(V) \subseteqq V$, so we can drop this from our requirements on φ. The result is the following.

THEOREM 1. *Let A be a closed subgroup of the center C of $GL(n_1) \times \cdots \times GL(n_s)$, and let $G = \prod GL(n_i)/A$. Let $V \subseteqq \boldsymbol{Z}^s = \mathrm{Hom}\,(C, G_m)$ be the characters vanishing on A. Then the radial algebraic automorphisms of G correspond to the linear maps $\psi\colon V \to \boldsymbol{Z}^s$ for which $v \mapsto v + \mathrm{diag}\,(n_1, \cdots, n_s)\psi(v)$ is an automorphism of V.* $\square$

The rank of the abelian group V is equal to the dimension of C/A, the center of G. Whenever this rank is bigger than one, it is easy to see (using the theorem) that there are infinitely many radial automorphisms (cf. [1, p. 141]). It is also true that in this case we have been stretching the meaning of "radial", because we have allowed arbitrary multipliers from the center, and they are not really pure scalars. From now on, therefore, we consider only the case where the center of G is (one-dimensional and hence) isomorphic to the multiplicative group G_m. In this case V is specified by giving one spanning element $w = \sum r_i e_i$. The map φ is determined by $\varphi(w)$, which must have the form $\sum q_i n_i e_i$ for some integers q_i. If $v \mapsto v + \varphi(v)$ is to be an automorphism of V, then $w + \varphi(w)$ must be either w or $-w$. The first possibility implies $\varphi \equiv 0$ and corresponds to the trivial automorphism of G. The second possibility is $\varphi(w) = -2w$, which means $q_i n_i = -2r_i$ for each i. The q_i here are thus uniquely determined (as $-2r_i/n_i$), and the possibility for φ is realized only if these numbers are all integers. Translating everything back into group terms, we have reached the following result.

THEOREM 2. *Let A be a central subgroup of $\prod GL(n_i)$, and assume that $G = \prod GL(n_i)/A$ has a center of dimension one. Identify that center with G_m, and let the map $\prod G_m \to G_m$ induced on centers by $\prod GL(n_i) \to G$ be given by $\langle a_i \rangle \mapsto \prod a_i^{r_i}$. Then G has at* *most one nontrivial radial algebraic automorphism. Such an automorphism exists iff $2r_i$ is divisible by n_i for each index i. When it exists, it is given by*

$$[\langle g_i \rangle] \longmapsto (\det (g_i)^{-2r_i/n_i})[\langle g_i \rangle] \, ,$$

where the scalar is interpreted as an element of the central G_m in G. □

The simplest example is $G = GL(n)/\mu_r$, where μ_r is the rth roots of unity; here there is a radial automorphism when $2r$ is divisible by n. The case $n = 1$ is included, and there we just get the inverse map on G.

2. The outer automorphism group. In this section we assume still that our algebraic group $G = \prod GL(n_i)/A$ has one-dimensional center, so the characters of C vanishing on A are the multiples of some single $w = \sum r_i e_i$. Under this assumption we will compute the exact automorphism group of our algebraic group.

DEFINITION. An *automorphism type* for G is a family $\langle \sigma, \{d_i\}_{i=1}^s, d \rangle$ where

(1) σ is a permutation of $\{1, \cdots, s\}$ satisfying $n_{\sigma(i)} = n_i$ for all i and $\sigma(i) = i$ when $n_i = 1$,

(2) the values of d_i and d are ± 1 with $d_i = 1$ when n_i is 1 or 2, and

(3) $r_{\sigma(i)} \equiv d d_i r_i \pmod{n_i}$ for all i.

The *product* of two automorphism types $\langle \sigma, \{d_i\}, d \rangle$ and $\langle \tau, \{c_i\}, c \rangle$ is $\langle \sigma\tau, \{c_i d_{\tau(i)}\}, cd \rangle$.

Simple computation gives the following result:

LEMMA. *The automorphism types for G form a group. Mapping each type to its permutation component σ is a homomorphism; the kernel is an elementary abelian 2-group of order 2^{t+1}, where t is the number of indices i with $n_i > 2$ and $2r_i \equiv 0 \pmod{n_i}$. There is a complementary subgroup formed by all types that have $d = 1$ and $d_i = 1$ for each i satisfying $2r_i \equiv 0 \pmod{n_i}$.* □

DEFINITION. The *standard outer automorphism* of G of type $\langle \sigma, \{d_i\}, d \rangle$ is the map sending $[\langle g_i \rangle]$ to $\lambda[\langle g_i \rangle]\, [\langle h_i \rangle]$ where

$$h_{\sigma(i)} = \begin{cases} g_i & \text{if } \ d_i = 1 \\ (g_i^{tr})^{-1} & \text{if } \ d_i = -1 \end{cases}$$

and $\lambda[\langle g_i \rangle]$ is the scalar $\prod \det(g_i)^{q_i}$ with $q_i = (dr_i - d_i r_{\sigma(i)})/n_i$.

Of course it is not clear in advance that these maps are automorphisms, or even that they are well defined. That is part of our main theorem, which we are now ready to state.

THEOREM 3. *Let G be as in Theorem 2.*

(1) *The inner automorphisms of G form a group isomorphic to $\prod PGL(n_i, k)$.*

(2) *The standard outer automorphisms are indeed automorphisms, and they form a group isomorphic to the group of automorphism types for G.*

(3) *The group of all algebraic automorphisms of G is the semidirect product of the inner automorphisms and the standard outer automorphisms.*

Proof. Consider first the inner automorphisms. An element of $G(k)$ gives a trivial automorphism iff it lies in the center C/A. We have $G/(C/A) \cong \prod GL(n_i)/C = \prod PGL(n_i)$: that is, we have the exact sequence

$$1 \longrightarrow C/A \longrightarrow G \longrightarrow \prod PGL(n_i) \longrightarrow 1$$

of algebraic groups. In general this would not imply that $G(k) \to \prod PGL(n_i, k)$ is surjective, but by [5, §18.1] it is so here because $C/A \cong G_m$ and $H^1(\bar{k}/k, G_m)$ is trivial. Thus (1) is proved. We see then that every inner automorphism of $\prod PGL(n_i)$ lifts in just one way to an inner automorphism of G.

Every automorphism of G preserves its center and thus induces an automorphism of $\prod PGL(n_i)$. This product is semisimple, and its automorphisms are well known: the outer ones correspond to "graph automorphisms" of the root system. Explicitly, they are given by permuting factors of the same dimension and by taking transpose inverses of various factors (for $n_i > 2$; on $PGL(2)$ the transpose inverse map is an inner automorphism). Such an automorphism then is described by a permutation σ (with $n_{\sigma(i)} = n_i$ for all i and $\sigma(i) = i$ when $n_i = 1$) and a set of values $d_i = \pm 1$ with $d_i = -1$ representing the transpose inverse operation on the ith factor. We do the transpose inverse operations before the permutation of factors, though of course we could equally well adopt the convention of doing them in the other order.

As we have already seen, if we change an automorphism of G

by an appropriate inner automorphism, we can change its effect on $\prod PGL(n_i)$ by an arbitrary inner automorphism there. Hence if any automorphism of G induces an automorphism of $\prod PGL(n_i)$ that is in the outer automorphism class $\langle\sigma, \{d_i\}\rangle$, we can change it to make it induce precisely the explicit outer automorphism described above. Our problem then is to determine the liftings (if any) of such explicit outer automorphisms to automorphisms of G. They of course have obvious explicit liftings to $\prod GL(n_i)$, and we denote those again by $\langle\sigma, \{d_i\}\rangle$. Identifying the center of G with G_m, we see that a lifting to G will be given by a homomorphism λ from $\prod GL(n_i)$ to G_m such that the map $\prod GL(n_i) \to G$ given by

$$g \longmapsto \lambda(g)[\langle\sigma, \{d_i\}\rangle g]$$

has kernel precisely A. As in the previous section, the kernel is obviously central, and we analyze it on character groups.

For $g = \langle g_i\rangle$ in the center C of $\prod GL(n_i)$, transpose inverse on a factor is simply inverse. Thus the $\sigma(i)$-coordinate of $\langle\sigma, \{d_i\}\rangle g$ is $g_i^{d_i}$. Hence the dual map on character groups sends $e_{\sigma(i)}$ to $d_i e_i$. A homomorphism $\lambda(g) = \prod \det(g_i)^{q_i}$ corresponds to $\varphi(w) = \sum q_i n_i e_i$. Thus for our map to have kernel precisely A we need

$$\sum q_i n_i e_i + \sum r_{\sigma(i)} d_i e_i = \pm\sum r_i e_i \ .$$

If we set d equal to the ± 1 on the right, we see that our lifting is determined by the data $\langle\sigma, \{d_i\}, d\rangle$, and that it exists provided $r_{\sigma(i)} d_i \equiv d r_i \pmod{n_i}$. As $d_i = \pm 1$, this agrees with the condition defining automorphism types for G, and the automorphism thus determined is what we called the standard outer automorphism of this type.

All that remains is to compute the composite of two standard outer automorphisms, say type $\langle\tau, \{c_i\}, c\rangle$ followed by type $\langle\sigma, \{d_i\}, d\rangle$. Take an element $[\langle g_i\rangle]$ in G. To apply the first automorphism, we begin by forming the element that in the $\tau(i)$ place has $g_i^{c_i}$ (where for brevity we indicate transpose inverse action just by the exponent); then we multiply the class of this by the scalar

$$\prod (\det g_i)^{q_i} = \prod (\det g_i)^{(cr_i - c_i r_{\tau(i)})/n_i} \ .$$

Now it suffices to do this computation on elements defined over the algebraic closure $\bar{k}$, and there we can take roots of scalars and thus absorb them into the $GL(n_i)$-factors. Specifically, recalling that our projection to G raises a scalar in the ith factor to the r_ith power, we absorb $(\det g_i)^{cr_i/n_i}$ into the ith factor and $(\det g_i)^{-c_i r_{\tau(i)}/n_i}$ into the $\tau(i)$th factor. Thus we can say that our standard automorphism applied to $[\langle g_i \rangle]$ gives $[\langle h_i \rangle]$ where

$$h_{\tau(i)} = g_i^{c_i}(\det g_{\tau(i)})^{c/n_i} (\det g_i)^{-c_i/n_i} .$$

We have then

$$\begin{aligned} \det h_{\tau(i)} &= (\det g_i)^{c_i} (\det g_{\tau(i)})^c (\det g_i)^{-c_i} \\ &= (\det g_{\tau(i)})^c . \end{aligned}$$

We can now apply the standard outer automorphism of type $\langle \sigma, \{d_i\}, d \rangle$ to $[\langle h_i \rangle]$. The same computation shows that we get $[\langle f_i \rangle]$ with

$$f_{\sigma\tau(i)} = h_{\tau(i)}^{d_{\tau(i)}} (\det h_{\sigma\tau(i)})^{d/n_i} (\det h_{\tau(i)})^{-d_{\tau(i)}/n_i} .$$

This is the product of

$$g_i^{c_i d_{\tau(i)}} (\det g_{\tau(i)})^{cd_{\tau(i)}/n_i} (\det g_i)^{-c_i d_{\tau(i)}/n_i}$$

and

$$(\det g_{\sigma\tau(i)})^{cd/n_i} (\det g_{\tau(i)})^{-cd_{\tau(i)}/n_i} ,$$

which is

$$g_i^{c_i d_{\tau(i)}} (\det g_{\sigma\tau(i)})^{cd/n_i} (\det g_i)^{-c_i d_{\tau(i)}/n_i} .$$

The same computation once more shows that this is the effect of the standard outer automorphism for the product of the two types. □

REMARKS AND EXAMPLES.

(1) In this proof we of course used our knowledge of the automorphisms of the semisimple group $\prod PGL(n_i)$. It is true in fact that any reductive group (like our G) has a "root system" in a generalized sense (where the roots may not span the ambient space), and from this one can determine its outer automorphisms [1, p. 328]. But though the computations in Theorems 2 and 3 can thus be rephrased as results on such root systems, this does not seem to introduce any notable simplifications in the proofs.

(2) As a simple example of the theorem we can take $G = GL(n)/\mu_r$; its outer automorphisms are a group of order 2 except when n is bigger than 2 and divides $2r$, where we get $\boldsymbol{Z}/2\boldsymbol{Z} \times \boldsymbol{Z}/2\boldsymbol{Z}$. Another useful example is

$$G = GL(n) \times GL(p)/\{\langle b, b^{-1} \rangle \mid b \text{ scalar}\} ,$$

which has $r_1 = r_2 = 1$. If $2 < n < p$, the only nontrivial outer automorphism is given by

$$[g_1, g_2] \longmapsto [(g_1^{tr})^{-1}, (g_2^{tr})^{-1})] .$$

If $2 = n < p$, again there is just one nontrivial outer automorphism, but now it is given by

$$[g_1, g_2] \longrightarrow (\det g_1)^{-1}[g_1, (g_2^{tr})^{-1}] .$$

If $n = p > 2$, the group of outer automorphisms has order 4, with

$$[g_1, g_2] \longmapsto [g_2, g_1]$$

and

$$[g_1, g_2] \longmapsto [(g_1^{tr})^{-1}, (g_2^{tr})^{-1}]$$

as generators of order 2. If finally $n = p = 2$, we again get the four group, but now with generators

$$[g_1, g_2] \longmapsto [g_2, g_1]$$

and

$$[g_1, g_2] \longmapsto (\det g_1)^{-1} (\det g_2)^{-1}[g_1, g_2] .$$

(3) The type of an automorphism has a very simple meaning: d describes how the automorphism acts on the central G_m in G, while σ and the d_i describe which outer automorphism class it occupies down on G modulo its center. What is not clear in advance is the compatibility condition these data must satisfy.

(4) Even when G has no radial automorphisms, its automorphisms need not all be induced by automorphisms of $\prod GL(n_i)$. We can see this in an example by taking $n_1 = n_2 = 6$ with $r_1 = 2$ and $r_2 = 8$. Here the interchange of factors in $PGL(6) \times PGL(6)$ lifts back uniquely to an automorphism of G, the standard outer automorphism for $\sigma = (12)$ and $d_1 = d_2 = d = 1$; explicitly, this lifting is

$$[g_1, g_2] \longmapsto (\det g_1)^{-1}(\det g_2)[g_2, g_1] .$$

Now any automorphism of $GL(6) \times GL(6)$ that has the same effect as this down on $PGL(6) \times PGL(6)$ must have the form

$$(g_1, g_2) \longmapsto ((\det g_2)^a(\det g_1)^b g_2, (\det g_2)^c(\det g_1)^d g_1) .$$

I claim that no such map sends the kernel A of $GL(6) \times GL(6) \to G$ to itself. Indeed, A consists of the scalar (g_1, g_2) with $g_1^2 g_2^8 = 1$, and contains in particular all the elements $(1, \zeta)$ with $\zeta^8 = 1$. The map on $GL(6) \times GL(6)$ sends such an element to (ζ^{6a+1}, ζ^c). If this image lies in A, then $1 = (\zeta^{6a+1})^2(\zeta^c)^8 = \zeta^{12a+2}$, and hence $1 = (\zeta^{12a+2})^2 = \zeta^4$. Thus for some ζ the image is not in A.

3. A radial automorphism in disguise. In this Section I want to run quickly through the analysis of a classical example, one that first suggested to me the results of this paper. Specifically, it implies the existence of a radial automorphism of $GL(4)/\mu_2$, at least for char $(k) \neq 2$. The necessary information is all contained in [2], though I will restate it here in a more old-fashioned style befitting the problem that led me to it [6].

Let V be the six-dimensional space of 4×4 alternating matrices $X = (X_{ij})$. On V the Pfaffian $Pf(X) = X_{12}X_{34} - X_{13}X_{24} + X_{14}X_{23}$ is a nondegenerate hyperbolic quadratic form. For any invertible 4×4 matrix A we can define $T(A)$: $V \to V$ by $T(A)(X) = AXA^{tr}$; we have then $\det T(A) = (\det A)^3$ and $Pf(T(A)X) = (\det A)Pf(X)$. If we restricted to $\det A = 1$, we would get an epimorphism of the special linear group SL (4) to the special orthogonal group SO (6); this corresponds to one of the familiar low-dimension isomorphisms of simple Lie algebras [3, p. 142].

Since 6 is even, the behavior of elements with arbitrary determinant is slightly more complicated. One defines the general orthogonal group $GO(6)$ to be all invertible linear g with $Pf(gX) = aPf(X)$ for some multiplier $a = a(g)$. We have then $(\det g)^2 = a^6$, so $\det g = \pm a^3$, and one defines $GO(6)^+$ to be those g where $\det g = a^3$. This is a proper subgroup (hence of index 2), since for instance the reflection R defined by

$$\begin{aligned} X_{12} &\longleftrightarrow X_{34} \\ X_{13} &\longleftrightarrow X_{24} \\ X_{14} &\longleftrightarrow X_{23} \end{aligned}$$

clearly preserves the Pfaffian but has determinant -1. All maps of the form $T(A)$ are in $GO(6)^+$, and in fact we have $GL(4)/\mu_2 \xrightarrow{\sim} GO(6)^+$.

Now the element R acts by conjugation on the normal subgroup $GO(6)^+ \cong GL(4)/\mu_2$. What is this action? It cannot be an inner automorphism; for if $RT(A)R = T(B)T(A)T(B)^{-1}$ for some B, then $RT(B)$ commutes with all $T(A)$ and hence is scalar, which is impossible because the scalars are all in $GO(6)^+$. Nor can we have $RT(A)R = T(B)T((A^{tr})^{-1})T(B)^{-1}$, since $\det(RT(A)R) = \det T(A) = (\det A)^3$ and $\det(T(B)T((A^{tr})^{-1})T(B)^{-1}) = \det T((A^{tr})^{-1}) = (\det (A^{tr})^{-1})^3 = (\det A)^{-3}$. (This would not be a restriction if we were looking just at $SO(6)$, but here it is impossible.) Thus conjugation by R has to represent some outer automorphism class of $GL(4)/\mu_2$ other than transpose inverse. Computation reveals in fact that

$$RT(A)R = \det(A)T(D)T((A^{tr})^{-1})T(D)$$

with $D = \operatorname{diag}(1, -1, 1, -1)$. This differs only by an inner automorphism from $[A] \mapsto (\det A)[(A^{tr})^{-1}]$.

4. Application to normalizer computations. Knowing the precise automorphism group of G can be useful when we want to find the normalizer of G inside some larger algebraic group. The point is that any element h of the normalizer induces an algebraic automorphism of G by conjugation. If for instance it gives an inner automorphism, so $hgh^{-1} = bgb^{-1}$ for some b in G, then $b^{-1}h$ is in the centralizer of G; and the centralizer should be relatively easy to compute. Any other normalizer h can be changed by something in G to make it induce one of the standard outer automorphisms, and computing the elements having precisely that effect on G may not be much harder than computing the centralizer.

Here is a specific example, one that (along with similar results) is used in [6]. I doubt that it is basically new, but I do not know a reference for it. Let $e_1, \cdots, e_n$ be a basis of $V = k^n$, and let T denote the induced action of $GL(n)$ on $\Lambda^r V$. For $I = \{i_1, \cdots, i_r\}$ with $i_1 < \cdots < i_r$, let e_I denote $e_{i_1} \wedge \cdots \wedge e_{i_r}$. Let $B\colon \Lambda^r V \to \Lambda^{n-r} V$ be the linear map defined by the condition $e_I \wedge Be_I = e_1 \wedge \cdots \wedge e_n$.

THEOREM 4. *For $1 \leqq r \leqq n-1$, the representation T of $GL(n)$ on $\Lambda^r V$ has kernel μ_r. The image algebraic group G acts irreducibly. It is its own normalizer except in the case when n is even $\geqq 4$ and $r = n/2$; in that case G is of index 2 in its normalizer, the other coset being generated by B.*

Proof. The kernel is trivial to compute, particularly if we remember that normal subgroups of $GL(n)$ not containing $SL(n)$ must consist of scalars. Irreducibility is a well known Lie algebra result in characteristic zero [3, p. 226–7]; essentially the same proof works in general, and we can easily go through it. The idea is to look at the action of the diagonal subgroup H of $GL(n)$. We have

$$T(\operatorname{diag}(a_1, \cdots, a_n))e_I = (\prod_{i \in I} a_i)e_I ,$$

and these characters of H are all distinct, so the only H-eigenvectors in $\Lambda^r V$ are scalar multiples of the e_I. Any nonzero G-invariant subspace will have H acting on it diagonalizably, so it will contain some e_I. As there are elements of G taking any one e_I to any other, the invariant subspace must be all of $\Lambda^r V$. It follows abstractly that only the scalars centralize G; we can also see this directly, because any map $\Lambda^r V \to \Lambda^r V$ commuting with the H-action must send each eigenvector e_I to a multiple of itself, and if it commutes with the G-action, all the multipliers must be the same.

Now we can compute the normalizer, following the outline given at the start of this section. As the centralizer of G is contained in G, it is enough to determine which elements C in $GL(\Lambda^r V)$ induce standard outer automorphisms of G under conjugation.

The case $n = 2$ is of course trivial. Suppose then $n > 2$ and $2r \neq n$. The only standard outer automorphism of $G = GL(n)/\mu_r$ then is $T(g) \mapsto T((g^{tr})^{-1})$. We observe now that $\det T(g)$ must be some power of $\det(g)$, since det generates the character group of $GL(n)$. To tell which power we have, it is enough to check it on scalars. For a scalar $g = aI$ we have $\det(g) = a^n$, while $T(g) = a^r I$ will have determinant given by a^r raised to a power equal to the dimension of $\Lambda^r V$. Thus we have

$$\det T(g) = (\det g)^{r/n\binom{n}{r}} = (\det g)^{\binom{n-1}{r-1}}$$

(for all g). In particular $T(g)$ and $T((g^{tr})^{-1})$ will in general have different determinants, so they cannot be conjugate in $GL(\Lambda^r V)$.

Suppose now that $2r = n$. We have then two more standard outer automorphisms, $T(g) \mapsto (\det g)^{-1} T(g)$ and $T(g) \mapsto (\det g) T((g^{tr})^{-1})$. The first one is again ruled out because it does not preserve determinants, but the second is not excluded in this way, and so we proceed to study in more detail what an element C inducing this automorphism must be. Again we look first at the action of the diagonal subgroup. If $g = \operatorname{diag}(a_1, \cdots, a_n)$, then

$$(\det g) T((g^{tr})^{-1}) = (\prod a_i) T(\operatorname{diag}(a_1^{-1}, \cdots, a_n^{-1})) ;$$

applied to an element e_I this gives

$$(\prod_{i \in J} a_i) e_I ,$$

where J is the complement of the subset I. We are supposed to have

$$T(g)C = C \det(g) T((g^{tr})^{-1}) ,$$

so

$$T(g)Ce_I = (\prod_{i \in J} a_i) Ce_I$$

for all g in the diagonal subgroup. As we saw earlier, this eigenvector behavior forces Ce_I to be a multiple of e_J.

Changing C by a scalar, we can assume that $Ce_{\{1,\cdots,r\}} = e_{\{r+1,\cdots,n\}}$. For arbitrary I with complement J, write the elements of I and J in increasing order, and let π be the permutation sending $(1, 2, \cdots, r, r+1, \cdots, n)$ to (I, J). We identify π with the corresponding linear map permuting $e_1, \cdots, e_n$, which gives us $(\pi^{tr})^{-1} = \pi$ and $\det(\pi) = \operatorname{sgn}(\pi)$. Our hypothesis on C tells us then that

$$\begin{aligned} e_J &= T(\pi)e_{\{r+1,\cdots,n\}} = T(\pi)Ce_{\{1,\cdots,r\}} \\ &= C(\det \pi)T(\pi)e_{\{1,\cdots,r\}} = (\operatorname{sgn} \pi)Ce_I \,. \end{aligned}$$

In different notation this is precisely the definition we gave for the map we called B.

The only step remaining is to check that B does indeed normalize G. In fact, of course, we know that we should have specifically

$$B^{-1}T(g)B = T((g^{tr})^{-1}) \det(g) \,.$$

It is enough to check this for scalars and for elementary g (fixing all basis elements but e_i and sending e_i to $e_i + ae_j$), and straightforward computation there shows that the two sides do agree on each e_I. $\square$

The existence of the normalizing map B is clearly related to Chow's theorem [2, p. 81] on adjacency-preserving maps of Grassmannians. But the combinatorial arguments needed for that theorem are unnecessary in our context: the analysis by standard outer automorphisms led us directly to B as the one and only possibility.

As a last remark, we observe that we can put Theorem 4 fully into the context of affine group schemes by proving that the group scheme normalizer of G is smooth. For this it suffices to show that the Lie algebra Lie(G) is its own normalizer. Now an element T in $M_n = \operatorname{Lie}(GL(n))$ acts on $\wedge^r V$ by

$$T(v_1 \wedge \cdots \wedge v_r) = (Tv_1) \wedge \cdots \wedge v_r + \cdots + v_1 \wedge \cdots \wedge (Tv_r) \,.$$

The matrix E_{ii} with sole entry 1 in the (i, i) place has $E_{ii}e_I = 0$ if $i \notin I$ and $E_{ii}e_I = e_I$ if $i \in I$. Hence any U commuting with the action of all E_{ii} has the e_I as eigenvectors. If we change I to J by replacing i by j, then $E_{ij}e_I = \pm e_J$; so if U commutes with all E_{ij} actions, it is a scalar. Now any U' normalizing Lie(G) induces a derivation of it; but all its derivations are inner, so some T in Lie(G) induces the same derivation. Then $U = U' - T$ commutes with Lie(G) and hence is scalar. But the scalars are in Lie$^{\cdot}(G)$, so we are done.

REFERENCES

1. M. Demazure, A. Grothendieck et al., SGA3: *Schêmas en Groupes,* vol. 3. Lecture Notes in Math. #153, Springer-Verlag, New York, 1970.
2. J. Dieudonné, *La Géométrie des Groupes Classiques,* 3^e ed. Springer-Verlag, New York, 1971.
3. N. Jacobson, *Lie Algebras,* Interscience, New York, 1962.
4. O. T. O'Meara, *Lectures on Linear Groups,* CBMS Regional Conference Series #22, Amer. Math. Soc., Providence, 1974.
5. W. Waterhouse, *Introduction to Affine Group Schemes,* Springer-Verlag, New York, 1979.
6. ———, *Automorphisms of* $\det(X_{ij})$: *the group scheme approach,* Advances in Math., to appear.

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