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AUTOMORPHISMS AND NONSELFADJOINT CROSSED PRODUCTS

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We are interested in the invariant subspace structure of the nonselfadjoint crossed product determined by a finite von Neumann algebra M and a trace preserving automorphism α. In this paper we investigate the form of two-sided invariant subspaces for the case that α is ergodic on the center of M.

1. **Introduction.** In this paper, we consider the typical finite maximal subdiagonal algebras which are called nonselfadjoint crossed products. These algebras are constructed as certain subalgebras of crossed products of finite von Neumann algebras by trace preserving automorphisms. Recently, McAsey, Muhly and the author studied the invariant subspace structure and the maximality of these algebras (cf. [4], [5], [6], [7]).

Let M be a von Neumann algebra with a faithful normal tracial state τ and let α be a *-automorphism of M such that $\tau\circ\alpha = \tau$. We regard M as acting on the noncommutative Lebesgue space $L^2(M, \tau)$ (cf. [10]) and consider the Hilbert space

$$L^2 = \{f: Z \longrightarrow L^2(M, \tau) \mid \Sigma \| f(n)\|_2^2 < \infty\}$$

which may be identified with $l^2(Z) \otimes L^2(M, \tau)$. Let $\mathfrak{L}$ (resp. $\mathfrak{R}$) be the left (resp. right) crossed product of M and α, and let $\mathfrak{L}_+$ (resp. $\mathfrak{R}_+$) be the left (resp. right) nonselfadjoint crossed product of $\mathfrak{L}$ (resp. $\mathfrak{R}$) (cf. §2). In [6], we showed that the following three conditions are equivalent; (i) M is a factor; (ii) a conditioned form of the Beurling-Lax-Halmos theorem is valid; and (iii) $\mathfrak{L}_+$ is a maximal σ-weakly closed subalgebra of $\mathfrak{L}$. Furthermore, in [7], we proved that α fixes the center $\mathfrak{Z}(M)$ of M elementwise if and only if the Beurling-Lax-Halmos theorem is valid. However, if α does not fix the center $\mathfrak{Z}(M)$ of M elementwise, then the form of invariant subspace is very complicated. Considering the reduction theory with respect to the abelian subalgebra $\{z \in \mathfrak{Z}(M): \alpha(z) = z\}$ of $\mathfrak{Z}(M)$, it seems to be sufficient to investigate the case that α is ergodic on $\mathfrak{Z}(M)$. Therefore, our aim in this paper is to study the invariant subspace structure of L^2 when α is ergodic on $\mathfrak{Z}(M)$. We now suppose that α is ergodic on $\mathfrak{L}(M)$. Then every two-sided invariant subspace of L^2 which is not left-reducing is left-pure, left-full, right-pure and right-full (Theorems 3.2 and 4.5). Further, if $\mathfrak{L}$ is a factor, then every proper two-sided invariant subspace of L^2 is of the form $\{f \in L^2: \sum_{k=-\infty}^{n} e_k f(n) = f(n),\ n \in Z\}$, where $\{e_n\}_{n=-\infty}^{\infty}$ is a family of mutually orthogonal central projections of M such that $\sum_{n=-\infty}^{\infty} e_n = 1$ and $\alpha(e_n) \leqq \sum_{k=-\infty}^{n+1} e_k$ (Theorems 3.3 and 4.6). However, if $\mathfrak{Z}(M)$ is atomic and there is some $k > 0$ such that α^k is inner, then we present a two-sided invariant subspace of L^2 which is not the above form (Example 4.7). In case $M = L^\infty(X)$, McAsey in [4] and [5] studied about these results.

In the next section, we define the nonselfadjoint crossed products. In §3, we consider the case that $\mathfrak{Z}(M)$ is nonatomic. Finally, in §4, we study two-sided invariant subspace of L^2 when $\mathfrak{Z}(M)$ is atomic.

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2. Preliminaries. We suppose that M is a von Neumann algebra with a faithful normal tracial state τ and α is a *-automorphism of M which preserves τ; i.e., $\tau\circ\alpha = \tau$. Let $L^2(M, \tau)$ be the noncommutative Lebesgue space associated with M and τ in Segal [10]. We denote the operators in the left regular representation of M on $L^2(M, \tau)$ by l_x, $x \in M$, and those in the right regular representation by r_x. Put $l(M) = \{l_x: x \in M\}$ and $r(M) = \{r_x: x \in M\}$. Since $\tau\circ\alpha = \tau$, there is a unitary operator u on $L^2(M, \tau)$ induced by α. To construct a crossed product, we consider the Hilbert space L^2 defined by

$$\left\{f: Z \longrightarrow L^2(M, \tau) \,\middle|\, \sum_{n \in Z} \|f(n)\|_2^2 < \infty\right\}$$

where $\|\cdot\|_2^2$ is the norm of $L^2(M, \tau)$. For $x \in M$, we define operators L_x, R_x, L_δ and R_δ on L^2 by the formulae $(L_x f)(n) = xf(n)$, $(R_x f)(n) = f(n)\alpha^n(x)$, $(L_\delta f)(n) = uf(n-1)$ and $(R_\delta f)(n) f(n-1)$, $g \in L^2$, $n \in Z$. Put $L(M) = \{L_x: x \in M\}$ and $R(M) = \{R_x: x \in M\}$. We set $\mathfrak{L} = \{L(M), L_\delta\}''$ and $\mathfrak{R} = \{R(M), R_\delta\}''$ and define the left (resp. right) nonselfadjoint crossed product $\mathfrak{L}_+$ (resp. $\mathfrak{R}_+$) to be the σ-weakly closed subalgebra of $\mathfrak{L}$ (resp. $\mathfrak{R}$) generated by $L(M)$ (resp. $R(M)$) and L_δ (resp. R_δ).

The automorphism group $\{\beta_t\}_{t \in R}$ of $\mathfrak{L}$ dual to α in the sense of Takesaki [9] is implemented by the unitary representation of R, $\{W_t\}_{t \in R}$, defined by the formula $(W_t f)(n) = e^{2\pi i n t} f(n)$, $f \in L^2$; that is, $\beta_t(T) = W_t T W_t^*$, $T \in \mathfrak{L}$, by definition. Similarly, we define $\beta_t(T) = W_t T W_t^*$, $T \in \mathfrak{R}$. It is elementary to check that the spectral resolution of $\{W_t\}_{t \in R}$ is given by the formula $W_t = \sum_{n=-\infty}^{\infty} e^{2\pi i n t} E_n$, where E_n is the projection on L^2 defined by the formula

$$(E_n f)(k) = \begin{cases} f(n)\,, & k = n\,, \\ 0\,, & k \neq n\,. \end{cases}$$

We also define the integral

$$\varepsilon_n(T) = \int_0^1 e^{-2\pi int}\beta_t(T)dt \ , \quad T \in \mathfrak{L} \ .$$

Furthermore, we define $H^2 = \{f \in L^2 : f(n) = 0, n < 0\}$. We refer the reader to [6] and [7] for discussions of these algebras including some of their elementary properties.

DEFINITION 2.1. Let $\mathfrak{M}$ be a closed subspace of L^2. We say that $\mathfrak{M}$ is: left-invariant, if $\mathfrak{L}_+\mathfrak{M} \subseteqq \mathfrak{M}$; left-reducing, if $\mathfrak{L}\mathfrak{M} \subset \mathfrak{M}$; left-pure, if $\bigcap_{n \in Z} L_\delta^n \mathfrak{M} = \{0\}$; and left-full, if $\bigvee_{n \in Z} L_\delta^n \mathfrak{M} = L^2$. The right hand versions of these concepts are defined similarly, and a closed subspace which is both left and right is called two-sided invariant. If $\mathfrak{M}$ is both left-reducing and right-reducing, $\mathfrak{M}$ is said to be two-sided reducing.

Throughout this paper, we suppose that α is ergodic on the center $\mathfrak{Z}(M)$ of M. By the ergodicity of α on $\mathfrak{Z}(M)$, $\mathfrak{Z}(M)$ is either nonatomic or atomic. Therefore, in §§3 and 4, we consider the invariant subspaces of L^2 in two cases, respectively.

3. Case $\mathfrak{Z}(M)$ is nonatomic. In this section we investigate the structure of two-sided invariant subspaces of L^2 for the case when $\mathfrak{Z}(M)$ is nonatomic. To prove this, we need the following lemma. We believe that it is known. But, for completeness, we have included a proof.

LEMMA 3.1. $\{L(M), R(M)\}' = \{L(\mathfrak{Z}(M)), \{E_n\}_{n=-\infty}^{\infty}\}''$.

Proof. On L^2, which we identify with a direct sum of copies of $L^2(M, \tau)$, the operators L_x and R_x, $x \in M$, have these matricial representations;

$$L_x = \begin{bmatrix} \ddots & & & & 0 \\ & l_x & & & \\ & & l_x & & \\ & & & l_x & \\ 0 & & & & \ddots \end{bmatrix}$$

and

$$R_x = \begin{bmatrix} \ddots & & & & 0 \\ & r_{\alpha^{-1}(x)} & & & \\ & & r_x & & \\ & & & r_{\alpha(x)} & \\ 0 & & & & \ddots \end{bmatrix}.$$

Any operator A in $\{L(M), R(M)\}'$ is in $L(M)'$ and so has a matricial representation $A = [r_{x_{n,m}}]$ for suitable $x_{n,m}$ in M. In order for A to commute with $R(M)$, it is necessary and sufficient that for each pair (n, m), the equation $\alpha^n(y)x_{n,m} = x_{n,m}\alpha^m(y)$ holds for all y in M. This is equivalent to the validity of the equation

$$(3.1) \qquad y\alpha^{-n}(x_{n,m}) = \alpha^{-n}(x_{n,m})\alpha^{m-n}(y)\ , \quad \text{for all } y \text{ in } M\ .$$

If $n = m$, $x_{n,n}$ lies in $\mathfrak{Z}(M)$. Suppose that $n \neq m$ and $x_{n,m} \neq 0$. Let q be the central support projection of $\alpha^{-n}(x_{n,m})$. Since α is ergodic on $\mathfrak{Z}(M)$, it is well-known that α^n is freely-acting on $\mathfrak{Z}(M)$ for $n \neq 0$. Thus there exists a nonzero projection $p \in \mathfrak{Z}(M)$ such that $\alpha^{m-n}(p)p = 0$ and $0 < p \leqq q$. By (3.1), $p\alpha^{-n}(x_{n,m}) = \alpha^{-n}(x_{n,m})\alpha^{m-n}(p) = 0$. This is a contradiction and so $x_{n,m} = 0$. Therefore $\{L(M), R(M)\}' \subset \{L(\mathfrak{Z}(M)), \{E_n\}_{n=-\infty}^{\infty}\}''$. The converse is clear. This completes the proof.

By [5, Corollary 4.3], every two-sided invariant subspace which is left- (or right-) reducing is two-sided reducing. Therefore, since $\mathfrak{L}$ is a factor by the ergodicity of α on $\mathfrak{Z}(M)$, such a space is $\{0\}$ or L^2.

THEOREM 3.2. *Every proper two-sided invariant subspace of L^2 is left-pure, left-full, right-pure and right-full.*

Proof. Put $\mathfrak{M}_1 = \bigcap_{n=1}^{\infty} L_\delta^n\mathfrak{M}$ and let P be the projection of L^2 onto $\mathfrak{M}_1$. Since $\mathfrak{M}_1$ is left-reducing, $P \in \mathfrak{L}' = \mathfrak{R}$. Since $\mathfrak{M}_1$ is right-invariant, $R_\delta P R_\delta^* \leqq P$ and $P \in R(M)'$. By the finiteness of $\mathfrak{R}$, $P \in \mathfrak{L} \cap \mathfrak{R} = \mathfrak{Z}(\mathfrak{L})$. Since $\mathfrak{L}$ is a factor and $P \neq 1$, $\mathfrak{M}_1 = \{0\}$. The rest are analogously proved.

Let $\{e_n\}_{n=-\infty}^{\infty}$ be a family of mutually orthogonal central projections in M such that $\sum_{n=-\infty}^{\infty} e_n = 1$ and $\alpha(e_n) \leqq \sum_{m=-\infty}^{n+1} e_m$. Put

$$L^2(\{e_n\}_{n=-\infty}^{\infty}) = \left\{ f \in L^2 \colon \sum_{m=-\infty}^{n} e_m f(n) = f(n)\ , \quad \text{for all } n \right\}\ .$$

Then it is clear that $L^2(\{e_n\}_{n=-\infty}^{\infty})$ is a two-sided invariant subspace of L^2 which is not left-reducing. Conversely, we have the following theorem.

THEOREM 3.3. *Suppose that $\mathfrak{Z}(M)$ is nonatomic. Then every proper two-sided invariant subspace of L^2 is of the form $L^2(\{e_n\}_{n=-\infty}^{\infty})$ where $\{e_n\}_{n=-\infty}^{\infty}$ is a family of mutually orthogonal central projections in M such that $\sum_{n=-\infty}^{\infty} e_n = 1$ and $\alpha(e_n) \leqq \sum_{m=-\infty}^{n+1} e_m$.*

Proof. Let $\mathfrak{M}$ be a proper two-sided invariant subspace of L^2. By Theorem 3.2, $\mathfrak{M}$ is right-pure. Put $\mathfrak{F} = \mathfrak{M} \ominus R_\delta \mathfrak{M}$ and let P (resp. $P_{\mathfrak{M}}$) be the projection of L^2 onto $\mathfrak{F}$ (resp. $\mathfrak{M}$). It is clear that $P \in \{L(M), R(M)\}'$. By Lemma 3.1, there is a family $\{e_n\}_{n=-\infty}^{\infty}$ of central projections of M such that $(Pf)(n) = e_n f(n)$. Thus we have for all n,

$$e_n f(0) = e_n(R_\delta^n f)(n) = (PR_\delta^n f)(n) = (R_\delta^{*n} PR_\delta^n f)(0)$$

and so, for every $m, n (m \neq n)$,

$$e_m e_n f(0) = ((R_\delta^{*m} PR_\delta^m)(R_\delta^{*n} PR_\delta^n) f)(0) = 0 ,$$

because $R_\delta^{*n} PR_\delta^n$ and $R_\delta^{*m} PR_\delta^m$ are orthogonal. This implies that $e_m e_n = 0$, $m \neq n$. Further, since $(R_\delta^k PR_\delta^{*k} f)(n) = e_{n-k} f(n)$, for all k and n, we have

$$(P_{\mathfrak{M}} f)(n) = \left(\left(\sum_{k=0}^{\infty} R_\delta^k PR_\delta^{*k}\right) f\right)(n) = \sum_{k=0}^{\infty} e_{n-k} f(n) = \sum_{k=-\infty}^{n} e_k f(n) .$$

Hence $f \in \mathfrak{M}$ if and only if $f(n) = \sum_{k=-\infty}^{n} e_k f(n)$. Now, if $f \in L^2$, then

$$\begin{aligned}(L_\delta P_{\mathfrak{M}} L_\delta^* f)(n) &= u(P_{\mathfrak{M}} L_\delta^* f)(n-1) = u \sum_{k=-\infty}^{n-1} e_k (L_\delta^* f)(n-1) \\ &= u \sum_{k=-\infty}^{n-1} e_k u^* f(n) = \sum_{k=-\infty}^{n-1} \alpha(e_k) f(n) .\end{aligned}$$

Since $L_\delta \mathfrak{M} \subseteqq \mathfrak{M}$, this implies that $\sum_{k=-\infty}^{n-1} \alpha(e_k) \leqq \sum_{k=-\infty}^{n} e_k$. Since α is ergodic on $\mathfrak{Z}(\mathfrak{M})$ and $\alpha(\sum_{n=-\infty}^{\infty} e_n) \leqq \sum_{n=-\infty}^{\infty} e_n$, $\sum_{n=-\infty}^{\infty} e_k = 1$. Therefore $\mathfrak{M} = L^2(\{e_n\}_{n=-\infty}^{\infty})$. This completes the proof.

4. Case $\mathfrak{Z}(M)$ is atomic. In this section we investigate the structure of two-sided invariant subspaces of L^2 for the case when $\mathfrak{Z}(M)$ is atomic. We suppose that α is ergodic on $\mathfrak{Z}(M)$ and $\mathfrak{Z}(M)$ is atomic. Since M is finite, there is a family $\{p_n\}_{n=0}^{N-1}$ of mutually orthogonal minimal projections in $\mathfrak{Z}(M)$ such that $\sum_{n=0}^{N-1} p_n = 1$, $\alpha(p_n) = p_{n+1}$, $n = 0, 1, \cdots, N-2$, and $\alpha(p_{N-1}) = p_0$. Hence Mp_n is a factor and $\alpha^{kN}|_{Mp_n}$ is a *-automorphism of Mp_n. In this section we keep the notations.

To prove Theorems 4.5 and 4.6, we need the following lemmas. As may be well-known, we include them for completenss in our version. At first, we have the following lemma easily and so the proof will be omitted.

LEMMA 4.1. *The following conditions are equivalent.*

(i) α^k *is outer for all* $k \neq 0$;

(ii) *for every* $n = 0, 1, \cdots, N-1$, $\alpha^{kN}|_{Mp_n}$ *is outer for all* $k \neq 0$; *and*

(iii) *for some* n, $\alpha^{kN}|_{Mp_n}$ *is outer for all* $k \neq 0$.

As in Lemma 3.1, we have the following lemma.

LEMMA 4.2. *If* α^k *is outer for all* $k \neq 0$, *then* $\{L(M), R(M)\}' = \{L(\mathfrak{Z}(M)) \cup \{E_n\}_{n=-\infty}^{\infty}\}''$.

Proof. As in the proof of Lemma 3.1, take $A = [r_{x_{n,m}}] \in L(M)' \cap R(M)'$. Then $y\alpha^{-n}(x_{n,m}) = \alpha^{-n}(x_{n,m})\alpha^{m-n}(y)$, $y \in M$. If $n = m$, $x_{n,n} \in \mathfrak{Z}(M)$ and, if $m - n \neq kN$, then $x_{n,m} = 0$. Thus, suppose that $x_{n,m} \neq 0$, $m - n = kN$. Put $z = \alpha^{-kN}(x_{n,n+kN})$. Then there is a j such that $zp_j \neq 0$ and so $yz = z\alpha^{kN}(y)$, $y \in Mp_j$. Hence $l_y l_z = l_z v l_y v^*$, where $v = u^{kN}$, and so $l_z v \in l(M)' = r(M)$. Since $(l_z v)(l_z v)^* \in l(M) \cap r(M) = l(\mathfrak{Z}(M))$, $zz^* \in \mathfrak{Z}(M)$. Hence we have $zz^* p_j = \|zp_j\|^2 p_j$. If w is then chosen $w = zp_j/\|zp_j\|$, then w is a partial isometry which is an element of Mp_j. Since Mp_j is finite, w is a unitary operator when viewed as an element of Mp_j and implements $\alpha^{-kN}|_{Mp_j}$. By Lemma 4.1, this is a contradiction and so $z = 0$. This completes the proof.

It is well-known that if M is a factor and α^k is outer for all $k \neq 0$, then $\mathfrak{L}$ is a factor. In this case, the converse is true and we have the following.

LEMMA 4.3. α^k *is outer for all* $k \neq 0$ *if and only if* $\mathfrak{L}$ *is a factor.*

Proof. ($\leftarrow$). If α^{kN} is inner for some $k \neq 0$, then there is a unitary operator $v \in M$ such that $\alpha^{kN}(x) = vxv^*$. Thus we have $v\alpha(x)v^* = \alpha^{kN+1}(x) = \alpha(v)\alpha(x)\alpha(v^*)$. Hence we have that, for all n, $\alpha^n(v)$ and v induce the same automorphism by conjugation. So $L_{\alpha^n(v)} L_x L_{\alpha^n(v^*)} = L_{\alpha^{kN}(x)}$, hence $L_x L_{\alpha^n(v^*)} = L_{\alpha^n(v^*)} L_{\alpha^{kN}(x)}$. From $L_\delta^* L_x L_\delta = L_{\alpha^{-1}(x)}$, $L_{\alpha^{kN}(x)} L_\delta^{kN} = L_\delta^{kN} L_x$. Thus $L_x L_{\alpha^n(v^*)} L_\delta^{kN} = L_{\alpha^n(v^*)} L_{\alpha^{kN}(x)} L_\delta^{kN} = L_{\alpha^n(v^*)} L_\delta^{kN} L_x$ and $L_{\alpha^n(v^*)} L_\delta^{kN} \in L(M)'$. Since $L_\delta L_x L_\delta^* = L_{\alpha(x)}$, for all $x \in M$, we have $L_\delta^{kN} L_x L_\delta^{*kN} = L_{\alpha^{kN}(x)}$. Since $\alpha^{kN}(v) = v$, we have also $\alpha^{kN}(\alpha^n(v)) = \alpha^n(v)$ and $\alpha^{kN}(\alpha^n(v^*)) = \alpha^n(v^*)$. Hence L_δ^{kN} commutes with $L_{\alpha^n(v)}$ and $L_{\alpha^n(v^*)}$. Put $w = v\alpha(v) \cdots \alpha^{kN-1}(v)$. Since $\alpha(w) = w$, we have $L_{w^*}(L_\delta^{kN})^{kN} \in L(M)'$. On the other hand, since $\alpha(w^*) = w^*$, L_{w^*} commutes with L_δ and $L_{w^*}(L_\delta^{kN})^{kN}$ commutes with L_δ. Thus we have $L_{w^*}(L_\delta^{kN})^{kN} \in \mathfrak{Z}(\mathfrak{L})$. Therefore $\mathfrak{L}$ is not a factor. This completes the proof.

($\rightarrow$). Suppose that α^k is outer for all $k \neq 0$. Take $A \in \mathfrak{Z}(\mathfrak{L}) \subset L(M)' \cap R(M)'$. By Lemma 4.2, there is a sequence $\{x_n\} \in \mathfrak{Z}(M)$ such that $A = [l_{x_n}]$. Since A commutes with L_δ and R_δ, $x_n = x_0$ and $\alpha(x_0) = x_0$. Since α is ergodic, $A = \lambda 1$ for some λ. Therefore $\mathfrak{L}$ is a factor. This completes the proof.

Next we investigate the center of crossed products when α^k is inner for some $k \neq 0$.

LEMMA 4.4. *Suppose that $\mathfrak{L}$ is not a factor. Then there are a unitary operator $v \in M$ and $k > 0$ such that $\mathfrak{Z}(\mathfrak{L}) = \{L_v L_\delta^{kN}\}''$.*

Proof. Put $\tilde{\beta}_t = \beta_{t|\mathfrak{Z}(\mathfrak{L})}$. Then $\{\tilde{\beta}_t\}_{t \in R}$ is a σ-weakly continuous one-parameter group of *-automorphisms of $\mathfrak{Z}(\mathfrak{L})$ with period 1 and is ergodic on $\mathfrak{Z}(\mathfrak{L})$ in the sense that, if $T \in \mathfrak{Z}(\mathfrak{L})$ such that $\tilde{\beta}_t(T) = T$, $t \in R$, then $T = \lambda 1$ for some complex number λ. For every $n \in Z$, put $K_n = \{T \in \mathfrak{Z}(\mathfrak{L}): \beta_t(T) = e^{2\pi i n t} T,\ t \in R\}$. Then it is clear that $\varepsilon_n(\mathfrak{Z}(\mathfrak{L})) = K_n$. Let $Z_1 = \{n \in Z: K_n \neq \{0\}\}$. We claim that Z_1 is a subgroup of the additive group Z. Let T_n be a nonzero element in K_n such that $\|T_n\| = 1$ for a fixed $n \in Z_1$. Then $T_n^* T_n$, $T_n T_n^*$ is nonzero elements of K_0 (cf. [9, Lemma 1(a)]). Since $\{\tilde{\beta}_t\}_{t \in R}$ is ergodic on $\mathfrak{Z}(\mathfrak{L})$, T_n is a unitary operator. By [9, Lemma 1(a)], we have $K_n = \boldsymbol{C} T_n$ for every $n \in Z_1$. Therefore, Z_1 is a subgroup of Z. Let m be the smallest positive integer in Z_1. By the group property of Z_1, we have $Z_1 = mZ$. Hence, by [9, Lemma 1(a)], $K_{nm} = \boldsymbol{C} T_m^n$, $n \in Z$. By [9, Theorem 1], $\mathfrak{Z}(\mathfrak{L})$ is generated by T_m. Since $\varepsilon_m(\mathfrak{L}) = L(M) L_\delta^m$ (cf. [3, Corollary 4.3.2]), there is a unitary operator v in M such that $T_m = L_v L_\delta^m$. Since $T_m \in \mathfrak{Z}(\mathfrak{L})$, we have, for $x \in M$,

$$L_{\alpha^m(x)} = L_\delta^m L_x L_\delta^{*m} = L_v^* T_m L_x (L_v^* T_m)^* = L_{v^*} L_x L_v = L_{v^* x v}$$

and so α^m is inner. Since α^n is not inner for all $n \neq jN$, there is a $k > 0$ such that $m = kN$. This completes the proof.

The following theorem is proved by McAsey [5] in case $M = l^\infty(X)$, $(X) = \{x_0, x_1, \cdots, x_{N-1}\}$. We present the simple proof in more general setting.

THEOREM 4.5. *Every two-sided invariant subspace which is not left-reducing is left-pure, left-full, right-pure and right-full.*

Proof. If α^k is outer for all $k \neq 0$, by Lemma 4.3, $\mathfrak{L}$ is a factor. Then we have this theorem as in the proof of Theorem 3.2. Suppose now that $\mathfrak{L}$ is not a factor. Let $\mathfrak{M}$ be a two-sided invariant subspace which is not left-reducing and let P be the projection of L^2 onto

$\bigcap_{n>0} L_\delta^n \mathfrak{M}$. Put $B = \{T \in \mathfrak{L}: T\mathfrak{M} \subset \mathfrak{M}\}$. As in the proof of Theorem 3.2 and [6, Theorem 4.1], $P \in B \cap \mathfrak{Z}(\mathfrak{L})$. Since $\mathfrak{Z}(\mathfrak{L}) \cap \mathfrak{L}_+$ is a maximal σ-weakly closed subalgebra of $\mathfrak{Z}(\mathfrak{L})$ ([6, Theorem 2.3]), we find $\mathfrak{Z}(\mathfrak{L}) \cap B = \mathfrak{Z}(\mathfrak{L}) \cap \mathfrak{L}_+$, in which case $P = 0$, or $\mathfrak{Z}(\mathfrak{L}) \cap B = \mathfrak{Z}(\mathfrak{L})$. But if $\mathfrak{Z}(\mathfrak{L})$ were contained in B, by Lemma 4.4, $L_v L_\delta^{*kN} \in B$ for some unitary $v \in M$ and some $k > 0$. Since $\mathfrak{M}$ is left-invariant, $L_\delta^* \in B$ and so $B = \mathfrak{L}$. This is a contradiction. Therefore we conclude once more $P = 0$ and so $\mathfrak{M}$ is left-pure.

Next, let P be the projection of L^2 onto $\bigcap_{n>0} R_\delta^n \mathfrak{M}$. Put $B = \{T \in \mathfrak{R}: T\mathfrak{M} \subset \mathfrak{M}\}$. As before, $P \in \mathfrak{Z}(\mathfrak{L}) \cap B$ and we find that $\mathfrak{Z}(\mathfrak{L}) \cap B = \mathfrak{Z}(\mathfrak{L}) \cap \mathfrak{R}_+$, in which case $P = 0$, or $\mathfrak{Z}(\mathfrak{L}) \cap B = \mathfrak{Z}(\mathfrak{L})$. But if $\mathfrak{Z}(\mathfrak{L})$ were contained in B, then there exists a unitary $v \in M$ and $k > 0$ such that $R_v R_\delta^{*kN} \in \mathfrak{Z}(\mathfrak{L})$, as in the proof of Lemma 4.4. Thus $B = \mathfrak{R}$. Therefore $\mathfrak{M}$ is right-reducing. By [6, Corollary 4.3], $\mathfrak{M}$ is two-sided reducing. This is a contradiction. The rest is analogously proved. This completes the proof.

As in §2, we define $L^2(\{e_n\}_{n=-\infty}^{\infty}) = \{f \in L^2: \sum_{m=-\infty}^{n} e_m f(n) = f(n)$, for all $n\}$ for a family $\{e_n\}_{n=-\infty}^{\infty}$ of mutually orthogonal central projections in M such that $\sum_{n=-\infty}^{\infty} e_n = 1$ and $\alpha(e_n) \leqq \sum_{m=-\infty}^{n+1} e_m$. Then it is clear that $L^2(\{e_n\}_{n=-\infty}^{\infty})$ is a two-sided invariant subspace of L^2 which is not left-reducing. Observe that all but finitely many of e_n are zero. Conversely, we have the following theorem by Lemmas 4.2, 4.3 and Theorem 4.5.

THEOREM 4.6. *Suppose that α^k is outer for all $k \neq 0$. Then every proper two-sided invariant subspace of L^2 is of the form $L^2(\{e_n\}_{n=-\infty}^{\infty})$ where $\{e_n\}_{n=-\infty}^{\infty}$ is a family of mutually orthogonal central projections in M such that $\sum_{n=-\infty}^{\infty} e_n = 1$ and $\alpha(e_n) \leqq \sum_{m=-\infty}^{n+1} e_m$.*

Finally, if $\mathfrak{L}$ is not a factor, then Theorem 4.6 is not valid. That is, there is a two-sided invariant subspace of L^2 which is not of the form $L^2(\{e_n\}_{n=-\infty}^{\infty})$.

EXAMPLE 4.7. Suppose that $\mathfrak{Z}(\mathfrak{L}) = \{L_v L_\delta^{kN}\}''$ for some unitary v in M and some $k > 0$. Let θ be a finite Blaschke product with zeros $\{\lambda_1, \lambda_2, \cdots, \lambda_s\}$ such that $0 < |\lambda_j| < 1$. This θ has the form

$$\prod_{j=1}^{s} (|\lambda_j|(\lambda_j - z))/(\lambda_j(1 - \bar{\lambda}_j z)) .$$

Let $V = \theta(L_v L_\delta^{kN})$ be the unitary operator in $\mathfrak{Z}(\mathfrak{L})$ defined by θ and the operator $L_v L_\delta^{kN}$ via the functional calculus. Let $\sum_{i=0}^{\infty} a_i z^i$ be the power series for θ. Since the power series converges absolutely, the series $\sum_{i=0}^{\infty} a_i (L_v L_\delta^{kN})^i$ converges in norm to the operator V. Observe that $a_0 \neq 0$ and $V \in \mathfrak{L}_+$. Put $\mathfrak{M} = VH^2$. It is clear that $\mathfrak{M}$

is a two-sided invariant subspace of H^2 which is not left-reducing. We now suppose that $\mathfrak{M}$ is of the form $L^2(\{e_n\}_{n=-\infty}^{\infty})$. Since $Vf \in \mathfrak{M}$, $f \in H^2$, we have $\sum_{n=-\infty}^{m} e_n(Vf)(m) = (Vf)(m)$, $(Vf)(-m) = 0$, $m > 0$, and

$$\begin{aligned}(Vf)(0) &= \sum_{n=0}^{\infty} a_n((L_v L_\delta^{kN})^n f)(0) = \sum_{n=0}^{\infty} a_n v^n u^{nkN} f(-nkN) \\ &= a_0 f(0) .\end{aligned}$$

Thus this implies that $\sum_{m=-\infty}^{0} e_m = 1$ and $\sum_{m=-\infty}^{-1} e_m = 0$. Therefore $e_0 = 1$ and $e_n = 0$, $n \neq 0$. Hence $\mathfrak{M} = H^2$ and so it is clear that $V^* \in \mathfrak{L}_+$ which is clearly impossible for V constructed above. Hence $\mathfrak{M} \neq L^2(\{e_n\}_{n=-\infty}^{\infty})$.

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