

THE EXACT SEQUENCE OF A LOCALIZATION FOR WITT GROUPS II: NUMERICAL INVARIANTS OF ODD-DIMENSIONAL SURGERY OBSTRUCTIONS

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The propose of this paper is to define numerical invariants of odd-dimensional surgery obstructions, computable in a way similar to that used to compute the index and Arf invariants of even-dimensional surgery obstructions. The main result is that a system of integral congruences ("numerical invariants") suffices, modulo the projective class group, to determine whether or not an odd-dimensional surgery obstruction vanishes, when the fundamental group is a finite 2-group. In addition, the numerical invariants turn out to be Euler characteristics in certain cases of topological interest, including the existence of product formulas.

Let π be a group and $\mathbb{Z}\pi$ its integral group ring, with the involution induced by $g \rightarrow g^{-1}$, $g \in \pi$. The even-dimensional surgery obstruction group $L_{2n}(\mathbb{Z}\pi)$ is, roughly speaking, the Grothendieck group on isometry classes of hermitian forms over $\mathbb{Z}\pi$, modulo the subgroup generated by hyperbolic forms. A striking fact, discovered by C. T. C. Wall ([56, §6]), is that the odd-dimensional surgery obstruction group, $L_{2n+1}(\mathbb{Z}\pi)$, is (again roughly) the commutator quotient of the group of isometries of the stable hyperbolic form. An important consequence of this result is that the obvious analogy between L_{2n} and L_{2n+1} on the one hand, and K_0 and K_1 on the other, can be used to translate techniques from algebraic K -theory to unitary K -theory. This has been done by many authors.

In spite of this conceptual connection between L_{2n} and L_{2n+1} , however, there remains an important difference between them. Classical invariants of quadratic forms, such as the index or Arf invariant, have been easier to compute than any known algebraic invariants of the unitary group; and, on the geometric side, the braid diagram (in [56, §6]) necessary to construct the odd-dimensional obstruction seems to contain more delicate geometric information than the intersection and self-intersection forms of the even-dimensional case. The purpose of this paper is to define algebraic invariants of odd-dimensional surgery, by a procedure analogous to the one furnishing the signature of a quadratic form.

To see what is meant by this, recall the ingredients necessary for the computation of the signatures of a hermitian form over $\mathbb{Z}\pi$. Let π be a finite group and $R\pi$ its real group ring. Any element

of $L_{2n}(\mathbf{Z}\pi)$ yields, by extension of scalars, an element of $L_{2n}(\mathbf{R}\pi)$ which is determined by its collection of classical signatures, usually called the multisignature ([56, p. 165]). In order to compute the multisignature one needs to know, first, the matrix components of the product decomposition

$$(0.1) \quad \mathbf{R}\pi \cong \coprod M_{n_i}(D_i)$$

(furnished by the Wedderburn theorem) where each D_i is a real division algebra (only $D_i = \mathbf{R}, \mathbf{C},$ or \mathbf{H} are possible); and, second, one must understand how the involution on $\mathbf{R}\pi$, induced by $g \rightarrow g^{-1}$, is translated to an involution

$$(0.2) \quad \sigma_i: M_{n_i}(D_i) \longrightarrow M_{n_i}(D_i)$$

on each of the factors in (0.1). With this information, the given element of $L_{2n}(\mathbf{R}\pi)$ is projected into each factor $L_{2n}(M_{n_i}(D_i))$, is then translated by “Morita theory” to an element of $L_{2m}(D_i)$, where $m = n$ or $n + 1$ depending on σ_i , and, finally, the classical signature is evaluated if $D_i = \mathbf{C}$ or if m is even and $D_i = \mathbf{R}$ or \mathbf{H} . The subject of this paper is the construction of invariants of $L_{2n+1}(\mathbf{Z}\pi)$ from similar, but somewhat more delicate information about $\mathbf{Q}\pi$, where π is a finite 2-group. This is Theorem B below. A very special case, $\pi = \mathbf{Z}/2$, exemplifies the method.

Suppose given a degree—one normal map $(f, b): (M^{4k+3}, \nu) \rightarrow (X, \xi)$ ($\pi_1 X = \mathbf{Z}/2, k \geq 1$), for which the kernel groups $K_i(f) = 0, i \neq 2k + 1$, and

$$(0.3) \quad S := K_{2k+1}(f) \text{ is odd torsion.}$$

Let $\phi: S \times S \rightarrow \mathbf{Q}[\mathbf{Z}/2]/\mathbf{Z}[\mathbf{Z}/2]$ be the linking form. (See [57]; we neglect the self-linking form because S is odd torsion so that it is determined by ϕ .) It follows from [57, 5.6] that (f, b) is normally cobordant to a homotopy equivalence if and only if there is a free $\mathbf{Z}[\mathbf{Z}/2]$ -module Q and an even hermitian form $g: Q \times Q \rightarrow \mathbf{Z}[\mathbf{Z}/2]$ such that there exists a short exact sequence, where $\bar{Q} = \text{Hom}(Q, \mathbf{Z}[\mathbf{Z}/2])$ and d_g is the adjoint of g ,

$$(0.4) \quad Q \xrightarrow{d_g} \bar{Q} \xrightarrow{j} S$$

and such that if $s_1, s_2 \in S, q_1, q_2 \in \bar{Q}$ satisfy $j(q_i) = s_i$, and $n \in \mathbf{Z}$ satisfies $ns_i = 0$, then

$$(0.5) \quad \phi(s_1, s_2) = \frac{1}{n^2} g((d_g)^{-1}(nq_1), (d_g)^{-1}(nq_2)) \pmod{\mathbf{Z}[\mathbf{Z}/2]}.$$

The pair (Q, g) will be called a *resolution* of (S, ϕ) . (This is just a translation of the geometric data: If $(F, B): (W^{4k+4}, \nu) \rightarrow (X \times I,$

$\xi \times I$) is a highly-connected normal cobordism of f to a homotopy equivalence, then $Q = K_{2k+2}(F)$, $\bar{Q} = K_{2k+2}(F, \partial F)$, and g is the intersection form of $H_{2k+2}(W^{4k+4})$, restricted to $K_{2k+2}(F)$. Thus, (0.4) is a homology exact sequence and (0.5) is an easily derived relation between linking numbers on the boundary and intersection numbers in the interior of a manifold.) To test whether $\sigma(f, b) \in L_{4k+3}^h(\mathbb{Z}[\mathbb{Z}/2])$ is trivial, it suffices to analyze the obstruction to finding a pair (Q, g) satisfying (0.4) and (0.5). (For the definition of L_*^h , see (1.4).)

Let $p_{\pm}: \mathbb{Z}[\mathbb{Z}/2] \rightarrow \mathbb{Z}$ be defined by $p_{\pm}(a + bt) = a \pm bt$, where $a, b \in \mathbb{Z}$ and $\langle t \rangle = \mathbb{Z}/2$. Applying p_{\pm} to $\sigma(f, b)$, one obtains $(p_{\pm})_*\sigma(f, b) \in L_{4k+3}^h(\mathbb{Z})$, represented by $(S_{\pm}, \phi_{\pm}) := (S, \phi) \otimes_{\mathbb{Z}[\mathbb{Z}/2]} \mathbb{Z}_{\pm}$, where \mathbb{Z}_{\pm} has $\mathbb{Z}[\mathbb{Z}/2]$ -module structure given by $t \cdot n = \pm n$, $n \in \mathbb{Z}$. It is well-known that $L_{4k+3}^h(\mathbb{Z}) = 0$, so that to each linking pair (S_+, ϕ_+) and (S_-, ϕ_-) we may associate a pair (Q_+, g_+) and (Q_-, g_-) satisfying (0.4), (0.5).

Consider the cartesian square (pull-back diagram) of rings

$$(0.6) \quad \begin{array}{ccc} \mathbb{Z}[\mathbb{Z}/2] & \xrightarrow{p_+} & \mathbb{Z} \\ p_- \downarrow & & \downarrow r_2 \\ \mathbb{Z} & \xrightarrow{r_2} & \mathbb{F}_2 \end{array}$$

where r_2 is reduction mod 2. In terms of this diagram, we have started with (S, ϕ) over $\mathbb{Z}[\mathbb{Z}/2]$ and found resolutions (Q_{\pm}, g_{\pm}) of $(p_{\pm})_*(S, \phi)$ over the anti-diagonal copies of \mathbb{Z} . A standard (“glueing”) argument now shows that a resolution (Q, g) of (S, ϕ) can be found satisfying $(p_{\pm})_*(Q, g) = (Q_{\pm}, g)$ if and only if the mod 2 reductions are isometric:

$$(0.7) \quad (r_2)_*(Q_+, g_+) \cong (r_2)_*(Q_-, g_-) .$$

But, as $\text{cok}(d_{g_{\pm}})$ is odd torsion, $(r_2)_*(Q_{\pm}, g_{\pm})$ is nonsingular over \mathbb{F}_2 , and so, possibly after a rank adjustment, (0.7) holds if and only if the Arf invariants agree:

$$\text{Arf}((r_2)_*(Q_+, g_+)) = \text{Arf}((r_2)_*(Q_-, g_-)) .$$

Now a remarkable theorem of Levine ([29]) asserts that these Arf invariants depend only on $|S_{\pm}|$, the number of elements in S_{\pm} :

$$(0.8) \quad \text{Arf}((r_2)_*(Q_{\pm}, g_{\pm})) = 0 \iff |S_{\pm}| \equiv \pm 1 \pmod{8} .$$

Putting these results together yields

(0.9) PROPOSITION. $\sigma(f, b) = 0$ if and only if $|S_+| \equiv \pm |S_-| \pmod{8}$. (This has a more intrinsic formulation using the fact that $|S_+| \equiv \pm |S_-| \iff |S| \equiv \pm |S_-|^2 \iff |S| \equiv \pm 1 \pmod{8}$.)

From the observation that the map $(p_+, p_-): \mathbf{Z}[\mathbf{Z}/2] \rightarrow \mathbf{Z} \times \mathbf{Z}$ is the inclusion of $\mathbf{Z}[\mathbf{Z}/2]$ into a maximal \mathbf{Z} -order (see [42]) in $Q[\mathbf{Z}/2]$, one is led to generalize the construction leading to (0.9) as follows. Let π be an arbitrary finite 2-group. A theorem of J.-M. Fontaine [13] permits a description of the matrix algebras $M_{m_i}(D_i)$ over division algebras D_i appearing in the product decomposition $Q\pi \cong \prod M_{m_i}(D_i)$ (compare (0.1)); and a more careful analysis yields a precise description of a maximal order \mathcal{M}_i in each $M_{m_i}(D_i)$ and of the involution induced on it (compare (0.2)). Then

$$(0.10) \quad \mathcal{M} = \prod \mathcal{M}_i$$

is a maximal order in $Q\pi$ containing $\mathbf{Z}\pi$. Also, Levine's theorem can be generalized to cover forms over certain rings of algebraic integers.

To state the main theorem of this paper say a factor $M_{m_i}(D_i)$ of $Q\pi$ has type (m_i, n_i) if D_i is the real subfield $Q(\zeta + \zeta^{-1})$ of $Q\zeta$, where $\zeta = \zeta_{2^{n_i}}$ is a primitive 2^{n_i} th root of unity, and if the involution on $M_{m_i}(D_i)$ is matrix transpose. (In this case $\mathcal{M}_i = M_{m_i}(\mathbf{Z}(\zeta + \zeta^{-1}))$.) Suppose the factors corresponding to $i = 1, \dots, k$ have type (m_i, n_i) . If S is any odd torsion $\mathbf{Z}\pi$ -module, define $b_i \in \mathbf{Z}/2$, $i = 1, \dots, k$, by

$$b_i(S) = \begin{cases} 0, & \text{if } |S \otimes_{\mathbf{Z}\pi} \mathcal{M}_i| \equiv \pm 1 \pmod{m_i 2^{n_i+1}} \\ 1, & \text{otherwise.} \end{cases}$$

Recall from [38] the definition of $L_*^p(\mathbf{Z}\pi)$. (See also (1.4).)

THEOREM B. *Let $(f, b): (M^{4k+3}, \nu) \rightarrow (X, \xi)$ be a degree-one normal map where $K_i(f) = 0$, $i \neq 2k+1$, and $S := K_{2k+1}(f)$ is odd torsion. Then $\sigma(f, b) \in L_{4k+3}^b(\mathbf{Z}\pi)$ vanishes in $L_{4k+3}^p(\mathbf{Z}\pi)$ if and only if*

$$b_1(S) = b_2(S) = \dots = b_k(S).$$

Theorem B follows from Theorem A and the generalized Levine theorem.

THEOREM A (for $L_{4k+3}^p(\mathbf{Z}\pi)$). *If the integer k is as above, there is an isomorphism*

$$L_{4k+3}^p(\mathbf{Z}\pi) \xrightarrow{\cong} (\mathbf{Z}/2)^{k-1}.$$

Theorem A is found in (3.9) below, where $L_{4k+1}^p(\mathbf{Z}\pi)$ is also calculated. There is a version of Theorem B in (3.16) for $L_{4k+1}^p(\mathbf{Z}\pi)$, but it is weaker since a large part of these groups seem inaccessible using a generalization of Levine's theorem.

One weakness of Theorem B is the assumption that $K_{2k+1}(f)$ be odd torsion. At the end of §3, a method is given for converting any unitary matrix giving $\sigma(f, b)$, to one for which $K_{2k+1}(f)$ is odd torsion. The method is easy to carry out in practice. A more serious weakness, at least as Theorem B compares to the multisignature discussion above, is that (f, b) must be highly-connected. It seems likely that, given an explicit degree-one normal map, one may complete surgery to a $\mathbb{Z}/2$ -homology equivalence, keeping track of the remaining odd torsion in $K_*(f)$. If this is so, then Theorem B should be generalized by replacing the numbers $|S \otimes \mathcal{M}_i|$ by an analogously defined Euler characteristic. Indeed, we will carry out this procedure to derive a simple product formula in (3.22). Perhaps the most serious drawback is that Theorem B detects only L^p , not L^h , the group of greater geometric interest. However, since this paper was written I. Hambleton and R. J. Milgram [16] have used Rothenberg sequences and the calculations of $L_{\pm 1}^p(\mathbb{Z}\pi)$ to make fairly complete calculations of $L_{\pm 1}^h(\mathbb{Z}\pi)$.

The geometric considerations above motivated this work, but methods themselves are entirely algebraic. Here is an outline of the paper. In §1 definitions of the Witt groups are recalled, together with the localization sequence and the notion of resolution of a form; for the most part the reader is referred to [32] for details. In §2, some qualitative relations between Witt groups of \mathbb{Z} -orders, maximal \mathbb{Z} -order and their mod p reductions ($p \in \mathbb{Z}$) are studied; this leads naturally to the notion of Dickson and Arf invariants (mod 2 reductions) in (2.5). §3 begins with a statement of the theorem which describes the factors in (0.10) above and tabulates their Arf and Dickson invariants in (3.2). Assuming these results, the proof of Theorem A is given in (3.9) and that of Theorem B in (3.16). The product formula mentioned above is proved in (3.22). The remaining §§4–7 are devoted to proving (3.1) and (3.2). In §4, (3.1) is proved and (3.2) is reduced by Morita theory to calculations in cyclotomic extensions of \mathbb{Q} , their subfields, and quaternion algebras over them. Finally, these latter calculations are carried out in §§5–7.

Let us very briefly compare these results to those of other authors. First G. Carlsson and R. J. Milgram have independently proved Theorem A for $L_{\mathbb{Z}}^p(\mathbb{Z}\pi)$ in [16]. Second, A. Bak has announced computations of $L_{\mathbb{Z}}^{p,h,s}(\mathbb{Z}\pi)$ in [4], where π has abelian, normal 2-Sylow subgroup, and he has listed generators in many cases when n is odd. Theorem A is relatively easy when π is abelian; also Bak's class of groups excludes, for example, dihedral groups, where the semi-characteristics studied in [35] appear. However, Bak's list of computations is complete for $L_{\mathbb{Z}}^{s,h}$, groups which are not reached in this paper. Another major program for computation of

L -groups is [48]-[53]. In [53] Wall studies the “intermediate” L -groups $L_*^Y(\mathbb{Z}\pi)$, $Y = \ker\{K_1(\mathbb{Z}\pi) \rightarrow K_1(Q\pi)\}$. When π is a 2-group, fairly general results are obtained in [53, 5.2], but the lack of a good description of Y makes complete computations difficult. (Indeed, our success with 2-groups comes about partly because we ignore K_0 - and K_1 -difficulties, which would have to be confronted to compute L^h or L^s .) S. Cappell has pointed out to me that Wall’s technique of lifting elements of $L_*(\mathbb{Z}\pi)$ back to $L_*(\mathbb{Z}\pi \rightarrow \mathbb{Z}_2\pi)$ corresponds to our method of making $K_*(f)$ odd torsion. This is probably the way to see the relation between Theorem B and Wall’s results.

Since this paper was written, several further results have been obtained. As mentioned above the groups $L_{\pm 1}^h(\mathbb{Z}\pi)$, π a 2-group, were studied in [16]. Working independently, A. Bak and M. Kolster [5] and C. Wright [59] have further computed $L_{\pm 1}^h(\mathbb{Z}\pi)$ and $L_{\pm 1}^P(\mathbb{Z}\pi)$ when π is 2-hyerelementary.

This work has been underway for several years and I have profitted from conversations with several people, including Hyman Bass, John Morgan, Andrew Ranicki, David Carter, Ted Petrie, Julius Shaverson, and Sylvain Cappell. I also thank the referee for many suggestions leading to the present complete revision of the original version of this paper.

Notational conventions. The word “prime” will mean a prime ideal or a valuation, unless otherwise specified. A dyadic prime is one dividing the principal ideal generated by 2. A finite (infinite) prime is one which is nonarchimedean (archimedean). If \mathfrak{p} is a prime ideal in the ring R , then $R_{\mathfrak{p}}$ denotes completion at \mathfrak{p} , $R_{(\mathfrak{p})}$ denotes localization, and R/\mathfrak{p} is the quotient ring. F_q denotes the field of q elements. “ ζ_m ” always denotes a primitive m th root of unity.

The symbol $\langle a, b, c, \dots \rangle$ denotes the quadratic form whose matrix is diagonal, with entries a, b, c, \dots .

Direct sum is denoted by “+”, unless \oplus is used to avoid confusion; [*] denotes bibliographical reference to *; (*) denotes reference to (*) in this paper.

1. Review of basic definitions, localization sequence, resolution of forms.

(1.1) Let A be a ring-with-involution containing 1, where the involution is denoted “ $-$ ”: $\overline{a+b} = \bar{a} + \bar{b}$, $\overline{a\bar{b}} = \bar{b}\bar{a}$, $\bar{1} = 1$, for all $a, b \in A$. All A -modules will be right A -modules, unless otherwise specified. Let $S \subseteq A$ be a central multiplicative subset, $S = \bar{S}$, containing 1 and no zero-divisors. Let $B := A[S^{-1}]$ be a semi-simple ring containing $1/2$ and inheriting an involution from A in the

obvious way. A projective A -module M is called B -free if $M \otimes_A B$ is B -free, and has *rank* n , if $M \otimes_A B$ is B -free of rank n ;

(1.2) DEFINITION. Let $FW_0^\lambda(A)$, $FB_0^\lambda(B/A)$ denote the groups defined in [32, 1.13] (denoted (op. cit.) $W_0^\lambda(A)$, $W_0^\lambda(B/A)$). By replacing free modules of even rank (resp. S -torsion modules with short free resolution) in [32, 1.13] by projective modules of even rank (resp. S -torsion modules with short projective resolution) define groups $fW_0^\lambda(A)$, $fW_0^\lambda(B/A)$. By removing the even rank hypothesis from the definition of $fW_0^\lambda(A)$, define $L_{1-\lambda}^\lambda(A)$. Finally, by removing the quadratic forms from the definitions of $fW_0^\lambda(A)$, $fW_0^\lambda(B/A)$, and replacing hyperbolic forms by metabolic forms [60] in that of $fW_0^\lambda(A)$, one obtains groups denoted $fW_{\text{herm}}^\lambda(A)$, $fW_{\text{herm}}^\lambda(B/A)$.

It is understood that all definitions above involving A alone apply to B in place of A . The objects underlying $fW_0^\lambda(A)$ or $FW_0^\lambda(A)$ (resp. $fW_0^\lambda(B/A)$ or $FW_0^\lambda(B/A)$) will be called λ -quadratic forms over A (resp., over B/A).

(1.3) DEFINITION. Let $FW_1^\lambda(A)$ (resp. $FW_1^\lambda(B/A)$) denote the groups defined in [32, 1.23] (resp. in [32, 1.34]). By replacing in [32, 1.34] S -torsion modules having short free resolution by those having short projective resolution, define $fW_1^\lambda(B/A)$; if $A \rightarrow B$ as in (1.1), the group $fW_0^\lambda(A)$ is obtained by replacing in [32, (1.28–34)] torsion modules by projective modules of even rank using relations (i)–(iv) (with projectives of even rank) in [32, 1.34] (cf. [32, 1.35]). Finally, $L_\lambda^\lambda(A)$ is defined as $fW_1^\lambda(A)$, this time using arbitrary projectives, modulo relations (i)–(iv) in [32, 1.34].

(1.4) REMARKS. (a) When π is a finite group, then

$$FW_1^\lambda(\mathbb{Z}\pi) = L_{1-\lambda}^h(\mathbb{Z}\pi)$$

$$FW_0^\lambda(\mathbb{Z}\pi) \left/ \left\langle \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} \right\rangle \right. = L_\lambda^h(\mathbb{Z}\pi)$$

for the groups $L_*^h(\mathbb{Z}\pi)$ of [56, §17D]; and

$$fW_0^\lambda(\mathbb{Z}\pi) = L_{1-\lambda}^p(\mathbb{Z}\pi)$$

for the groups $L_*^p(\mathbb{Z}\pi)$ of (1.2) or [38].

(b) A triple $(P, Q, (\alpha, \gamma))$ is called a λ -formation over A (see [32, 1.30]) if P and Q are projective and $(\alpha, \gamma): P \rightarrow Q + \bar{Q}$ ($\bar{Q} = \text{Hom}_A(Q, A)$) is the inclusion of a subkernel [32, 1.13] (or sublagrangian in [38]) into the λ -quadratic hyperbolic form on $Q + \bar{Q}$. These are the objects underlying the groups $fW_1^\lambda(\mathbb{Z}\pi)$ and $L_\lambda^\lambda(\mathbb{Z}\pi)$, and

$$(1.5) \quad fW_1^\lambda(\mathbb{Z}\pi) \left\langle \left\langle A^2, A^2, \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) \right\rangle \right\rangle = L_1^\lambda(\mathbb{Z}\pi);$$

moreover, this group agrees with that of [38].

More precisely, the following holds.

(1.6) PROPOSITION. *If A is a ring-with-involution, and there is a surjection of rings-with-involution, $A \rightarrow F_2$, then there is an isomorphism*

$$L_1^\lambda(A) \oplus \mathbb{Z}/2 \xrightarrow{\cong} fW_1^\lambda(A)$$

where the $\mathbb{Z}/2$ -summand of $fW_1^\lambda(A)$ is represented by

$$\theta = \left(A^2, A^2, \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) \right).$$

Proof. It is left to the reader to show that $\theta \in fW_1^\lambda(A)$ has order at most two (compute in $FW_1^\lambda(A)$ using [32, 4.9]). Thus, there is a sequence

$$\mathbb{Z}/2 \xrightarrow{i} fW_1^\lambda(A) \longrightarrow L_1^\lambda(A)$$

which is claimed to be split exact. For by [39, 5.4], if $\sigma \in fW_1^\lambda(A)$ vanishes in $L_1^\lambda(A)$, it is stably isomorphic to a graph formation [38], $\gamma = (P, \bar{P}, (\alpha, \text{Id}))$. Stable isomorphism corresponds to operations (i) and (iv) in [32, 1.34]; in using stabilization the ranks of the projectives used in the definition of σ may change from even to odd. If P has odd rank, then add to γ the λ -formation $(A, A, (1, 0))$ (stabilization): if P has even rank leave γ unchanged. By an operation of type [32, 1.34 (iii)], γ is either

$$(Q, \bar{Q}, (0, \text{Id})), \quad \text{or} \quad (P \oplus A, \bar{P} \oplus A, (0 \oplus 1, \text{Id} \oplus 0))$$

where Q has even rank, and P has odd rank. By an operation of type [32, 1.34 (ii)], the first type is trivial in $fW_1^\lambda(A)$; the second type may be written, for sufficiently stable P , as $\theta + (R, \bar{R}, (0, \text{Id}))$ where $P = R \oplus A$. Hence the sequence above is exact at $fW_1^\lambda(A)$. Define an inverse to i by the induced $fW_1^\lambda(A) \rightarrow fW_1^\lambda(F_2) \cong \mathbb{Z}/2$, where the isomorphism is by [33, (4.1)] and the generator of $fW_1^\lambda(F_2)$ is precisely θ .

(1.7) There are several reasons for the even rank hypotheses in (1.2) and (1.3) above. The first is that the discriminant

$$(1.8) \quad \text{dis}: fW_0^\lambda(B) \longrightarrow F^\times / NK^\times$$

becomes a homomorphism for B a central simple algebra over K , F the fixed field of the involution on K . In fact, define, for a λ -hermitian form $g: B^{2n} \times B^{2n} \rightarrow B$,

$$\text{dis}(g) = (-\lambda)^n nr(G) \in F^\times$$

where G is a matrix for g and $nr: M_n(B) \rightarrow F^\times$ is the reduced norm ([42, §9]). Since nr is a homomorphism and $\text{dis}(g) \in NK^\times$ if g is hyperbolic, (1.8) is a well-defined homomorphism.

Also the Morita equivalences of (4.3) increase ranks in general, while if the even rank hypotheses above are made, no change is caused on the Witt group level. Finally, the rank distinction between $fW_1^\lambda(A)$ and $L_1^\lambda(A)$ is essentially detected by the Dickson invariant (2.5), which is central to the proof of Theorem A in §3.

(1.9) The groups $fW_*^\lambda(A)$, $fW_*^\lambda(B)$, etc. are essentially classical. The following results show the same is true of $fW_*^\lambda(B/A)$, under appropriate conditions. Let R be Dedekind ring, K its fraction field; assume A (as in (1.1)) is an R -algebra, B a K -algebra, $S = R - \{0\}$. Each S -torsion A -module M splits uniquely as a direct sum of \mathfrak{q} -torsion A -modules, $M_{(\mathfrak{q})}$, $\mathfrak{q} \in \text{Spec}(R)$. It follows from [19, Thm. B, p. 124] that M has homological dimension 1 if and only if each $M_{(\mathfrak{q})}$ does, and from [20, App. 5, Lemme] if and only if each completion $M_{\mathfrak{q}}$ does. There are similar splittings $B/A = \coprod (B/A)_{(\mathfrak{q})} \cong \coprod B/A_{(\mathfrak{q})} \cong \coprod B_{\mathfrak{q}}/A_{\mathfrak{q}}$, such that the involution on B/A induces one on each $B/A_{(\mathfrak{q})}$ and $B_{\mathfrak{q}}/A_{\mathfrak{q}}$ if $\mathfrak{q} = \bar{\mathfrak{q}}$, and on $B/A_{(\mathfrak{q})} + B/A_{(\bar{\mathfrak{q}})}$ and $B_{\mathfrak{q}}/A_{\mathfrak{q}} + B_{\bar{\mathfrak{q}}}/A_{\bar{\mathfrak{q}}}$ (switching the summands) if $\mathfrak{q} \neq \bar{\mathfrak{q}}$. The following is now clear.

(1.10) PROPOSITION. *With the above notation, there are isomorphisms*

$$fW_*^\lambda(B/A) \cong \coprod_{\mathfrak{q}=\bar{\mathfrak{q}}} fW_*^\lambda(B/A_{(\mathfrak{q})}) \cong \coprod_{\mathfrak{q}=\bar{\mathfrak{q}}} fW_*^\lambda(B_{\mathfrak{q}}/A_{\mathfrak{q}})$$

induced by localization, splitting and completion. (It is easy to show there is no contribution from those summands of B/A for which $\mathfrak{q} \neq \bar{\mathfrak{q}}$.)

(1.11) PROPOSITION. *Keep the notation of (1.10) and assume in addition that K is a number field, and $\mathfrak{q} = \bar{\mathfrak{q}} \subseteq R$ is a nonradial prime for which $A_{(\mathfrak{q})}$ is a maximal $R_{(\mathfrak{q})}$ -order in B . Then there is a natural isomorphism*

$$L_{i+\lambda-1}^p(A/\mathfrak{q}) \cong fW_i^\lambda(B_{\mathfrak{q}}/A_{\mathfrak{q}}).$$

If, in addition, \mathfrak{q} ramifies in K/F , F = fixed field of the involution,

then

$$fW_i^{-\lambda}(B_q/A_q) \cong fW_i^{\lambda}(B_q/A_q) .$$

Proof. The first statement for $i = 0$ and the trivial involution is proved in [40, 4.2.3 (iv)] and the argument works for any involution; for $i = 1$, one must use relations (i), (ii) in [32, 1.34], which is left as an exercise. The second statement follows from “scaling” [3], if a skew-symmetric unit in R can be found. Let S_p = ring of integers in F_p , where q lies over $p \subseteq S$ = ring of integers of F . By [44, I. 6] $R_q = S_p \sqrt{\pi'}$, for some uniformizer π' of p . Setting $\pi = \sqrt{\pi'}$ yields $\bar{\pi} = \sqrt{\pi'} = -\sqrt{\pi'} = -\pi$, so that $\bar{\pi}/\pi$ is a skew-symmetric unit of R_q .

For any ring-with-involution A , let $W(A)$ denote the group studied in [60], where symmetric bilinear forms are replaced by hermitian forms. Then completely analogous arguments work, under more general circumstances, to prove the following.

(1.12) PROPOSITION. *With the notation of (1.10), there are natural isomorphisms*

- (a) $W(A/q) \cong fW_{\text{herm}}^1(B_q/A_q)$, q finite
- (b) $fW_{\text{herm}}^{\lambda}(B/A) \cong \coprod_{q \text{ finite}} fW_{\text{herm}}^{\lambda}(B_q/A_q)$.

Here is a result which will be used often and is stated here for the reader's convenience.

(1.13) PROPOSITION [50, Lemma 5]. *Let A be a ring-with-involution and $I \subseteq A$ an involution invariant ideal such that A is complete in the I -adic topology. Then the map $A \rightarrow A/I$ induces isomorphisms*

$$fW_{*}^{\lambda}(A) \cong fW_{*}^{\lambda}(A/I) .$$

(1.14) *The localization sequence.* The following is a variation on Theorem (2.1) of [32]. The proof given there was for FW_{*}^{λ} (denoted op. cit. “ W_{*}^{λ} ”); except at one very important point it is routine to modify to work for the groups fW_{*}^{λ} . Namely, Sharpes normal form [46] used in [32, §5] must be replaced by a projective version, due to Ranicki [39, 5.4]. Or one may refer to Ranicki's proof in [40].

(1.15) THEOREM. *Let A be a ring-with-involution and B a ring of quotients as in (1.1). Then there is a long exact sequence of abelian groups*

$$\cdots \longrightarrow fW_1^{\lambda}(A) \xrightarrow{\mathcal{K}_1^{\lambda}} fW_1^{\lambda}(B) \xrightarrow{\mathcal{L}_1^{\lambda}} fW_1^{\lambda}(B/A) \xrightarrow{\mathcal{D}_1^{\lambda}} fW_0^{\lambda}(A) \xrightarrow{\mathcal{K}_1^{\lambda}}$$

$$fW_0^\lambda(B) \xrightarrow{\mathcal{L}_0^\lambda} fW_0^\lambda(B/A) \xrightarrow{\mathcal{L}_0^\lambda} fW_1^{-\lambda}(A) \xrightarrow{\mathcal{K}_1^{-\lambda}} fW_1^{-\lambda}(B) \longrightarrow \dots$$

(1.16) The map \mathcal{L}_1^λ in (1.15) has to be discussed in detail (see [32, §3]). Let $g: B^{2m} \times B^{2m} \rightarrow B$ be λ -hermitian. There is a projective A -submodule (*integral lattice*) $L \subseteq B^{2m}$, $L \otimes B = B^{2m}$, such that $g(L \times L) \subseteq A$ (in fact, $g|L \times L \in \text{Sesq}_\lambda(L)$, c.f. [2, I. 3.3]). Let $L' := \{x \in B^{2m} | g(L, x) \subseteq A\}$ (the *dual lattice*), $S := \text{cok}(L \rightarrow L')$, $R :=$ the resolution of S , $\{L \rightarrow L' \xrightarrow{j} S\}$, and $\tau := g|L' \times L': L' \times L' \rightarrow B$. Then the class $[(S, \phi, \psi)] \in fW_0^\lambda(B/A)$ is by definition $\mathcal{L}_0^\lambda([B^{2m}, g])$, where $\phi: S \times S \rightarrow B/A$ and $\psi: S \rightarrow B_\lambda(A)$ are defined by

$$(1.17) \quad \phi(jm, jn) \equiv \tau(m, n) \bmod A, \quad \psi(jm) \equiv \tau(m, m) \bmod S_\lambda(A)$$

where $S_\lambda(A) := \{a \in A | a = b + \lambda \bar{b}, b \in A\}$.

(1.18) DEFINITION. If the λ -quadratic form over B/A arises as above from (B^{2m}, g) and L , then (L, g) (or equivalently (R, τ)) is said to be a *resolution* of (S, ϕ, ψ) . (This notion is also studied in [12], where “lifting” is used for “resolution”.)

(1.19) PROPOSITION. *With the notation above, a λ -quadratic form over B/A , (S, ϕ, ψ) , is resolvable if and only if $[S, \phi, \psi] \in \text{im}(\mathcal{L}_0^\lambda)$.*

Proof. “Only if” is definition; so suppose given $[V, g] \in fW_0^\lambda(B)$ such that $V \cong B^{2n}$ and

$$\mathcal{L}_0^\lambda[V, g] = [S, \phi, \psi].$$

Choosing an integral lattice L , it follows that there is a resolvable form (S', ϕ', ψ') such that $(S', \phi', \psi') \perp (H_1, \phi_1, \psi_1) \cong (S, \phi, \psi) \perp (H_2, \phi_2, \psi_2)$, where the (H_i, ϕ_i, ψ_i) are kernels ([32, 1.13(b)]). Since kernels are resolvable (see [32, 5.2]), one can take $H_1 = 0$. The proof is finished by taking $K = \text{subkernel of } H_2$ in the following lemma.

(1.20) LEMMA. *Suppose given (T, ν, μ) , a resolvable λ -quadratic form over B/A , and $K \subseteq T$ such that $\nu|K \times K \equiv 0 \equiv \mu|K$ and K has a short projective resolution. Then the naturally induced form on K^\perp/K is resolvable.*

Proof. Let (R, τ) be the resolution, $R = (L \rightarrow L' \xrightarrow{j} S)$, $\tau: L' \times L' \rightarrow B$, and let $k: S \rightarrow S/K$ be the quotient map. Defining $M := \ker(kj)$, it follows from the hypotheses on K that $\tau(M \times M) \subseteq A$ and $L \subseteq M \subseteq M' \subseteq L'$. It is now easily shown that $M'/M \cong K^\perp/K$

and that $(M \twoheadrightarrow M' \rightarrow K^\perp/K, \tau|M')$ is the desired resolution.

2. Qualitative properties; Arf and Dickson invariants.

(2.1) Let A be a maximal \mathbf{Z} -order in the semi-simple \mathbf{Q} -algebra B . Then

$$\mathcal{K}_0^\lambda + r_2: fW_0^\lambda(A) \longrightarrow fW_0^\lambda(B) + fW_0^\lambda(A/2A)$$

is injective, where \mathcal{K}_0^λ is from (1.15) and r_2 is induced by $A \rightarrow A/2A$.

Proof. If $p \in \mathbf{Z}$ is prime, then modulo its (nilpotent) radical, A/pA is semi-simple, whence, by (1.13) and [33, 4.1], $L_1^p(A/pA) = 0$ if p is odd. By (1.10), $fW_1^\lambda(B/A) \cong fW_1^\lambda(B/A_{(2)}) \cong fW_1^\lambda(B_2/A_2)$. Thus, there is a commutative diagram

$$\begin{array}{ccccc} fW_1^\lambda(B/A_{(2)}) & \longrightarrow & fW_0^\lambda(A) & \xrightarrow{\mathcal{K}_0^\lambda} & fW_0^\lambda(B) \\ \downarrow & & \downarrow & \searrow & \\ fW_1^\lambda(B_2/A_2) & \xrightarrow{\beta} & fW_0^\lambda(A_2) & & \\ & & \downarrow \cong & \nearrow r_2 & \\ & & fW_0^\lambda(A/2A) & & \end{array}$$

from which the result follows if β is injective. But $fW_1^\lambda(A_2) \rightarrow fW_1^\lambda(B_2)$ is surjective, because [33, 4.1] gives representatives for the elements of $fW_1^\lambda(B_2)$; and the maximality of A_2 means that if $B_2 = \prod \mathcal{B}_i$ is a product of simple algebras, then $A_2 = \prod \mathcal{A}_i$ where each \mathcal{A}_i is maximal in \mathcal{B}_i so that representatives can be pulled back.

(2.2) REMARK. When A is not maximal, $\mathcal{K}_0^\lambda + r_2$ is no longer injective. In fact, there is an exact sequence

$$\hat{H}^0(\mathbf{Z}/2; K_{-1}(A)) \xrightarrow{i} fW_0^\lambda(A) \longrightarrow fW_0^\lambda(B) + fW_0^\lambda(A/2A)$$

valid when A is any \mathbf{Z} -order in B ; and i is nontrivial, for example, when $A = \mathbf{Z}Q_{16}$, Q_{16} = the generalized quaternion group of order 16 and $\lambda = -1$.

The following sort of result is important in Petrie's theory [37] and was also useful in [35].

(2.3) Let A be a \mathbf{Z} -order in the \mathbf{Q} -algebra B . Then for any prime p , the map

$$r_p: FW_1^\lambda(A_{(p)}) \longrightarrow FW_1^\lambda(A/pA)$$

is injective.

Proof. Suppose $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in U_{2n}^\lambda(A_{(p)})$ is given, representing $[\sigma] \in FW_1^\lambda(A_{(p)})$, and $\sigma_p = \begin{pmatrix} \alpha_p & \beta_p \\ \gamma_p & \delta_p \end{pmatrix} \in U_{2n}^\lambda(A/pA)$ represents zero in $FW_1^\lambda(A/pA)$. It follows from the normal form of [46] that after multiplying σ_p (on the left and right) by matrices X_+ , X_- , and w_n^λ , n even (see [32] for this notation), it becomes $H(\rho)$, for some $\rho \in GL_n(A/pA)$ (it may also be necessary to stabilize). But each matrix of type X_+ , X_- , or w_n^λ can be lifted to a matrix of the same type over $A_{(p)}$; this uses the fact that $S_{-\lambda}(A_{(p)}) \rightarrow S_{-\lambda}(A/pA)$ is surjective. Thus, one may assume σ has the property that α is invertible mod p ; by Nakayama's lemma, this means α is invertible. By [2, II. 2.5(b)], $[\sigma] = 0$.

The next result is central to the present style of computation.

(2.4) COROLLARY. *Let A be a \mathbf{Z} -order in the \mathbf{Q} -algebra $B = A[S^{-1}]$, $S = \mathbf{Z} - \{0\}$. Let $fW_0^\lambda(B/A[1/2]) \subseteq fW_0^\lambda(B/A)$ be the subgroup consisting of forms supported on odd \mathbf{Z} -torsion A -modules, and $\mathcal{R} \subseteq fW_0^\lambda(B/A[1/2])$ the subgroup generated by resolvable forms. Then \mathcal{D}_0^λ (from (1.15)) induces an isomorphism*

$$fW_0^\lambda\left(B/A\left[\frac{1}{2}\right]\right)/\mathcal{R} \cong \ker \{r_2: fW_1^{-\lambda}(A) \longrightarrow fW_1^{-\lambda}(A/2A)\}$$

and by (1.6) this equals $L_{-\lambda}^2(A)$ if there is a surjection $A \rightarrow \mathbf{F}_2$ of rings with involution.

Proof. By definition, if (S, ϕ, ψ) is a λ -quadratic form over B/A , then $\mathcal{D}_0^\lambda[S, \phi, \psi] = [P, Q, (\alpha, \gamma)] = [\sigma]$ where $\text{cok}\{\alpha: P \rightarrow Q\} = S$. Thus $r_2[\sigma]$ has corresponding α (denoted α_2 in the proof of (2.3)) invertible, so by [33, 3.1, 4.1] represents zero in $fW_1^{-\lambda}(A/2A)$. Conversely, noting that, if P is A -projective, then $S_\lambda(P) \rightarrow S_\lambda(P/2P)$ is surjective, and replacing the use of Sharpe's normal form in (2.3) by [39, 5.4], it follows that if $r_2[\sigma] = 0$, then $\sigma = (P, Q, (\alpha, \gamma))$ may be assumed to satisfy $\text{cok}(\alpha_{(2)}) = 0$. Again by the definition of \mathcal{D}_0^λ , $[\sigma] = \mathcal{D}_0^\lambda(S, \phi, \psi)$, where S is odd torsion. Thus,

$$\ker r_2 = \text{im} \left(\mathcal{D}_0^\lambda | W_0^\lambda\left(B/A\left[\frac{1}{2}\right]\right) \right).$$

By (1.19), the proof is complete.

The preceding results (2.3) and (2.4) show there are no non-trivial invariants to be found by reducing representatives of $fW_1^\lambda(\mathbf{Z}\pi)$

modulo odd primes, π a 2-group (even though for p odd the groups $fW_1^i(F_p\pi)$ are, ingeneral, nontrivial—c.f. [35, 3.9], [33, 3.1]). For suppose $[\sigma] \in fW_1^i(\mathbb{Z}\pi)$ has nontrivial image in $fW_1^i(F_p\pi)$, p an odd prime. By results of Swan [10, 77.2], (2.3) applies to show $[\sigma_{(p)}] \neq 0$. By [35, (3.1), (3.9)], $[\sigma \otimes Q] \neq 0 \in fW_1^i(\mathbb{Z}\pi)$; but this contradicts the main result of [18].

To single out mod 2 reduction, the following definition is made.

(2.5) DEFINITION. Let B be a finite-dimensional Q -algebra, $A \subseteq B$ a \mathbb{Z} -order or any localization, completion or quotient of such an A . Given $x \in fW_*^i(A)$ its mod 2 reduction $r_2(x)$ in $fW_*^i(A/2A)$ is called the *Arf invariant* of x if $*$ = 0 and the *Dickson invariant* if $*$ = 1.

Modulo its radical, $A/2A$ is a product of matrix rings over finite fields of characteristic 2. By Morita theory and reduction (1.13), to compute $fW_*^i(A/2A)$ the following suffices.

(2.6) PROPOSITION. Let F_q be the field with q elements. Then

$$fW_*^i(F_q) = \begin{cases} \mathbb{Z}/2, & \text{if the involution is trivial} \\ 0, & \text{if the involution is nontrivial,} \end{cases}$$

where the nonzero representative is that given in (1.6), if $*$ is odd.

Proof. If $*$ = 1, results follow from [33, 4.1]; if $*$ = 0 and the involution is nontrivial this is the Arf invariant; if $*$ = 0 with nontrivial involution, see [60, p. 117, Ex. 1].

REMARK. In his study [11] of the orthogonal group of a quadratic form over a finite field F , $\text{char}(F) = 2$, Dickson proved (among many other things) that $fW_1^i(F) \cong \mathbb{Z}/2$ and derived a “normal form” (the generalization of which was used in (2.4)). The invariant of [33] is a generalization of Dickson’s to the case of semi-simple algebras with involution. An interesting historical point is that Dickson also classified quadratic forms over F , using what is now called the Arf invariant. This was 40 years before Arf’s work.

3. Proof of Theorem A and B.

(3.1) THEOREM. Let π be a finite 2-group. There is an involution-invariant maximal order \mathcal{M} ,

$$\mathbb{Z}\pi \subseteq \mathcal{M} \subseteq Q\pi, \quad \mathcal{M} = \prod \mathcal{M}_i$$

where each \mathcal{M}_i is involution-invariant and maximal in some simple

component of $\mathbf{Q}\pi$ (from which it inherits its involution) and has one of the following four forms

- (I) $M_{2^m}(\mathbf{Z}\zeta_{2^n})$, some $m \geq 0$, $n \geq 1$
- (II) $M_{2^m}(\mathbf{Z}(\zeta_{2^n} - \zeta_{2^n}^{-1}))$, some $m \geq 1$, $n \geq 3$
- (III) $M_{2^m}(\mathbf{Z}(\zeta_{2^n} + \zeta_{2^n}^{-1}))$, some $m \geq 1$, $n \geq 2$
- (IV) $M_{2^m}(\mathcal{N}_n)$, some $m \geq 0$, $n \geq 2$, where \mathcal{N}_n is a maximal order in the quaternion algebra $(-1, -1/\mathbf{Q}(\zeta_{2^n} + \zeta_{2^n}^{-1}))$ (see [25] for this notation).

Each type has a uniquely determined involution, which need not be specified until the theorem is proved in (4.16). The following table summarizes the calculations found in (5.3), (5.4), (6.18), (6.19), (7.14), (7.15), (4.3), and (4.16). Notice that, because of (2.4), the second column is the kernel of the Dickson invariant. Thus, the Arf and Dickson invariants over the \mathcal{M}_i in (3.1) are precisely what is needed to compute $L_\lambda^p(\mathbf{Z}\pi)$.

(3.2) Table.

\mathcal{M}_i \ Properties	$r_2: fW_0^2(\mathcal{M}_i) \rightarrow fW_0^2(\mathcal{M}_i/2\mathcal{M}_i)$	$fW_0^2\left(\mathcal{B}_i/\mathcal{M}_i\left[\frac{1}{2}\right]\right)/\mathcal{P}$ (c.f. (2.4))
(I) $M_{2^m}(\mathbf{Z}\zeta)$	$fW_0^2(\mathcal{M}_i) \rightarrow \mathbf{Z}/2$	trivial
(II) $M_{2^m}(\mathbf{Z}(\zeta - \zeta^{-1}))$	$fW_0^2(\mathcal{M}_i) \rightarrow \mathbf{Z}/2$	trivial
(III) $M_{2^m}(\mathbf{Z}(\zeta + \zeta^{-1}))$	$fW_0^2(\mathcal{M}_i) \rightarrow \mathbf{Z}/2 \begin{cases} \text{trivial, } \lambda = 1 \\ \text{surjective, } \lambda = -1 \end{cases}$	trivial
(IV) $M_{2^m}(\mathcal{N}_n)$	$n = 2: fW_0^2(\mathcal{N}_i) \rightarrow 0$ $n \geq 3: fW_0^2(\mathcal{N}_i) \rightarrow \mathbf{Z}/2 \begin{cases} \text{trivial, } \lambda = 1 \\ \text{surjective, } \lambda = -1 \end{cases}$	$n = 2: \begin{cases} \mathbf{Z}/2 + \mathbf{Z}/2, \lambda = -1 \\ \text{trivial, } \lambda = 1 \end{cases}$ $n \geq 3: \begin{cases} (\mathbf{Z}/2)^{2^n-2}, \lambda = -1 \\ \text{trivial, } \lambda = 1 \end{cases}$

(3.3) PROPOSITION. For rings A, B as in (1.1), let $\mathcal{Q}^\lambda(B/A)$ denote the set of isometry classes of λ -quadratic forms (S, ϕ, ψ) over B/A . If π, \mathcal{M} and \mathcal{M}_i are as in (3.1), then there is a set isomorphism

$$\mathcal{Q}^\lambda\left(\mathbf{Q}\pi/\mathbf{Z}\left[\frac{1}{2}\right]\pi\right) \longrightarrow \prod_i \mathcal{Q}^\lambda\left(\mathcal{B}_i/\mathcal{M}_i\left[\frac{1}{2}\right]\right)$$

$$(S, \phi) \longmapsto \{(S_{\mathcal{M}_i}, \phi_{\mathcal{M}_i})\}$$

where $S_{\mathcal{M}_i} := S \otimes_{\mathbf{Z}\pi} \mathcal{M}_i$, $\phi_{\mathcal{M}_i} := \phi \otimes \mathcal{M}_i$ and $\mathcal{B}_i = \mathcal{M}_i \otimes \mathbf{Q}$ is the simple component of $\mathbf{Q}\pi$ containing \mathcal{M}_i . (Since S and the $S_{\mathcal{M}_i}$ are odd torsion, the quadratic part ψ of (S, ϕ, ψ) is omitted from the notation here and below.)

Proof. By [42, 41.1], $2^s \mathcal{M} \subseteq \mathbb{Z}\pi$, $2^s = |\pi|$. Thus, inclusion induces $\mathbb{Z}[1/2]\pi \xrightarrow{\cong} \mathcal{M}[1/2] := \mathcal{M} \otimes \mathbb{Z}[1/2]$, from which the result is immediate.

Given $x \in L_1^p(\mathbb{Z}\pi)$, let $\langle S, \phi \rangle$ denote the corresponding coset (cf. (2.4)) in $fW_0^{-1}(\mathbb{Q}\pi/\mathbb{Z}[1/2]\pi)/\mathcal{R}$, and $(S_{\mathcal{M}_i}, \phi_{\mathcal{M}_i})$ the elements of $\mathcal{C}^{-1}(\mathcal{B}_i/\mathcal{M}_i)$ corresponding to $\langle S, \phi \rangle$. Let $\text{Sp}_n(\pi)$ denote the number of factors in \mathcal{M} of type (3.1) (IV). (These are of type Sp, in the language of [48].)

(3.4) PROPOSITION. *With the above notation, there is a surjection*

$$L_1^p(\mathbb{Z}\pi) \xrightarrow{R} (\mathbb{Z}/2)^{2\text{Sp}_2(\pi)} + \sum_{n \geq 2} (\mathbb{Z}/2)^{2^n - 2\text{Sp}_n(\pi)}$$

such that $R(x)=0$ if and only if $[S_{\mathcal{M}_i}, \phi_{\mathcal{M}_i}]=0$ in $fW_0^{-1}(\mathcal{B}_i/\mathcal{M}_i[1/2])/\mathcal{R}$, for each i (i.e., each $(S_{\mathcal{M}_i}, \phi_{\mathcal{M}_i})$ is resolvable).

Proof. R is defined by taking the class $[S_{\mathcal{M}_i}, \phi_{\mathcal{M}_i}]$ in $fW_0^{-1}(\mathcal{B}_i/\mathcal{M}_i[1/2])/\mathcal{R}$ and using the Table. It is well-defined because of (3.3) and the fact that if $\langle S, \phi \rangle$ is resolvable over $\mathbb{Z}\pi$, then each $(S_{\mathcal{M}_i}, \phi_{\mathcal{M}_i})$ is resolvable over \mathcal{M}_i ; it is surjective because of (3.3).

If $|\pi| = 2^s$, then $2^s \mathcal{M} \subseteq \mathbb{Z}\pi$ by [42, 41.1]. The following is a Cartesian square of rings-with-involution, the “conductor situation” of [1, p. 535].

$$(3.5) \quad \begin{array}{ccc} \mathbb{Z}\pi & \longrightarrow & \mathcal{M} \quad (= \Pi \mathcal{M}_i) \\ \downarrow & & \downarrow r_2^s \\ \mathbb{Z}\pi/2^s \mathcal{M} & \longrightarrow & \mathcal{M}/2^s \mathcal{M} \quad (= \Pi \mathcal{M}_i/2^s \mathcal{M}_i) . \end{array}$$

It is not difficult to show that for some r , $2^r \mathbb{Z}\pi \subseteq 2^s \mathcal{M}$, from which it follows there are surjections

$$\mathbb{Z}\pi/2^s \mathcal{M} \longrightarrow \mathbb{Z}\pi/2^r \mathbb{Z}\pi \longrightarrow F_2\pi \longrightarrow F_2$$

where the last map is the augmentation. The kernel of the composite is nilpotent by [47, 4.3] and because $2^s \mathcal{M} \subseteq 2^s \mathbb{Z}\pi$. By reduction (1.13), the induced map

$$fW_0^\lambda(\mathbb{Z}\pi/2^s \mathcal{M}) \longrightarrow fW_0^\lambda(F_2) \xrightarrow{\cong} \mathbb{Z}/2$$

is an isomorphism. Thus, the isometry class of a nonsingular λ -quadratic form over $\mathbb{Z}\pi/2^s \mathcal{M}$ is determined by its rank and Arf invariant.

(3.6) PROPOSITION. *Referring to the rings and maps of (3.5),*

the isometry class of a λ -quadratic form over $\mathbf{Z}\pi/2^s\mathcal{M}$ or $\mathcal{M}_i/2^s\mathcal{M}_i$ is determined by rank and Arf invariant; if \mathcal{M}_i is of type (3.1) (IV) with $n = 2$, then the Arf invariant is always trivial. A nonsingular λ -quadratic form over $\mathbf{Z}\pi/2^s\mathcal{M}$ has nontrivial Arf invariant iff its image does, in each component $\mathcal{M}_i/2^s\mathcal{M}_i$ of $\mathcal{M}/2^s\mathcal{M}$, except for those of type (3.1) (IV) with $n = 2$.

Proof. The first statement for $\mathbf{Z}\pi/2^s\mathcal{M}$ was proved above; for $\mathcal{M}_i/2^s\mathcal{M}_i$ it follows from the Table. Let (N, g, q) be the nonsingular λ -quadratic form over N , a free rank 2 $\mathbf{Z}\pi/2^s\mathcal{M}$ -module, where if N has basis $\{e, f\}$, g has matrix $\begin{pmatrix} 1+\lambda & 1 \\ \lambda & 1+\lambda \end{pmatrix}$ and $q(e) = 1 = q(f)$. Clearly, both it and its image in $\mathcal{M}_i/2^s\mathcal{M}_i$ have Arf invariant 1, for each i .

(3.7) DEFINITION. A nonsingular λ -quadratic form over $\mathcal{M}/2^s\mathcal{M} = \coprod \mathcal{M}_i/2^s\mathcal{M}_i$ is said to have equal Arf invariants if either the Arf invariants of its components in each $\mathcal{M}_i/2^s\mathcal{M}_i$ are all zero; or are all equal to 1, except in components of type (3.1) (IV) with $n = 2$.

(3.8) DEFINITION. Let π be a finite 2-group. Define $O(\pi)$ to be the number of components in \mathcal{M} of type (3.1) (III), and $\text{Sp}(\pi)$ to be the number of type (3.1) (IV) with $n > 2$.

(3.9) THEOREM A. Let π be a finite 2-group, and let R be as in (3.4). Then there is an isomorphism

$$(a) \quad E: L_{-1}^p(\mathbf{Z}\pi) \xrightarrow{\cong} (\mathbf{Z}/2)^{O(\pi)-1}$$

and a split short exact sequence

$$(b) \quad (\mathbf{Z}/2)^{\text{Sp}(\pi)} \longrightarrow L_1^p(\mathbf{Z}\pi) \longrightarrow (\mathbf{Z}/2)^{2\text{Sp}_2(\pi)} + \sum_{n>2} (\mathbf{Z}/2)^{2^n-2\text{Sp}_n(\pi)}.$$

Proof. By the Table (3.2), given $x \in fW_0^1(Q\pi/\mathbf{Z}[1/2]\pi)/\mathcal{R}$ ($\cong L_{-1}^p(\mathbf{Z}\pi)$, by (2.4)), the corresponding coset $\langle S, \phi \rangle$, S an odd \mathbf{Z} -torsion $\mathbf{Z}\pi$ -module, is such that each $(S_{\mathcal{M}_i}, \phi_{\mathcal{M}_i})$ is resolvable for all i , say, by (L_i, g_i) over \mathcal{M}_i . By [55] (or exactness at $fW_0^1(\mathcal{B}_i)$ in (1.15)), (L_i, g_i) is uniquely determined by $(S_{\mathcal{M}_i}, \phi_{\mathcal{M}_i})$, up to orthogonal sum with a nonsingular $(+1)$ -quadratic form over \mathcal{M}_i .

Each $(L_i, g_i)_{(2)}$ is nonsingular because, by construction, $\text{cok}(\text{Ad}(g_i)) = S_{\mathcal{M}_i}$ is odd torsion. Thus, the mod 2^s reduction $r_{2^s}(L_i, g_i)$ is nonsingular over $\mathcal{M}_i/2^s\mathcal{M}_i$. By the result of Wall referred to above, together with the data in the first column of (3.2), the Arf invariant of $r_{2^s}(L_i, g_i)$ is determined by $(S_{\mathcal{M}_i}, \phi_{\mathcal{M}_i})$ if and only

if \mathcal{M}_i is of type III in (3.1); otherwise it can be changed by adding to (L_i, g_i) a nonsingular form over \mathcal{M}_i , whose Arf invariant is 1, without changing the form $(S_{\mathcal{M}_i}, \phi_{\mathcal{M}_i})$ being resolved.

Let $\Delta: \mathbf{Z}/2 \rightarrow (\mathbf{Z}/2)^{O(\pi)}$ be the diagonal inclusion, and define $E(x) \in (\mathbf{Z}/2)^{O(\pi)}/\text{im } \Delta$ to be the coset in $(\mathbf{Z}/2)^{O(\pi)}$ whose components are the Arf invariants of the $r_{2^i}(L_i, g_i)$ for \mathcal{M}_i of type III. Thus $E(x) = 0$ iff the forms $r_{2^i}(L_i, g_i)$ have equal Arf invariants, for all i .

Finally, by (3.6) and a theorem of Bass [2, III. 2.2], $E(x) = 0$ iff the collection of forms (L_i, g_i) (for all i) lift back to a form (L, g) over $\mathbf{Z}\pi$, in which case it is easily seen that (L, g) resolves (S, ϕ) . This means $x = \langle S, \phi \rangle$ represents zero in $fW_0^1(Q\pi/\mathbf{Z}[1/2])/\mathcal{R} \cong L_1^2(\mathbf{Z}\pi)$.

Exactly the same argument, applied to $\ker R$, replacing type (3.1) (III) factors by type (3.1) (IV) factors of \mathcal{M} , shows $\ker R \cong \text{Sp}(\pi)$. A splitting will be exhibited in (3.16).

The esthetic and practical difficulties in the proof of Theorem A are evident. What will be shown next is that, in the construction of E (3.9) (a), the Arf invariant of a form resolving $(S_{\mathcal{M}_i}, \phi_{\mathcal{M}_i})$ (\mathcal{M}_i of type (3.1) (III)) depends only on the number of elements $|S_{\mathcal{M}_i}|$ of $S_{\mathcal{M}_i}$; in particular, it is independent both of the structure of $S_{\mathcal{M}_i}$ as an \mathcal{M}_i - or $\mathbf{Z}\pi$ -module, and of the hermitian form ϕ . This generalizes a well-known theorem of Levine [29].

(3.10) LEMMA [28, 2.7]. *Let R denote the \mathfrak{p} -adic completion of $\mathbf{Z}(\zeta + \zeta^{-1})$, $\zeta = \zeta_{2^n}$, $n \geq 2$, where \mathfrak{p} is its unique dyadic prime. The map $R \rightarrow R^\times/R^{\times 2}$ defined by $r \rightarrow 1 + 4r$, $r \in R$, induces an isomorphism*

$$j: \mathbf{Z}/2 = R/\mathfrak{p} \xrightarrow{\cong} \ker \alpha_2$$

where $\alpha_2: R^\times/R^{\times 2} \rightarrow (R/4R)^\times/(R/4R)^{\times 2}$ is induced by reduction mod 4. There is a commutative diagram of isomorphisms.

$$\begin{array}{ccc} FW_0^1(R) & \xrightarrow[\cong]{\text{dis}} & \ker \alpha_2 \\ (1.13) \downarrow \cong & & \cong \uparrow j \\ fW_0^1(R/\mathfrak{p}) & \xrightarrow{\text{Arf}} & \mathbf{Z}/2 \end{array}$$

EXAMPLE. When $n = 2$ in (3.10), $R = \mathbf{Q}_2$, the 2-adic rationals. Let $g: \mathbf{Z}^n \times \mathbf{Z}^n \rightarrow \mathbf{Z}$ arise as $g|L \times L$ in the construction of (1.16) and suppose $g_{(2)}: (\mathbf{Z}_{(2)})^n \times (\mathbf{Z}_{(2)})^n \rightarrow \mathbf{Z}_{(2)}$ is nonsingular. Then (\mathbf{Z}^n, g) resolves (in the sense of (1.18)) a symmetric form (S, ϕ) , $\phi: S \times S \rightarrow \mathbf{Q}/\mathbf{Z}$, where S is odd torsion (because $g_{(2)}$ is nonsingular). It is well-known that $\text{dis } g = \pm |S|$, where $|S|$ is the number of elements in S (because S is the cokernel of $\text{Ad } (g): \mathbf{Z}^n \rightarrow \text{Hom}(\mathbf{Z}^n, \mathbf{Z})$, the adjoint of g). As $\ker \alpha_2$ is represented by the class of 5 in $\mathbf{Z}_2^\times/\mathbf{Z}_2^{\times 2}$, and

$a \in \mathbf{Z}_2$ is a square if and only if $a \equiv 1 \pmod{8}$, Lemma (3.10) asserts that the Arf invariant of (\mathbf{Z}^n, g) is nontrivial if and only if $|S| \equiv \pm 5 \pmod{8}$. This is the theorem of Levine referred to above.

Another way of starting Levine's result uses the fact that for $n \in \mathbf{Z}$, $n \equiv \pm 5 \pmod{8}$ if and only if the Legendre symbol $(2/n) = -1$. Lannes [28] generalized this replacing \mathbf{Z} by a ring of integers in a number field, 2 by an arbitrary unramified (over \mathbf{Z}) dyadic prime, and the Legendre symbol by the Artin symbol. In the case of present interest, of course, the ring of integers $\mathbf{Z}(\zeta + \zeta^{-1})$, $\zeta = \zeta_{2^n}$, is totally ramified over $(2) \subseteq \mathbf{Z}$, so that the generalization of Levine's result given below in (3.13) seems to be new.

(3.11) LEMMA. Let $R_n = \mathbf{Z}(\zeta + \zeta^{-1})_{\mathfrak{p}}$, $\zeta = \zeta_{2^n}$, $n \geq 2$, where \mathfrak{p} is the unique dynamic prime of $\mathbf{Z}(\zeta + \zeta^{-1})$. Let $N: R_n \rightarrow \mathbf{Z}_2$ be the norm. Then

$$(a) \quad N(R_n^{\times 2}) \subseteq (1 + 2^{n+1}\mathbf{Z}_2)^{\times}$$

$$(b) \quad \text{If } \bar{N}: R_n^{\times}/R_n^{\times 2} \rightarrow \mathbf{Z}_2^{\times}/(1 + 2^{n+1}\mathbf{Z}_2)^{\times}$$

is the induced map, then $\bar{N}|_{\ker \alpha_2}$ is injective, where $\alpha_2: R_n^{\times}/R_n^{\times 2} \rightarrow (R_n/4R_n)^{\times}/(R_n/4R_n)^{\times 2}$ is reduction mod 4.

Proof. (2) Since $N = N_3 \circ \dots \circ N_{n-1} \circ N_n$, where $N_k: R_k^{\times} \rightarrow R_{k-1}^{\times}$ is the norm, since $R_n^{\times 2} = (1 + 4\pi_n R_n)^{\times}$, where π_n is the uniformizer of R_n ([31, 63:1a]), and since π_2 , the uniformizer of $R_2 = \mathbf{Z}_2$, is (2), it suffices to show that

$$(3.12) \quad N_k(1 + 2^l u \pi_k) \equiv 1 \pmod{2^{l+1} \pi_{k-1}},$$

for each $u \in R_k$, $3 \leq k \leq n$, and $l \geq 2$. By [44, V. 3, Lemma 5],

$$N_k(1 + 2^l u \pi_k) \equiv 1 + N_k(2^l u \pi_k) + \text{Tr}_k(2^l u \pi_k) \pmod{\text{Tr}_k(4^l \pi_k^2)},$$

where $\text{Tr}_k: R_k \rightarrow R_{k-1}$ is the trace. It follows from [44, V. 3, Lemma 4], that

$$\text{Tr}_k(2^l u \pi_k) \in 2^{l+1} \pi_{k-1} R_{k-1} \quad \text{and} \quad \text{Tr}_k(4^l \pi_k^2) \in 2^{2l+1} \pi_{k-1} R_{k-1}.$$

Evidently $N_k(2^l u \pi_k) = 4^l u \bar{u} \pi_k \bar{\pi}_k \in 4^l \pi_{k-1} R_{k-1}$. These facts prove (3.12) and hence part (a).

Part (b) follows from the fact that $\ker \alpha_2 \cong \mathbf{Z}/2$ (see (3.10)), is represented by 5, and has norm $N(5) = 5^{2^n-2} \not\equiv 1 \pmod{2^{n+1}}$.

Let $A = \mathbf{Z}(\zeta + \zeta^{-1})$, $\zeta = \zeta_{2^n}$, $n \geq 2$, and let $g: A^n \times A^n \rightarrow A$ be a symmetric form such that $g \in \text{Sesq}_1(A)$ (i.e., $g = h + \bar{h} = h + h^t$, for some sesquilinear h), and $g_{(2)}: A_{(2)}^n \times A_{(2)}^n \rightarrow A_{(2)}$ is nonsingular. By the construction of (1.16), (A^n, g) resolves some hermitian form (S, ϕ) where S is a nondyadic torsion A -module. Let $|S|$ denote the number of elements in S .

(3.13) THEOREM. *With the notation above, the Arf invariant of (A^n, g) is trivial if and only if $|S| \equiv \pm 1 \pmod{2^{n+1}}$, and is non-trivial if and only if $|S| \equiv 2^n \pm 1 \pmod{2^{n+1}}$.*

Proof. By [44, I, §5], $|S| = \pm N \pmod{2^{n+1}}$, where $N: A_{(2)} \rightarrow Z_{(2)}$ is the norm. By (3.10) and (3.11) the first statement is proved. The second follows from [58, 7.2.4].

Unfortunately, the ring of integers $Z(\zeta + \zeta^{-1})$ does not itself appear as a factor of the maximal order \mathcal{M} containing $Z\pi$ in (3.1). However, its Morita equivalent, $M_{2^m}(Z(\zeta + \zeta^{-1}))$, does, so (3.13) will be translated to this context.

Let \mathcal{M}_i be a component of \mathcal{M} of type (3.11) (III) where $\mathcal{M}_i \cong M_{2^m}(Z(\zeta + \zeta^{-1}))$, $\zeta = \zeta_{2^n}$, $n \geq 2$. Let $(S_{\mathcal{M}_i}, \phi_{\mathcal{M}_i})$ be a hermitian form over B_i/\mathcal{M}_i , where B_i is the corresponding simple component of $Q\pi$ and $S_{\mathcal{M}_i} \otimes Z_{(2)} = 0$ ($S_{\mathcal{M}_i}$ is odd torsion). Suppose (\mathcal{M}_i^n, g) is a resolution of $(S_{\mathcal{M}_i}, \phi_{\mathcal{M}_i})$. Let $|S_{\mathcal{M}_i}|$ denote the number of elements in $S_{\mathcal{M}_i}$.

(3.14) THEOREM. *With the above notation, the Arf invariant of (\mathcal{M}_i^n, g) is trivial if and only if $|S_{\mathcal{M}_i}| \equiv \pm 5 \pmod{m2^{n+1}}$.*

Proof. From [3], the inverse to the isomorphism $m(B/A)$ of (7.3) is given (without the quadratic form ψ since S is odd torsion) by $[S, \phi] \rightarrow [S \otimes_A A^{2^m}, \phi']$ where (S, ϕ) is λ -hermitian over B/A , S is odd torsion, $S \otimes_A A^{2^m}$ is given an $M_{2^m}(A)$ -module structure, ϕ' is λ -hermitian over $M_{2^m}(B)/M_{2^m}(A)$, and $A = Z(\zeta + \zeta^{-1})$, $B = Q(\zeta + \zeta^{-1})$. In particular, $|S| \equiv \pm 1 \pmod{2^{n+1}}$ if and only if $|S \otimes_A A^{2^m}| \equiv \pm 1 \pmod{m2^{n+1}}$; and the Arf invariant of a form resolving (S, ϕ) is trivial if and only if its Morita equivalent form over $M_{2^m}(A)$ (which, by the construction of $m(B/A)$, resolves $(S \otimes_A A^{2^m}, \phi')$) has trivial Arf invariant. This completes the proof.

Before stating the main result of this paper, Theorem B, fix the following notation. Let $x \in L_1^2(Z\pi)$ be given where π is a finite 2-group. Using (2.4) suppose $x = \mathcal{D}_0^{-\lambda}(y)$, $y \in fW_0^{-\lambda}(Q\pi/Z[1/2]\pi)$, where y is represented by (S, ϕ) and S is odd torsion. Let $\mathcal{M}_1^{-1}, \dots, \mathcal{M}_k^{-1}$ be the components of the maximal order \mathcal{M} which are of type (3.1) (III), and $\mathcal{M}_1^1, \dots, \mathcal{M}_l^1$ those of type (3.1) (IV) with $n \geq 3$. Then $\mathcal{M}_i^{-1} \cong M_{2^{m_i}}(Z(\zeta + \zeta^{-1}))$, $\zeta = \zeta_{2^{n_i}}$, $1 \leq i \leq k$, $m_i \geq 0$, $n_i \geq 2$; and $\mathcal{M}_i^1 = M_{2^{m_i-1}}(\mathcal{N}_{n_i})$, \mathcal{N}_{n_i} maximal in $(-1, -1/Q(\zeta + \zeta^{-1}))$, $\zeta = \zeta_{2^{n_i}}$, $m_i \geq 1$, $n_i \geq 3$. Define $b_i^2(S) \in Z/2$ by

$$(3.15) \quad b_i^2(S) = \begin{cases} 0, & |S \otimes \mathcal{M}_i| \equiv \pm 1 \pmod{m_i 2^{n_i+1}} \\ 1, & \text{otherwise.} \end{cases}$$

(3.16) THEOREM B. *Let π be a finite 2-group. Then (3.15)*

defines homomorphisms

$$b_i^\lambda: L_k^p(\mathbf{Z}\pi) \longrightarrow \mathbf{Z}/2$$

such that

(a) when $\lambda = -1$, $x \in L_k^p(\mathbf{Z}\pi)$ is zero if and only if $b_1^{-1}(x) = b_2^{-1}(x) = \dots = b_k^{-1}(x)$.

(b) when $\lambda = 1$, the homomorphism $(b_1^1, \dots, b_l^1): L_1^p(\mathbf{Z}\pi) \rightarrow \mathrm{Sp}(\pi)$ splits (3.9) (b).

Proof. Assume $\lambda = -1$. Recall from the proof of (3.9) (a) that $E(x) = 0$ if and only if the resolutions of the form $(S_{\mathcal{M}_i}, \phi_{\mathcal{M}_i}): = (S, \phi) \otimes \mathcal{M}_i$ have equal Arf invariants, \mathcal{M}_i of type (3.1) (III). (Arf invariants for other types can be chosen as desired.) By Theorem (3.14) these Arf invariants equal the corresponding b_i^{-1} defined above.

The proof in case $\lambda = 1$ is left as an exercise.

REMARK. (a) Given a geometric context, i.e., $x \in L_{-1}^h(\mathbf{Z}\pi)$, (3.16) gives a fairly strong necessary condition for the vanishing of x . For example, since $\tilde{K}_0(\mathbf{Z}D_n) = 0$ where $\pi = D_n$ is the dihedral group (see (7.4) (a) and [14]), $L_*^h(\mathbf{Z}\pi) = L_*^p(\mathbf{Z}\pi)$. If Q_n is generalized quaternion ((7.4) (c)), $\tilde{K}_0(\mathbf{Z}Q_n) = \mathbf{Z}/2$ by [14]. In general, it is necessary to understand the maps in the Rothenberg sequence to know how strong (3.16) is in any given case. For this, see [16].

(b) Given $x \in L_\lambda^h(\mathbf{Z}\pi)$, how difficult is it to find $(S, \phi) \in fW_0^{-\lambda}(\mathbf{Q}\pi/\mathbf{Z}\pi)$ such that $\mathcal{D}_0^{-\lambda}(S, \phi) = x$ and S is odd torsion? Suppose $x \in L_\lambda^h(\mathbf{Z}\pi)$ is represented by $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in U_{2n}^\lambda(\mathbf{Z}\pi)$ (see (1.4)). Let α_2, γ_2 denote the image of α, γ under the map $\mathbf{Z}\pi \xrightarrow{\text{aug}} \mathbf{Z} \rightarrow \mathbf{F}_2$, the mod 2 augmentation. Since $fW_1^\lambda(\mathbf{F}_2) = \mathbf{Z}/2$, represented by $w_1^\lambda = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} \in U_2(\mathbf{F}_2)$, either σ or $\sigma \perp w_1^\lambda$ has the property that there is a symmetric matrix $\tilde{\rho} \in M_n(\mathbf{F}_2)$ having zeros on the diagonal such that $\gamma_2 + \tilde{\rho}\alpha_2$ is invertible. (Finding $\tilde{\rho}$ in practice is not too difficult since one works over the field \mathbf{F}_2 .) Choose any $\rho \in M_n(\mathbf{Z}\pi)$ such that ρ is $(-\lambda)$ -quadratic and $\rho_2 = \tilde{\rho}$. This is also easy. Since $\gamma_2 + \tilde{\rho}\alpha_2 \in GL_n(\mathbf{Z}_2)$, $\mathrm{cok}(\gamma + \rho\alpha)$ is odd torsion. This cokernel is S (see the construction [32] of $\mathcal{D}_0^{-\lambda}$), whose $\mathbf{Z}\pi$ -module structure (actually just the order of $S \otimes \mathcal{M}_i^\lambda$, \mathcal{M}_i^1 of type (3.1) (IV) \mathcal{M}_i^{-1} of type (3.1) (III)) is what is needed to apply (3.16).

(3.17) If the reader is familiar with the difficulties encountered in finding the surgery obstruction of a nonhighly-connected surgery problem, he will recognize that the reduction to odd torsion used above allows him to hope for a simple definition of the surgery obstruction of such a problem. Moreover, the fact that in the

analysis above, the $\mathbb{Z}\pi$ -modules involved (not the quadratic forms on which they are supported) alone determine the surgery obstruction, leads to the conclusion that an Euler characteristic invariant ought to work. This will now be made relatively precise. Since we will give no applications of the product formula (3.22), the proofs will only be sketched.

(3.18) DEFINITION. Let π be a finite group. $G_o(\mathbb{Q}\pi/\mathbb{Z}[1/2]\pi)$ is the free abelian group on isomorphism classes $[M]$ of odd torsion $\mathbb{Z}\pi$ -modules M , modulo the subgroup generated by elements $[M'] + [M''] - [M]$, whenever there is a short exact sequence $M' \twoheadrightarrow M \twoheadrightarrow M''$. $GU_o^*(\mathbb{Q}\pi/\mathbb{Z}[1/2]\pi)$ is the quotient of $G_o(\mathbb{Q}\pi/\mathbb{Z}[1/2]\pi)$ by the subgroup generated by $[M]$ for which there is a nonsingular λ -form $M \times M \rightarrow \mathbb{Q}\pi/\mathbb{Z}[1/2]\pi$.

(3.19) PROPOSITION. Let π be a 2-group. Then

$$GU_o^*(\mathbb{Q}\pi/\mathbb{Z}[1/2]\pi) \cong \begin{cases} (\mathbb{Z}/2)^{O(\pi)}, & \lambda = -1 \\ (\mathbb{Z}/2)^{Sp(\pi)}, & \lambda = 1. \end{cases}$$

Next, let K be a finite complex with $\pi_1 K = \pi$. Denote by $L_n^{\mathbb{Q}/\mathbb{Z}[1/2]}(K)$ the cobordism group of normal maps $(g, b): (N, \partial N; \nu_N) \rightarrow (Y, X; \xi)$, and maps $\omega: Y \rightarrow K$ where $g|_{\partial N}$ is a homotopy equivalence and the homology kernels $K_*(g)$ with $\pi_1 K$ -coefficients are odd torsion; cobordisms are to have the same restriction on homology kernels. (For a precise definition see [33, 1.6]. This group is computed in [33, §§6, 7], replacing \mathbb{Z} by $\mathbb{Z}[1/2]$.)

Let n be odd, and if M is an odd torsion $\mathbb{Z}\pi$ -module let $\{M\}$ denote its class in $GU_o^*(\mathbb{Q}\pi/\mathbb{Z}[1/2]\pi)$. If (g, b) is a normal map as above, define (its Euler characteristic)

$$X(g) = \sum_{i=0}^n (-1)^i \{K_i(g)\}.$$

(3.20) PROPOSITION. Let $\pi_1 K = \pi$. Then X defines a homomorphism

$$X: L_{2k+1}^{\mathbb{Q}/\mathbb{Z}[1/2]}(K) \longrightarrow GU_o^{(-1)^k}(\mathbb{Q}\pi/\mathbb{Z}[\frac{1}{2}]\pi).$$

Proof. X is clearly additive so it suffices to show $X(g) = 0$ if (g, b) is null-cobordant. If (G, B) is a normal cobordism with boundary (g, b) , then from the exact sequence of the pair (G, g) it follows that $X(G) = X(g) + X(G, g)$, or $X(g) = X(G) - X(G, g)$. Since $K_i(G, g) \cong K_{2k+1-i}(G)^\wedge$ and $M + M^\wedge$ always supports the hyperbolic

form, $X(g) = 0$ as claimed.

To use these results, recall the product pairing $L_m^h(\mathbb{Z}\pi) \times \Omega_n(\rho) \rightarrow L_{m+n}^h(\mathbb{Z}[\pi \times \rho])$. Supposing $n = 2l$ and ρ is a finite 2-group, a given class in $\Omega_n(\rho)$ may be represented by $M^{2l} \rightarrow K(\rho, 1)$ where $H_*(\tilde{M})$ has no odd torsion (for example by doing surgery on $M^{2l} \rightarrow BSO \times K(\rho, 1)$ to make it an $(l-1)$ -equivalence, where $M \rightarrow BSO$ classifies the normal bundle of M). If $m = 2k+1$ and π is a finite 2-group, then any element of $L_m^h(\mathbb{Z}\pi)$ can be represented by normal map (g, b) with $K_i(g) = 0$, $i \neq k$; and $K_k(g)$ odd torsion (see remark following (3.16)). Thus, if $(f, c) = (g \times 1_N, b \times 1_M)$ is the product normal map and \tilde{M} is the universal cover of M ,

$$K_i(f) = K_k(g) \otimes H_{i-k}(\tilde{M}),$$

an odd torsion $\mathbb{Z}[\pi \times \rho]$ -module. Since $K_k(g)$ is odd torsion, $K_i(f) = K_k(g) \otimes H_{i-k}(\tilde{M}; \mathbb{Z}[1/2])$. Here $H_*(\tilde{M}; \mathbb{Z}[1/2])$ is a $\mathbb{Z}[1/2]\rho$ -module. If $[\cdot]$ denotes its isomorphism class, then an Euler characteristic $\chi(\tilde{M}; \mathbb{Z}[1/2]\rho) \in G_0(\mathbb{Z}[1/2]\rho)$, the Grothendieck group of $\mathbb{Z}[1/2]\rho$ -modules, is defined by $\chi(\tilde{M}; \mathbb{Z}[1/2]\rho) = \sum_{i=0}^{2l} (-1)^i [H_i(\tilde{M}; \mathbb{Z}[1/2])]$. The usual argument using the equality of the Euler characteristic of a chain complex with that of its homology shows

(3.21) PROPOSITION. *For M as above,*

$$\chi(\tilde{M}; \mathbb{Z}\left[\frac{1}{2}\right]\rho) = \chi(M)R$$

where $\chi(M)$ is the (usual) Euler characteristic of M and $R \in G_0(\mathbb{Z}[1/2]\rho)$ is the class of $\mathbb{Z}[1/2]\rho$.

(3.22) THEOREM. *With the notation above, suppose the surgery obstruction $\sigma(g, b) \in L_{2k+1}^h(\mathbb{Z}\pi)$ is nonzero in $L_{2k+1}^p(\mathbb{Z}\pi)$ and l is even. Then $\sigma(f, c)$ is zero in $L_{2(k+l)+1}^p(\mathbb{Z}[\pi \times \rho])$ if and only if $\chi(M)$ is even.*

Proof. (Sketch) Notice first that $\mathbb{Z}[1/2]\pi$ appears as ring factor of $\mathbb{Z}[1/2][\pi \times \rho]$, so that the invariants of (3.9) or (3.15) for (g, b) appear for the product (f, c) as well. By (3.21) and the fact that (with obvious notation) $X(f) = K_k(g) \cdot \chi(\tilde{M}; \mathbb{Z}[1/2]\rho)$, these invariants are multiplied by the Euler characteristic of M . Since the invariants are of order two, the proof is complete.

4. The structure of the rational group ring of a 2-group and the existence of an involution-invariant maximal order in it (Proof of Theorem 3.1). Let the ring-with-involution C be a

product of matrix algebras, $C = \coprod M_{n_i}(D_i)$. The involution either takes a given factor of C to itself, or interchanges two factors; if it preserves a given factor $M_n(D)$, denote it $b \rightarrow b^\tau$. According to a theorem of Bass [2, I. 8.3] (generalization of a theorem of Albert) there is an involution $\sigma: D \rightarrow D$ and $h \in GL_n(D)$ such that, for each $b \in B$

$$(4.1) \quad \begin{aligned} (a) \quad & b^\tau = h({}^t b^\sigma) h^{-1}, \quad {}^t b^\sigma = \sigma\text{-conjugate transpose} \\ (b) \quad & h = \eta({}^t h^\sigma), \quad \eta = \pm 1. \end{aligned}$$

The involution σ is uniquely determined by τ , but in general, h and η are not.

(4.2) Morita theory asserts that there is a $(1 - 1)$ -correspondence between isometry classes of rank r , λ -quadratic forms over $(M_n(D), \sigma)$ and rank nr , $(\eta\lambda)$ -quadratic forms over (D, τ) . (The involution is included in the notation for emphasis.) Suppose n is even. Then because of the even rank conventions in Def. (1.2) and (1.3), there are induced isomorphisms $m(D): fW_*^\lambda(M_n(D), \tau) \cong fW_*^{\eta\lambda}(D, \sigma)$. Let $D = A$ as in (1.1). Using the notion of covering from [32, 1.17], it is routine to show there are isomorphisms $m(B/A): fW_*^\lambda(M_n(B)/M_n(A), \tau) \cong fW_*^{\eta\lambda}(B/A, \sigma)$ induced by Morita equivalences; in fact the whole localization sequence is compatible with Morita equivalence.

(4.3) THEOREM. (a) *With the notation above, there is a commutative diagram of localization sequences,*

$$\begin{array}{ccccccc} \cdots \rightarrow & fW_0^\lambda(M_n(A)) & \rightarrow & fW_0^\lambda(M_n(B)) & \rightarrow & fW_0^\lambda(M_n(B)/M_n(A)) & \rightarrow & fW_1^{-\lambda}(M_n(A)) \rightarrow \cdots \\ & \cong \downarrow m(A) & & \cong \downarrow m(B) & & \cong \downarrow m(B/A) & & \cong \downarrow \\ \cdots \rightarrow & fW_0^{\eta\lambda}(A) & \longrightarrow & fW_0^{\eta\lambda}(B) & \longrightarrow & fW_0^{\eta\lambda}(B/A) & \longrightarrow & fW_1^{-\eta\lambda}(A) \rightarrow \cdots \end{array}$$

(b) *The Arf and Dickson invariants are compatible with Morita equivalence: there is a commutative diagram*

$$\begin{array}{ccc} fW_*^\lambda(M_n(A)) & \xrightarrow{r_2} & fW_*^\lambda(M_n(A/2A)) \\ \cong \downarrow m(A) & & \cong \downarrow m(A/2A) \\ fW_*^{\eta\lambda}(A) & \xrightarrow{r_2} & fW_*^{\eta\lambda}(A/2A). \end{array}$$

This theorem will be applied to the simple factors of $\mathcal{Q}\pi$ (π a finite 2-group) and to involution-invariant maximal orders in them, the construction of which will be taken up next.

To set notation, define groups

$$\begin{aligned}
& \text{(a) } D_{n+1} = \{x, y \mid x^{2^n} = 1 = y^2, yxy^{-1} = x^{-1}\}, \quad n \geq 2 \quad (\text{dihedral}) \\
& \text{(b) } SD_{n+1} = \{x, y \mid x^{2^n} = 1 = y^2, yxy^{-1} = x^{-1+2^{n-1}}\}, \quad n \geq 3 \\
& \hspace{15em} (\text{semi-dihedral}) \\
(4.4) \quad & \text{(c) } Q_{n+1} = \{x, y \mid x^{2^{n-1}} = y^2, x^{2^n} = 1, yxy^{-1} = x^{-1}\}, \quad n \geq 2 \\
& \hspace{15em} (\text{generalized quaternion}) \\
& \text{(d) } C_n = Z/2^n = \{x \mid x^{2^n} = 1\}, \quad n \geq 0 \quad (\text{cyclic}).
\end{aligned}$$

These two-groups are precisely those having a cyclic subgroup of index two.

To describe $Q\pi$, where π is in (4.4), define the “twisted group rings” (including $Q\zeta_{2^n}$ for completeness):

$$\begin{aligned}
& \text{(a) } \mathcal{A}_n := Q\zeta[y]/\{y^2 = 1, y\zeta y^{-1} = \zeta^{-1}\}, \quad \zeta = \zeta_{2^n}, \quad n \geq 2 \\
(4.5) \quad & \text{(b) } S\mathcal{A}_n := Q\zeta[y]/\{y^2 = 1, y\zeta y^{-1} = -\zeta^{-1}\}, \quad \zeta = \zeta_{2^n}, \quad n \geq 3 \\
& \text{(c) } \Gamma_n := Q\zeta[y]/\{y^2 = -1, y\zeta y^{-1} = \zeta^{-1}\}, \quad \zeta = \zeta_{2^n}, \quad n \geq 2 \\
& \text{(d) } Q\zeta_{2^n}, n \geq 0 \quad (\text{we set } Q\zeta_{2^0} = Q \text{ when } n = 0).
\end{aligned}$$

It is now not difficult to construct isomorphisms (for example by tensoring the cartesian squares in [14] with Q):

$$\begin{aligned}
& \text{(a) } QD_{n+1} \cong \prod_{i=2}^n \mathcal{A}_i \times Q \times Q \times Q \times Q \\
(4.6) \quad & \text{(b) } QSD_{n+1} \cong \prod_{i=3}^n S\mathcal{A}_i \times \mathcal{A}_2 \times Q \times Q \times Q \times Q \\
& \text{(c) } QQ_{n+1} \cong \Gamma_n \times QD_n \\
& \text{(d) } QC_n \cong \prod_{i=0}^n Q\zeta_{2^i}.
\end{aligned}$$

(4.7) **THEOREM (Fontaine).** *Let π be a 2-group and M a $Q\pi$ -irreducible. Then there exist subgroups $H \triangleleft G$ of π and an irreducible $Q[G/H]$ -module N such that*

- (a) G/H is in (4.4) and
- (b) if N is viewed as a QG -module, then there is an isomorphism $N \otimes_{QG} Q\pi \cong M$.

Finally, each simple component of $Q\pi$ is a matrix algebra over one of the algebras in (4.5). (I.e., the “induction” in (b) does not change the center.)

Fontaine’s theorem will now be extended to include a description of the involution on the components of $Q\pi$, in the following sense.

(4.8) **DEFINITION.** A matrix algebra-with-involution $(M_n(D), \tau)$ satisfying (4.1) will be described by the quadruple $(M_n(D), \sigma, h, \eta)$.

(4.9) PROPOSITION. Let Id denote the trivial involution, and “ \sim ” the involution induced by $\zeta \rightarrow \zeta^{-1}$ on any subfield of $\mathbf{Q}\zeta_{2^n}$. Then the following is a description of the algebras (4.5), as algebras-with-involution (4.8):

- (a) $\mathcal{A}_n \cong (M_2(\mathbf{Q}(\zeta + \zeta^{-1})), \text{Id}, h, 1)$, where $\zeta = \zeta_{2^n}$
- (b) $S\mathcal{A}_n \cong (M_2(\mathbf{Q}(\zeta - \zeta^{-1})), \sim, g, 1)$, where $\zeta = \zeta_{2^n}$
- (c) $\Gamma_n \cong (-1, -1/\mathbf{Q}(\zeta + \zeta^{-1}))$, the usual quaternion algebra over $\mathbf{Q}(\zeta + \zeta^{-1})$, $\{a + bi + cj + dk \mid i^2 = -1 = j^2, ij = -ji, ij = k; a, b, c, d \in \mathbf{Q}(\zeta + \zeta^{-1})\}$, with involution $\bar{i} = -i, \bar{j} = -j$; here $\zeta = \zeta_{2^n}$.
- (c)_p If p is the unique dyadic prime of $\mathbf{Q}(\zeta_{2^n} + \zeta_{2^n}^{-1})$ and $n \geq 3$, or if p is any such nondyadic prime, then

$$(\Gamma_n)_p \cong (M_2(\mathbf{Q}(\zeta + \zeta^{-1}))_p, \text{Id}, h, -1), \quad \zeta = \zeta_{2^n}.$$

The algebra $\Gamma_2 = (-1, -1/\mathbf{Q})$ is not split at the prime 2.

Proof. A \mathbf{Q} -algebra B is split (isomorphic to a matrix ring over its center) if and only if it is so with respect to every completion, by the Brauer-Hasse-Noether theorem (see [42]). A matrix algebra over C is always split, because C is algebraically closed. Next $B \otimes R$ is a matrix algebra over R, C or H (see [45, p. 123]), where the induced $\sigma: (R, C, \text{ or } H) \rightarrow (R, C, \text{ or } H)$ in (4.2) is, respectively, trivial, complex conjugation, or the usual involution on H (see [45, p. 122]).

By a well-known argument [33, 4.8] each $R\pi$ -irreducible supports a nonsingular, hermitian, $R\pi$ -valued form, for any π . It follows easily that, since each algebra in (4.5) occurs as a factor in some $\mathbf{Q}\pi$, it cannot happen that any real completion of the algebras (a)–(c) contains the product of two matrix algebras interchanged by the involution. Since the center of $S\mathcal{A}_n$ is $\mathbf{Q}(\zeta - \zeta^{-1})$ with nontrivial involution (induced by $\zeta \rightarrow \zeta^{-1}$) and it has degree four over its center, the above discussion shows that the only possibility is $S\mathcal{A}_n \otimes_K R \cong M_2(C)$, $K = \mathbf{Q}(\zeta - \zeta^{-1})$. In case (a), the center is $\mathbf{Q}(\zeta + \zeta^{-1})$, totally real field with trivial involution. One checks that the fixed point set of the involution on \mathcal{A}_n has dimension 3 over its center, so $\mathcal{A}_{n+1} \otimes R \not\cong H$. Thus $\mathcal{A}_{n+1} \otimes R \cong M_2(R)$, for every real completion of $\mathbf{Q}(\zeta + \zeta^{-1})$. Finally, since $\mathbf{Q}\zeta = \mathbf{Q}(\zeta + \zeta^{-1}) (\sqrt{-1})$, it follows easily that $\Gamma_n \cong (-1, -1/\mathbf{Q}(\zeta + \zeta^{-1}))$, $\zeta = \zeta_{2^n}$. The involution is trivial on the center, and $\Gamma_n \otimes R \cong (-1, -1/R) =: H$.

Now it is known that $\mathcal{A}_n, S\mathcal{A}_n$, and Γ_n are all split at nondyadic primes (see [42, 41.7]). Thus, since there is only one dyadic prime in any subfield of $\mathbf{Q}\zeta_{2^n}$ ([58, §7]), and since an algebra can be non-split with respect to at most a finite, even number of valuations (by reciprocity, see [42, §41]), it follows that \mathcal{A}_n and $S\mathcal{A}_n$ are everywhere locally split, hence split. Since the irreducibles over \mathcal{A}_n and

SD_n support nonsingular hermitian forms [33, 4.8], the descriptions (a) and (b) follow from [2, I. 8.8].

Since $\mathbf{Q}(\zeta_{2^n} + \zeta_{2^n}^{-1})$ is totally real, it has $[\mathbf{Q}(\zeta_{2^n} + \zeta_{2^n}^{-1}) : \mathbf{Q}]$ real valuations. This number is even if and only if $n \geq 3$, so the reciprocity argument above yields the splittings at dyadic \mathfrak{p} in (c)_p. Finally, since the fixed point set of the involution on Γ_n is the center, $\mathbf{Q}(\zeta + \zeta^{-1})$, and $\dim_{\mathbf{Q}(\zeta + \zeta^{-1})} \Gamma = 4$, it follows that $h = -h^t$ in (c)_p.

Next is the question of existence of maximal orders (in the algebras \mathcal{A}_n , SD_n and Γ_n), which are preserved by the involution. To motivate this rather tedious analysis an example is given which shows it is necessary. That this phenomenon could occur was first pointed out in [43].

(4.10) *Example* of an involution-invariant order (in a matrix algebra-with-involution) which cannot be extended to an involution-invariant maximal order.

Let $\zeta = \zeta_m$ be a primitive m th root of unity, $m > 2$, and let \mathcal{O} be the twisted group ring

$$(4.11) \quad \mathcal{O} = \mathbf{Z}\zeta \circ \mathbf{Z}/2 := \{\mathbf{Z}\zeta[y] \mid y^2 = 1, y\zeta y = \zeta^{-1}\}.$$

To imbed \mathcal{O} in $M_2(\mathbf{Z}(\zeta + \zeta^{-1}))$, note that $\{1, 1 - \zeta\}$ is a basis for $\mathbf{Z}\zeta$ as a free, rank 2, $\mathbf{Z}(\zeta + \zeta^{-1})$ -module. Now view \mathcal{O} as a ring of $\mathbf{Z}(\zeta + \zeta^{-1})$ -endomorphisms of this module by setting

$$y \cdot \zeta = \zeta^{-1}, \quad y \cdot z = z(z \in \mathbf{Z}), \quad r \cdot s = rs(r \in \mathbf{Z}\zeta \subseteq \mathcal{O}, s \in \mathbf{Z}\zeta).$$

With these conventions, one easily finds that

$$(4.12) \quad f: \mathcal{O} \longrightarrow \text{End}_{\mathbf{Z}(\zeta + \zeta^{-1})}(\mathbf{Z}\zeta) = M_2(\mathbf{Z}(\zeta + \zeta^{-1}))$$

is injective and hence that $f \otimes \mathbf{Q}$ is an isomorphism; it defines an involution on $M_2(\mathbf{Q}(\zeta + \zeta^{-1}))$, trivial on the center and satisfying $h = h^t$ in (4.8). Straightforward computation shows that, setting $\pi = 2 - (\zeta + \zeta^{-1})$,

$$\begin{aligned} f(1) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & f(y) &= \begin{pmatrix} 1 & \pi \\ 0 & -1 \end{pmatrix}, & f(1 - \zeta) &= \begin{pmatrix} 0 & -\pi \\ 1 & \pi \end{pmatrix}, \\ f(1 - \zeta^{-1}) &= \begin{pmatrix} \pi & \pi \\ -1 & 0 \end{pmatrix}, & \text{and } f(\zeta - y) &= \begin{pmatrix} 0 & 0 \\ -1 & 2 - \pi \end{pmatrix}. \end{aligned}$$

Now assume that m is an odd prime p . It is not difficult to deduce that (see [20]), setting $B := \text{im}(f)$,

$$B = \begin{pmatrix} R & \pi R \\ R & R \end{pmatrix}, \quad R = \mathbf{Z}(\zeta + \zeta^{-1}).$$

By [20, 2(i)] and [42, 40.13], B is hereditary. Now complete at the prime πR . From [42, 39.18(v)]

$$I_1 = \begin{pmatrix} \pi R & \pi R \\ R & R \end{pmatrix} \quad \text{and} \quad I_2 = \begin{pmatrix} R & \pi R \\ R & \pi R \end{pmatrix}$$

are the two 2-sided maximal ideals of B , and

$$M_{I_i} := \{x \in M_2(Q(\zeta + \zeta^{-1})) \mid xI_i \subseteq B\}$$

are the two maximal orders containing B . A computation shows that $M_{I_2} = \{x \in M_2(Q(\zeta + \zeta^{-1})) \mid I_1 x \subseteq B\}$. Now suppose that $\bar{I}_1 = I_1$. Then $\bar{M}_{I_1} = \{\bar{x} \in M_2(Q(\zeta + \zeta^{-1})) \mid xI_1 \subseteq B\} = \{\bar{x} \in M_2(Q(\zeta + \zeta^{-1})) \mid I_1 \bar{x} \in \bar{B} = B\} = M_{I_2}$. So it remains to show $\bar{I}_1 = I_1$.

First, by [42, 39.16], $J := I_1 \cap I_2$ is $\text{Rad } B$ and $B/J \cong B/I_1 \times B/I_2 \cong F_p \times F_p$, with idempotents represented by $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in B$. Thus, it suffices to show $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is taken to itself by the involution. But $f(\zeta - y) = \begin{pmatrix} 0 & 0 \\ -1 & 2 - \pi \end{pmatrix}$ and $\overline{f(\zeta - y)} = f(\overline{\zeta - y}) = \begin{pmatrix} -\pi & 2\pi \\ 1 & 2 \end{pmatrix}$ which both become, mod J , $\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$.

Define, for $\zeta = \zeta_{2^n}$, \mathbf{Z} -orders

$$(4.13) \quad \begin{aligned} (a) \quad \mathcal{O}(\Delta_n) &:= \mathbf{Z}\zeta[y]/\{y^2 = 1, y\zeta y = \zeta^{-1}\} \subseteq \Delta_n \\ (b) \quad \mathcal{O}(S\Delta_n) &:= \mathbf{Z}\zeta[y]/\{y^2 = 1, y\zeta y = -\zeta^{-1}\} \subseteq S\Delta_n \\ (c) \quad \mathcal{O}(\Gamma_n) &:= \mathbf{Z}\zeta[y]/\{y^2 = -1, y\zeta y^{-1} = \zeta^{-1}\} \subseteq \Gamma_n. \end{aligned}$$

None of these orders is hereditary (or maximal) by [42, 40.13], since the unique dyadic prime in $\mathbf{Z}(\zeta + \zeta^{-1})$ or $\mathbf{Z}(\zeta - \zeta^{-1})$ is wildly ramified in $\mathbf{Z}\zeta$.

(4.14) THEOREM. *The order $\mathcal{O}(\Delta_n)$ (resp. $\mathcal{O}(S\Delta_n)$, $\mathcal{O}(\Delta_n)$) extends to an involution-invariant maximal order in Δ_n (resp. $S\Delta_n$, Δ_n).*

Assume this theorem for now. It is easy to deduce from the discussion of Cartesian squares in [14] that, under the isomorphisms of (4.6), $\mathcal{O}(\Delta_n)$ is the image of $\mathbf{Z}D_{n+1}$ in

$$\mathbf{Z}D_{n+1} \longrightarrow \mathbf{Q}D_{n+1} \xrightarrow{\cong} \coprod \Delta_i \times \mathbf{Q} \times \mathbf{Q} \times \mathbf{Q} \times \mathbf{Q} \longrightarrow \Delta_n;$$

similarly for $S\Delta_{n+1}$ and \mathbf{Q}_{n+1} . From this and (4.14), it follows easily that

(4.15) THEOREM. *If π is one of the groups in (4.4), then $\mathbf{Z}\pi \subseteq \mathbf{Q}\pi$ extends to an involution-invariant maximal order in $\mathbf{Q}\pi$.*

Proof of (4.14). Consider first the inclusion $\mathcal{O}(\Delta_n) \rightarrow \Delta_n$. Recall from Example (4.10) imbedding (4.12)

$$f: \mathcal{O}(\Delta_n) \longrightarrow M_2(R), \quad R = \mathbb{Z}(\zeta_{2^n} + \zeta_{2^n}^{-1}).$$

Using the same procedure, change only the basis of $\mathbb{Z}\zeta_{2^n}$ over R , taken here to be $\{1, \zeta\}$. Then for $\zeta = \zeta_{2^n}$,

$$f(\zeta) = \begin{pmatrix} 0 & -1 \\ 1 & \zeta + \zeta^{-1} \end{pmatrix}, \quad f(\zeta^{-1}) = \begin{pmatrix} \zeta + \zeta^{-1} & 1 \\ -1 & 0 \end{pmatrix}, \quad f(y) = \begin{pmatrix} 1 & \zeta + \zeta^{-1} \\ 0 & -1 \end{pmatrix}.$$

The involution on $M_2(R)$ induced by f from that of $\mathcal{O}(\Delta_n)$ is complicated; to simplify it, let $i = \zeta^{2^{n-2}}$, and define

$$A = \frac{1}{\mu} \begin{pmatrix} 2 & -(\zeta + \zeta^{-1}) \\ -(\zeta + \zeta^{-1}) & 2 \end{pmatrix}, \quad \mu = i(\zeta - \zeta^{-1}).$$

Then $\det A = 1$, $A = A^t$, and all its entries lie in R . (I.e., $2\mu^{-1}$, $(\zeta + \zeta^{-1})\mu^{-1} \in R$.) The same is true of A^{-1} . Let f' be the composition

$$f': \mathcal{O}(\Delta_n) \xrightarrow{f} M_2(R) \xrightarrow{mA^{-1}} M_2(R),$$

where mA^{-1} is left multiplication by A^{-1} . Then one checks that

$$(f'(\zeta))^t = f'(\bar{\zeta}) \quad \text{and} \quad (f'(y))^t = f'(\bar{y}).$$

But since ζ and y generate $\mathcal{O}(\Delta_n)$, this implies that the involution inherited by $M_2(R)$ is the transpose (i.e., $\sigma = \text{Id}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in (4.8)). It is now clear that the desired maximal order is $M_2(R)$.

The procedure for $\mathcal{O}(S\Delta_n) \subseteq S\Delta_n$ is similar and left to the reader.

Finally, consider $\mathcal{O}(\Gamma_n) \subseteq \Gamma_n$. When $n = 2$, $\mathcal{N}_2 := \mathcal{O}(\Gamma_2) + (1/2)(1 + i + j + k)\mathcal{O}(\Gamma_2)$ is maximal in Γ_2 by [42; Ex. 2, p. 152]; it is clearly involution-invariant. For $n \geq 3$, setting $\zeta = \zeta_{2^n}$, define

$$\mathcal{N}_n := \mathcal{O}(\Gamma_n) + a\mathcal{O}(\Gamma_n), \quad a = (1 + \zeta^2)^{-1}(1 + y).$$

Then the following equalities hold:

- (a) $a^2 = a + (2/\zeta^2 + \zeta^{-2} - 2) =$
 $\begin{cases} a - 1, & n = 3 \\ a + \rho, & \rho \in \mathfrak{p} = \text{the dyadic prime of } \mathbb{Z}(\zeta + \zeta^{-1}), n > 3. \end{cases}$
- (b) $a\zeta - \zeta a = \zeta^{-1}y.$
- (c) $a(\zeta y) - (\zeta y)a = \zeta(1 + \zeta^2/1 - \zeta^2)(y - 1) - \zeta^{-1}$ and $(1 + \zeta^2/1 - \zeta^2)$ is a unit in $\mathbb{Z}(\zeta + \zeta^{-1})$.
- (d) $a + \bar{a} = 1.$

From (d) it follows that \mathcal{N}_n is involution-invariant; and from (a)-(c) it follows that \mathcal{N}_n is \mathbb{Z} -finitely generated, hence a \mathbb{Z} -order.

Given $b \in \mathcal{N}_n$ denote by \hat{b} its class in $\mathcal{N}_n/\mathfrak{p}$, $\mathfrak{p} \in \mathbf{Z}(\zeta + \zeta^{-1})$ the dyadic prime. Define a map $f: \mathcal{N}_n/\mathfrak{p} \rightarrow M_2(A/\mathfrak{p})$, $A = \mathbf{Z}(\zeta + \zeta^{-1})$ by:

$$f(\hat{a}) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad f(\hat{\bar{a}}) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad f(\hat{\zeta}) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

$$f(\hat{y}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad n = 3$$

and

$$f(\hat{a}) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad f(\hat{\bar{a}}) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad f(\hat{\zeta}) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

$$f(\hat{y}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad n > 3.$$

Using the equations (a)–(d) above and making $M_2(A/\mathfrak{p})$ the algebra-with-involution $(M_2(A/\mathfrak{p}), \text{Id}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 1)$ (notation of (4.8)), one easily checks that f is an isomorphism of algebras with involution.

Now by [42, 41.1], $\mathcal{O}(\Gamma_n)_q$ is already maximal for q nondyadic; hence so is $(\mathcal{N}_n)_q$. But since $(\Gamma_n)_{\mathfrak{p}}$ is split when \mathfrak{p} is dyadic, the isomorphism f shows $(\mathcal{N}_n)_{\mathfrak{p}}$ is also maximal. Thus, \mathcal{N}_n is everywhere locally maximal, hence maximal.

Here is the main result of this section.

(4.16) THEOREM. *Let π be a finite 2-group. Then there is an involution-invariant maximal order \mathcal{M} ,*

$$\mathbf{Z}\pi \subseteq \mathcal{M} \subseteq \mathbf{Q}\pi$$

such that $\mathcal{M} = \Pi \mathcal{M}_i$ where each \mathcal{M}_i is maximal in some simple component of $\mathbf{Q}\pi$ and has one of the following four forms, as an algebra-with-involution in the sense of Definition (4.8).

- (I) $(M_{2^m}(\mathbf{Z}\zeta_{2^n}), \zeta \rightarrow \zeta^{-1}, f, 1), m \geq 0, n \geq 1.$
- (II) $(M_{2^m}(\mathbf{Z}(\zeta_{2^n} - \zeta_{2^n}^{-1}), \zeta \rightarrow \zeta^{-1}, g, 1)), m \geq 1, n \geq 3.$
- (III) $(M_{2^m}(\mathbf{Z}(\zeta_{2^n} + \zeta_{2^n}^{-1})), \text{Id}, h, 1), m \geq 1, n \geq 2.$
- (IV) $(M_{2^m}(\mathcal{N}_n), \sigma, I_{2^m}, 1), m \geq 0, n \geq 2,$

where \mathcal{N}_n is a maximal order in Γ_n , and $\sigma: \mathcal{N}_n \rightarrow \mathcal{N}_n$ is the restriction of the involution on Γ_n .

In addition, type (IV) completed at nondyadic q ; or at the dyadic prime \mathfrak{p} for $n \geq 3$, becomes

$$(IV)_{\mathfrak{p}} \quad (M_{2^{m+1}}(\mathbf{Z}(\zeta_{2^n} + \zeta_{2^n}^{-1})), \text{Id}, h, -1).$$

Proof. The theorem follows from (4.7), (4.15) the proof of [6, (5.2)] and [30, §1, Lemma 3]. Details are left to the reader.

5. **Arf and Dickson invariants over $Z\zeta_{2^n}$ and $Z(\zeta_{2^n} - \zeta_{2^n}^{-1})$.** Throughout this section, $A = Z\zeta_{2^n}$ ($n \geq 2$), or $Z(\zeta_{2^n} - \zeta_{2^n}^{-1})$ ($n \geq 3$), and K is the fraction field of A ; the (nontrivial) involution is induced by $\zeta \rightarrow \zeta^{-1}$, complex conjugation.

First recall that if $\mathfrak{p} = \bar{\mathfrak{p}}$ is a finite prime of K , then a theorem of [26] states that the discriminant induces

$$(5.1) \quad \text{dis}_{\mathfrak{p}}: fW_{\mathfrak{o}}^1(K_{\mathfrak{p}}) \xrightarrow{\cong} F_{\mathfrak{p}}^{\times}/NK_{\mathfrak{p}}^{\times}$$

where $F_{\mathfrak{p}}$ denotes the completion of the fixed field of the involution with respect to the prime under \mathfrak{p} and $N: K_{\mathfrak{p}}^{\times} \rightarrow F_{\mathfrak{p}}^{\times}$ is norm. Thus, from the commutative diagram, where \mathcal{L}_0^1 is from the localization sequence,

$$\begin{array}{ccccccc} & & & \mathcal{L}_0^1 & & & \\ & & & \curvearrowright & & & \\ fW_{\mathfrak{o}}^1(K) & \longrightarrow & \coprod_{\mathfrak{p}=\bar{\mathfrak{p}}} fW_{\mathfrak{o}}^1(K_{\mathfrak{p}}) & \xrightarrow{\coprod (\mathcal{L}_0^1)_{\mathfrak{p}}} & \coprod_{\mathfrak{p}=\bar{\mathfrak{p}}} fW_{\mathfrak{o}}^1(K_{\mathfrak{p}}/A_{\mathfrak{p}}) & \xrightarrow[\cong]{(1.10)} & fW_{\mathfrak{o}}^1(K/A) \\ \downarrow \text{dis} & & \downarrow \coprod \text{dis}_{\mathfrak{p}} & & & & \\ F^{\times}/NK^{\times} & \longrightarrow & \coprod_{\mathfrak{p}=\bar{\mathfrak{p}}} F_{\mathfrak{p}}^{\times}/NK_{\mathfrak{p}}^{\times} & & & & \end{array}$$

it follows that $\ker(\text{dis}) \subseteq \ker(\mathcal{L}_0^1)$. The argument of [60, III. 5.2] shows that the inverse of the isomorphism $fW_{\mathfrak{o}}^1(K)/\ker(\text{dis}) \xrightarrow{\cong} F^{\times}/NK^{\times}$ is given by $\delta(f) = \langle f, -1 \rangle$. These remarks furnish the commutative diagram, whose top line is a version of the Artin reciprocity law (see [27, X. 3. Thm. 4] and [7, p. 177]),

$$(5.2) \quad \begin{array}{ccccccc} F^{\times}/NK^{\times} & \xrightarrow{L} & \coprod_{\mathfrak{p}=\bar{\mathfrak{p}}} F_{\mathfrak{p}}^{\times}/NK_{\mathfrak{p}}^{\times} & \xrightarrow{r} & Z/2 \\ \downarrow \cong \delta & & \downarrow \coprod \delta_{\mathfrak{p}}, \mathfrak{p} \text{ finite; trivial, } \mathfrak{p} \text{ infinite} & & \\ & & \coprod_{\mathfrak{p}=\bar{\mathfrak{p}}} fW_{\mathfrak{o}}^1(K_{\mathfrak{p}}) & & \\ & & \downarrow \coprod \delta_{\mathfrak{p}}, \mathfrak{p} \text{ finite} & & \\ & & \downarrow \coprod (\mathcal{L}_0^1)_{\mathfrak{p}} & & \\ & & \coprod_{\mathfrak{p}=\bar{\mathfrak{p}}} fW_{\mathfrak{o}}^1(K_{\mathfrak{p}}/A_{\mathfrak{p}}) & & \\ & & \downarrow \coprod \delta_{\mathfrak{p}}, \mathfrak{p} \text{ finite} & & \\ & & \downarrow \cong & & \\ & & \mathcal{L}_0^1 & & \\ fW_{\mathfrak{o}}^1(K)/\ker(\text{dis}) & \xrightarrow{\mathcal{L}_0^1} & fW_{\mathfrak{o}}^1(K/A) & \xrightarrow{\mathcal{L}_0^1} & fW_1^{-1}(A) . \end{array}$$

(5.3) **THEOREM.** *The Dickson invariant induces*

$$fW_1^1(A) \xrightarrow{\cong} fW_1^1(A/2A) = Z/2 .$$

Proof. Since $(A/2A)/\text{Rad} = F_2$ (A has a unique dyadic prime, which is totally ramified), the surjectivity follows from (1.6) and (2.6).

Let $\lambda = -1$. Referring to (2.4), it suffices to show that, given (S, ϕ) (representing an element $[S, \phi]$ of $fW_0^1(K/A)$) where S is non-dyadic-torsion A -module, there exists $x \in fW_0^1(K)$ such that $\mathcal{L}_0^1(x) = [S, \phi]$. Since K/F is unramified at nondyadic primes, $fW_*^\lambda(A_p) \equiv 0$, $\mathfrak{p} = \bar{\mathfrak{p}}$ nondyadic: $fW_*^\lambda(A_p) \xrightarrow{\cong} fW_*^\lambda(A/\mathfrak{p})$ by reduction (1.13) and this vanishes if $\lambda = 0$ by [60, p. 117], by [33, (4.1)] if $\lambda = 1$. Thus, for such \mathfrak{p} , $(\mathcal{L}_0^1)_\mathfrak{p}$ is an isomorphism, by the localization sequence. From (5.2) it follows that there is $\{f_\mathfrak{p}\} \in \coprod F_\mathfrak{p}^\times / NK_\mathfrak{p}^\times$ such that $\partial\{f_\mathfrak{p}\} = [S, \phi]$. If $r\{f_\mathfrak{p}\} = 0$, a diagram chase completes the proof. If not, then change $\{f_\mathfrak{p}\}$ at a ramified infinite prime. Then $\partial\{f_\mathfrak{p}\}$ is unchanged.

In case $\lambda = 1$, and $A = \mathbb{Z}\zeta_{2^n}$, “scaling” [3] with $i = -\bar{i}$, shows $fW_1^1(A) \cong fW_1^{-1}(A)$. In case $A = \mathbb{Z}(\zeta - \zeta^{-1})$, then $(\zeta - \zeta^{-1})$ generates the ramified dyadic prime of $\mathbb{Z}(\zeta - \zeta^{-1})$ and satisfies $(\zeta - \zeta^{-1}) = -(\overline{\zeta - \zeta^{-1}})$. Since the argument for $\lambda = 1$ used only fraction fields (or their completions at nondyadic \mathfrak{p}) and nondyadic torsion modules, it too can be scaled to give the result in this case.

(5.4) THEOREM. *The Arf invariant induces surjections*

$$fW_0^\lambda(A) \longrightarrow fW_0^\lambda(A/2A) \cong \mathbb{Z}/2.$$

Proof. The values of $fW_0^\lambda(A/2A)$ follow from reduction and the fact that $(A/2A)/\text{Rad} \cong F_2$. Surjectivity for $\lambda = -1$ uses the composition $\mathbb{Z} \rightarrow A \rightarrow A/2A \rightarrow F_2$ and the usual representative for the Arf invariant over \mathbb{Z} . If $A = \mathbb{Z}\zeta$ and $\lambda = 1$, use scaling, as in (5.3).

To prove the assertion if $\lambda = 1$ and $A = \mathbb{Z}(\zeta - \zeta^{-1})$, observe first that $(\mathcal{N}_0^1)_\mathfrak{p}: fW_0^1(A_\mathfrak{p}) \rightarrow fW_0^1(K_\mathfrak{p})$ is injective, where \mathfrak{p} is the dyadic prime of A . For by reduction (1.13) and (2.6), $fW_0^1(A_\mathfrak{p}) \xrightarrow{\cong} fW_0^1(A/\mathfrak{p}) \cong \mathbb{Z}/2$, with nonzero representative $((A_\mathfrak{p})^2, g, q)$ where g has matrix $\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$ and $q(e) = 1 = -q(f)$ for a basis $\{e, f\}$ of $(A_\mathfrak{p})^2$. The image of this element in $fW_0^1(K_\mathfrak{p})$ has discriminant 5 and (by (5.1)) $\text{dis}: fW_0^1(K_\mathfrak{p}) \xrightarrow{\cong} F_\mathfrak{p}^\times / NK_\mathfrak{p}^\times = \mathbb{Z}/2$. Thus to prove that $(\mathcal{N}_0^1)_\mathfrak{p}$ is injective, it suffices to show $5 \notin NK_\mathfrak{p}^\times$. By [31, 63.10], this is so if and only if the Hilbert symbol

$$(5.5) \quad \left(\frac{5, (\zeta - \zeta^{-1})^2}{Q(\zeta^2 + \zeta^{-2})} \right)_\mathfrak{p} = -1,$$

since $K = Q(\zeta - \zeta^{-1})$ is obtained from the fixed field $F = Q(\zeta^2 - \zeta^{-2})$ by adjoining $\sqrt{(\zeta - \zeta^{-1})^2} = \zeta - \zeta^{-1}$. But $(\zeta - \zeta^{-1})^2 = \zeta^2 + \zeta^{-2} - 2$ is the generator of the unique dyadic prime in $\mathbb{Z}(\zeta^2 + \zeta^{-2})$ and $5 = 1 +$

4(1) has quadratic defect 4A, so (5.5) follows from [31, 63.11a].

Now it remains to prove that there is a (+1)-quadratic form over A whose image in $fW_0^1(K_p)$ is nontrivial. This can be done by the reciprocity arguments of (5.3). Namely, referring to Daigram (5.2), let $\{f_q\} \in \prod_{q=p} F_q^\times / NK_q^\times$ be given by

$$f_q = \begin{cases} 5, & q = p \text{ dyadic} \\ -1, & q = \text{some fixed ramified prime} \\ 1, & \text{otherwise.} \end{cases}$$

Thus, in (5.2), $r\{f_q\} = 0 = \prod_q (\mathcal{L}_0^1)_q \circ (\prod_q \delta_q)\{f_q\}$, so we can find a (+1)-quadratic form (K^n, g) such that $\mathcal{L}_0^1(K^n, g) = 0$ and $\text{dis}_p(K^n, g) = 5 \in F_p^\times / NK_p^\times$. Exactness in the localization sequence furnishes a (+1)-quadratic form (P, h) over A such that $\mathcal{K}_0^1(P, h) = (K^n, g)$. This completes the proof.

6. Arf and Dickson invariants over $\mathbf{Z}(\zeta_{2^n} + \zeta_{2^n}^{-1})$. Throughout this chapter $A = \mathbf{Z}(\zeta_{2^n} + \zeta_{2^n}^{-1})$, $n \geq 2$, K is its fraction field, and the involution on A and K is trivial. A is totally real and has exactly one dyadic prime, which is totally ramified over $(2) \subseteq \mathbf{Z}$.

To begin, consider the diagram of localization sequences (cf. (1.15), and [60, IV. 3.4])

$$(6.0) \quad \begin{array}{ccccccc} fW_0^1(A) & \xrightarrow{\mathcal{K}_0^1} & fW_0^1(K) & \xrightarrow{\mathcal{L}_0^1} & fW_0^1(K/A) & \xrightarrow{\mathcal{D}_0^1} & fW_1^{-1}(A) \\ \downarrow h(A) & & \downarrow = & & \downarrow h(K/A) & & \\ fW_{\text{herm}}^1(A) & \xrightarrow{\mathcal{K}_{\text{herm}}^1} & fW_{\text{herm}}^1(K) & \xrightarrow{\mathcal{L}_{\text{herm}}^1} & fW_{\text{herm}}^1(K/A) & & \end{array}$$

Since $fW_1^{-1}(K) = 0$ by [33, (4.1)], \mathcal{D}_0^1 is surjective; since the ideal class group \mathcal{C} of A has odd order [17, Satz 38'], $\mathcal{L}_{\text{herm}}^1$ is surjective by [60, IV. 3.4]. Thus there is an exact sequence

$$(6.1) \quad fW_0^1(A) \xrightarrow{h(A)} fW_{\text{herm}}^1(A) \xrightarrow{L} V(K/A) \longrightarrow fW_1^{-1}(A)$$

where

$$V(K/A) := \ker(h(K/A)), \quad L := \mathcal{L}_0^1 | \text{im}(fW_{\text{herm}}^1(A) \longrightarrow fW_0^1(K)).$$

To motivate the following procedure, recall that $W(K)$ (=the Witt group of symmetric bilinear forms over K , without rank restriction) is studied in [60, II. 5] by a filtration process, due essentially to Pfister (for any field K): the authors begin with the rank homomorphism $W(K) \rightarrow \mathbf{Z}/2$, and observe that the discriminant becomes a homomorphism on its kernel $I(K)$; next the sum of the Hasse-Witt invariants becomes a homomorphism on the kernel $I^2(K)$ of the discriminant; and, finally, the signatures (divided by 8) are

defined on $I^3(K)$, the kernel of the Hasse-Witt map (but the signatures are already a homomorphism on the larger $W(K)$). Thus, they obtain

(6.2) PROPOSITION. *There is a decreasing filtration of $W(K)$, where K is a number field,*

$$W(K) \supseteq I(K) \supseteq I^2(K) \supseteq I^3(K) \supseteq 0$$

with successive quotients and isomorphisms given by the above invariants (rank, discriminant, Hasse-Witt, signatures) $W(K)/I(K) \cong \mathbb{Z}/2$, $I(K)/I^2(K) \cong K^\times/K^{\times 2}$, $I^2(K)/I^3(K) \cong \coprod_{\mathfrak{p} \text{ finite}} \mathbb{Z}/2$, $I^3(K) \cong \coprod_{\substack{\mathfrak{p} \text{ real} \\ \text{infinite}}} \mathbb{Z}$.

Denote by $\partial_{\mathfrak{p}}$ the composition (see (1.12) (a))

$$(6.3) \quad \partial_{\mathfrak{p}} := \{W(K) \xrightarrow{\mathcal{L}_{\text{herm}}^1} fW_{\text{herm}}^1(K/A) \longrightarrow W(A/\mathfrak{p})\}$$

which by (1.12) equals the composition

$$\partial_{\mathfrak{p}} = \{W(K) \longrightarrow W(K_{\mathfrak{p}}) \xrightarrow{(\mathcal{L}_{\text{herm}}^1)_{\mathfrak{p}}} fW_{\text{herm}}^1(K_{\mathfrak{p}}/A_{\mathfrak{p}}) \xrightarrow[(1.12)(a)]{\cong} W(A/\mathfrak{p})\}$$

where the maps are the obvious ones. The following result was proved in [60, pp. 86, 96].

(6.4) PROPOSITION. (a) $\partial_{\mathfrak{p}}|I^3(K) \equiv 0$; (b) $\partial_{\mathfrak{p}}(I^2(K)) \subseteq I(A/\mathfrak{p})$ and $I(A/\mathfrak{p}) \xrightarrow{\text{dis}} (A/\mathfrak{p})^\times/(A/\mathfrak{p})^{\times 2} = \mathbb{Z}/2$ if \mathfrak{p} is nondyadic, and is zero otherwise; (c) $\partial_{\mathfrak{p}}|I^2(K): I^2(K) \rightarrow W(A/\mathfrak{p})$ may be identified with Hasse-Witt invariant at \mathfrak{p} , for \mathfrak{p} nondyadic; (d) the induced map $\partial_{\mathfrak{p}}: I(K)/I^2(K) \rightarrow W(A/\mathfrak{p})/I(A/\mathfrak{p}) \xrightarrow[\cong]{\text{rank}} \mathbb{Z}/2$ may be identified with the parity of the \mathfrak{p} -adic valuation, $v_{\mathfrak{p}}(\text{dis } \phi)$, of the discriminant of a form ϕ .

Since $fW_0^1(K) = \ker \{W(K) \xrightarrow{\text{rank}} \mathbb{Z}/2\}$ and $fW_{\text{herm}}^1(K_{\mathfrak{p}}/A_{\mathfrak{p}}) = fW_0^1(K_{\mathfrak{p}}/A_{\mathfrak{p}})$ for $\mathfrak{p} = \bar{\mathfrak{p}}$ nondyadic, the following is an immediate consequence of (6.4).

(6.5) PROPOSITION. *For \mathfrak{p} nondyadic, the result analogous to (6.4) holds, where $W(-)$ is replaced by $fW_0^1(-)$ and $\mathcal{L}_{\text{herm}}^1$ by \mathcal{L}_0^1 .*

It will turn out (see (6.17)(b)) that $(\mathcal{L}_0^1)_{\mathfrak{p}}$, unlike $(\mathcal{L}_{\text{herm}}^1)_{\mathfrak{p}}$, detects the \mathfrak{p} -adic Hasse-Witt invariant when \mathfrak{p} is dyadic

Lannes' idea is to filter $\mathcal{L}_0^1|V(K/A)$ so that \mathcal{L}_0^1 restricted to successive quotients is computed by invariants of $V(K/A)$, as was done in (6.5) for nondyadic \mathfrak{p} . The reader is reminded that fW_0^1 excludes the rank invariant.

Begin by observing that one may make the identification

$$V(K/A) = V(K/A_{(\mathfrak{p})}), \quad \mathfrak{p} \text{ dyadic}.$$

It is therefore sufficient to filter $V(K/A_{(\mathfrak{p})})$.

(6.6) [28, p. 543]. Let \mathfrak{p} be dyadic, and define a surjective homomorphism $\tau: V(K/A_{(\mathfrak{p})}) \rightarrow \mathbb{Z}/2$ by

$$\tau(S, \phi, \psi) = rk_{A/\mathfrak{p}}(S/\mathfrak{p}S),$$

where (S, ϕ, ψ) is a $(+1)$ -quadratic form over K/A . The map τ is closely related to the Dickson invariant.

(6.7) PROPOSITION. *There is a commutative diagram*

$$\begin{array}{ccc} V(K/A) & \xrightarrow{\mathcal{D}_0^1} & fW_1^{-1}(A) \\ \downarrow \tau & & \downarrow r_2 \\ & & fW_1^{-1}(A/2A) \\ & & \cong \downarrow (1.13) \\ \mathbb{Z}/2 & \xrightarrow[\omega(2.6)]{\cong} & fW_1^{-1}(A/\mathfrak{p}) \end{array} \quad \begin{array}{c} \curvearrowright \\ r_p \\ \curvearrowleft \end{array}$$

Proof. It is clearly sufficient to prove commutativity with $A_{(\mathfrak{p})}$ in place of A . So let $[S, \phi, \psi] \in V(K/A_{(\mathfrak{p})})$, let $(A_{(\mathfrak{p})})^{2n} \xrightarrow{\mu} (A_{(\mathfrak{p})})^{2n} \xrightarrow{j} S$ be a resolution, and let $\tau: (A_{(\mathfrak{p})})^{2n} \times (A_{(\mathfrak{p})})^{2n} \rightarrow K$ be chosen to satisfy (1.17) (see [32, 1.17, 1.18]). Then by definition $\mathcal{D}_0^1[S, \phi, \psi]$ is the class of the formation $\theta = ((A_{(\mathfrak{p})})^{2n}, (A_{(\mathfrak{p})})^{2n}, (\mu, \tau\mu))$ (see [32, §4]). By [33, (3.1), (4.1)], ω is defined on $r_p\theta$ to be $rk_{A/\mathfrak{p}}(\text{cok}(\mu/\mathfrak{p})) \bmod 2$, where μ/\mathfrak{p} denotes the reduction of $\mu \bmod \mathfrak{p}$. Since $A_{(\mathfrak{p})}$ is a principle ideal domain, $rk_{A/\mathfrak{p}}(\text{cok}(\mu/\mathfrak{p})) = rk_{A/\mathfrak{p}}(S/\mathfrak{p}S)$, so the diagram commutes.

(6.8) Next let

$$V^1(K/A_{(\mathfrak{p})}) := \ker \tau$$

and define a homomorphism $\bar{\mathcal{I}}$ on it as follows. By (2.3), $fW_1^{-1}(A_{(\mathfrak{p})}) \xrightarrow{r_p} fW_1^{-1}(A/\mathfrak{p}A)$ is injective so by (6.1) with $A_{(\mathfrak{p})}$ in place of A , $\mathcal{D}_0^1|_{\text{im}\{fW_1^{-1}(A_{(\mathfrak{p})}) \rightarrow fW_0^{-1}(K)\}}$ maps surjectively to $V^1(K/A_{(\mathfrak{p})})$. Thus suppose (P, g) is a nonsingular hermitian (i.e., bilinear) form over $A_{(\mathfrak{p})}$ and $\mathcal{D}_0^1(P, g) = x \in V^1(K/A_{(\mathfrak{p})})$. Setting $\bar{\mathcal{I}}(x)$ equal to the mod 4 reduction of $\text{dis}(P, g)$, defines a surjective homomorphism ([28, p. 544])

$$(6.9) \quad \bar{\mathcal{I}}: V^1(K/A) \longrightarrow (A/4A)^\times / (A/4A)^{\times 2} = (A_{(\mathfrak{p})}/4A_{(\mathfrak{p})})^\times / (A_{(\mathfrak{p})}/4A_{(\mathfrak{p})})^{\times 2}.$$

(6.10) Now let

$$(6.11) \quad V^2(K/A) := \ker \bar{J}.$$

There is a natural inclusion $A_{(\mathfrak{p})}/\mathfrak{p}A_{(\mathfrak{p})} \rightarrow K/A_{(\mathfrak{p})}$ given by $a \rightarrow a/\pi$, where π is a generator of $\mathfrak{p} \subseteq A_{(\mathfrak{p})}$. Using this, each quadratic form over $A_{(\mathfrak{p})}/\mathfrak{p}A_{(\mathfrak{p})} = A/\mathfrak{p}A$ gives rise to a $(+1)$ -quadratic form over $K/A_{(\mathfrak{p})}$. The resulting homomorphism $fW_0^1(A/\mathfrak{p}A) \rightarrow fW_0^1(K/A_{(\mathfrak{p})})$ is shown to induce an isomorphism [28, Prop. (2.2)],

$$(6.12) \quad j: fW_0^1(A/\mathfrak{p}A) \xrightarrow{\cong} V^2(K/A_{(\mathfrak{p})}).$$

The following theorem summarizes the above discussion and extends the filtration (6.4) of $W_{\text{herm}}^1(K/A)$.

(6.13) **THEOREM (Lannes).** *The homomorphisms $h(K/A)$, τ , \bar{J} , and j define a filtration of $fW_0^1(K/A)$*

$$fW_0^1(K/A) \supseteq V(K/A) \supseteq V^1(K/A) \supseteq V^2(K/A) \supseteq 0$$

and isomorphisms of the successive quotients with

$$fW_{\text{herm}}^1(K/A), \quad fW_1^{-1}(A/\mathfrak{p}A), \quad (A/4A)^\times/(A/4A)^{\times 2}, \quad fW_0^1(A/\mathfrak{p}A).$$

Set $V_\infty =$ the set of real valuations of A , $h_v(x) (\in \mathbb{Z}/2) =$ the Hasse-Witt invariant of $x \in fW_{\text{herm}}^1(K)$ at the prime \mathfrak{p} (see [60]) and $\sigma_v(x) =$ the signature at $v \in V_\infty$.

(6.14) **THEOREM (Lannes).** *Let the map \mathcal{L}_0^1 of (1.15) induce L in (6.1) and let $V(K/A)$ be filtered as in (6.13). Then there are commutative diagrams with exact rows and columns,*

$$(6.15) \quad \begin{array}{ccc} \frac{fW_0^1(A)}{fW_0^1(A) \cap I^2(K)} & \xrightarrow{\quad} & \frac{fW_{\text{herm}}^1(A)}{fW_{\text{herm}}^1(A) \cap I^2(K)} \xrightarrow{L} \frac{V^1(K/A)}{V^2(K/A)} \\ & \cong \downarrow \text{dis} & \cong \downarrow \bar{J} \\ & A^\times/A^{\times 2} & \xrightarrow[\cong]{\alpha_2} (A/4A)^\times/(A/4A)^{\times 2} \end{array}$$

and

$$(6.16) \quad \begin{array}{ccccc} fW_0^1(A) \cap I^2(K) & \longrightarrow & fW_{\text{herm}}^1(A) \cap I^2(K) & \xrightarrow{L|I(K)} & V^2(K/A) \\ \downarrow \amalg (\sigma_v/4) & & \downarrow \amalg \sigma_v/4 + h_{\mathfrak{p}} & & \cong \downarrow j^{-1} \\ \amalg_{v \in V_\infty} \mathbb{Z} & \longrightarrow & \amalg_{v \in V_\infty} \mathbb{Z} + \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/2 \\ \downarrow r_1 & & \downarrow r_2 & & \\ \mathbb{Z}/2 & \xrightarrow{=} & \mathbb{Z}/2 & & \end{array}$$

where the maps in the second rows of (6.15) and (6.16) are the obvious ones, and r_1 and r_2 are sums in $\mathbb{Z}/2$ ("reciprocity maps").

Proof. From [28, (4.8)], $\ker \alpha_2 \cong \text{Hom}(\mathcal{C}/\mathcal{C}^2, \mathbb{Z}/2)$, which vanishes since \mathcal{C} has odd order [17, Satz 38']; by the Dirichlet Unit Theorem, $rk_{F_2}(A^\times/A^{\times 2}) = 2^{n-2}$, which equals $rk_{F_2}((A/4A)^\times/(A/4A)^{\times 2})$ by [31, (63.8)]. Thus α_2 is an isomorphism. The map this is an isomorphism by an argument similar to, but simpler than that of [28, Prop. 1.12]. The commutativity of (6.15) follows from the description of \bar{I} in (6.8).

The exactness of the second column of (6.16) is from [60, IV. 4.5], that of the first from [28, Thm. 5.1]. The commutativity of the upper right square in (6.16) follows from the discussion in [28, 2.9].

(6.17) REMARKS. (a) The maps r_1 and r_2 are essentially restrictions of Hilbert reciprocity to V_∞ and $V_\infty \cup V_2$. (b) It is interesting to observe that (6.16) shows that \mathcal{L}_0^1 contains the dyadic Hasse-Witt invariants, while from (6.4), $\mathcal{L}_{\text{herm}}^1$ does not. Nondyadic Hasse-Witt invariants do not appear in (6.16) because they persist under both \mathcal{L}_0^1 and $\mathcal{L}_{\text{herm}}^1$ to $fW_0^1(K/A)$ (see (6.4), (6.5)). Thus, they cannot appear in $fW_{\text{herm}}^1(A)$ or $fW_0^1(A)$, by exactness of the localization sequence.

(6.18) COROLLARY. *The Dickson invariant induces an isomorphism*

$$fW_1^\lambda(A) \xrightarrow{\cong} fW_1^\lambda(A/2A).$$

Proof. $L(fW_{\text{herm}}^1(A)) \subseteq V^1(K/A)$ by the discussion in (6.8). This and the diagrams (6.15), (6.16) give the result if $\lambda = 1$.

The discussion of $fW_1^{-1}(A)$ is elementary (i.e., nonarithmetic). Namely, skew-symmetric forms on nondyadic A -torsion modules are always hyperbolic: the proof is essentially the same as for skew-symmetric forms over fields of characteristic $\neq 2$. Further, if (S, ϕ, ψ) is a (-1) -quadratic form on a dyadic A -torsion module, we claim it is also hyperbolic. For S may be assumed to be \mathfrak{p} -torsion, for some fixed dyadic \mathfrak{p} , and hence is an $A_{\mathfrak{p}}$ -module. In [54, §4], Wall classifies skew-hermitian forms (S, ϕ) over \mathbb{Q}/\mathbb{Z} by an argument which easily generalizes to $A_{\mathfrak{p}}$; and it is a consequence of the definition that (because of the presence of ψ) $\phi(x, x) = 0$, for all $x \in S$. Thus, by [54, Lemma 7], (S, ϕ, ψ) is hyperbolic, so that $fW_1^{-1}(K/A) = 0$. (However, $fW_{\text{herm}}^{-1}(K/A) \neq 0$ and is detected by de Rham invariants [54, §4].) From [33, (4.1)], $fW_1^1(A) \rightarrow fW_1^1(K) \cong \mathbb{Z}/2$ is surjective

(by lifting back a nonzero representative), hence by the localization sequence an isomorphism. Using the nonzero representative in (1.6), the proof is complete.

(6.19) THEOREM. *The Arf invariant*

(a) $fW_0^{-1}(A) \rightarrow fW_0^{-1}(A/2A) \cong \mathbf{Z}/2$ *is nontrivial.*

(b) $fW_0^1(A) \rightarrow fW_0^1(A/2A) \cong \mathbf{Z}/2$ *is trivial.*

Proof. The isomorphisms follow from reduction (1.13) and (2.6).

Part (a) is obvious. To prove (b) observe first that if \mathfrak{p} is dyadic, the $(+1)$ -quadratic form $((A_{\mathfrak{p}})^2, g, q)$, used in the proof of (5.4), has nontrivial image in $fW_0^1(A/\mathfrak{p})$, and discriminant 5 which is nontrivial in $A_{\mathfrak{p}}^{\times}/A_{\mathfrak{p}}^{\times^2}$ by [31, 63.2] and the fact that it is nontrivial in $(A\mathfrak{p}/4\mathfrak{p}A_{\mathfrak{p}})^{\times}/(A_{\mathfrak{p}}/4\mathfrak{p}A_{\mathfrak{p}})^{\times^2}$. Thus, it suffices to show that no element of $fW_0^1(A_{\mathfrak{p}})$ has discriminant, which, in $A_{\mathfrak{p}}^{\times}$ is congruent to 5 mod $A_{\mathfrak{p}}^{\times^2}$. But by (6.15) all such discriminants vanish.

Next it is necessary to work out the filtration of (6.13) for application in Chapter 7. First some lemmas.

(6.20) LEMMA. *Let \mathfrak{p} be the dyadic prime in A . Then the Hilbert symbols*

$$\left(\frac{-1, 5}{K_{\mathfrak{p}}}\right) = \left(\frac{-1, 2 - (\zeta + \zeta^{-1})}{K_{\mathfrak{p}}}\right) = 1, \quad \zeta = \zeta_{2^n}, \quad n \geq 3.$$

Proof. 5 is the sum of two squares, so the first symbol is 1 by definition [31, 63.10]. Since $Q(\zeta + \zeta^{-1})(\sqrt{-1}) = Q\zeta$, $N(1 - \zeta) = 2 - (\zeta + \zeta^{-1})$, where $N: Q\zeta^{\times} \rightarrow Q(\zeta - \zeta^{-1})^{\times}$ is the norm. By [31, 63.10] the second value follows and the proof is done.

(6.21) LEMMA. *The map $c: A^{\times}/A^{\times^2} \rightarrow A_{\mathfrak{p}}^{\times}/A_{\mathfrak{p}}^{\times^2}$ is injective (\mathfrak{p} the dyadic prime) and its cokernel is $\mathbf{Z}/2$, generated by the class of the \mathfrak{p} -adic unit 5.*

Proof. By the Dirichlet Unit Theorem, $rk_{F_2}(A^{\times}/A^{\times^2}) = 2^{n-2}$, while by [31, 63.9], $rk_{F_2}(A_{\mathfrak{p}}^{\times}/A_{\mathfrak{p}}^{\times^2}) = 2^{n-2} + 1$. Since α_2 in (6.15) factors through c , the proof is complete.

(6.22) *Let $S: A^{\times}/A^{\times^2} \rightarrow \prod_{v \in V_{\infty}} \{\pm 1\}$ be the map which assigns to $a \in A^{\times}$ its signs at real completions. Then S is an isomorphism.*

Proof. By [17, Satz 38'], S is surjective; it is an isomorphism because the ranks are equal, using the fact that K is totally real.

(6.23) PROPOSITION. Let $A = \mathbf{Z}(\zeta + \zeta^{-1})$, $\zeta = \zeta_{2^n}$, $n \geq 3$ and let \mathfrak{p} denote the dyadic prime. Then

(a) $V^2(K/A) \cong \mathbf{Z}/2$ with generator $\mathcal{L}_0^1\langle 1, 1, -\pi, -\pi \rangle$ where π is some generator for \mathfrak{p} .

(b) $V^1(K/A) \cong V^2(K/A) \oplus (V^1(K/A)/V^2(K/A)) = \mathbf{Z}/2 + (\mathbf{Z}/2)^{2^n-2}$, where V^1/V^2 has basis $\mathcal{L}_0^1\langle -1, u_i \rangle$, $\{u_i\}$ a basis for $A^\times/A^{\times 2}$ over \mathbf{F}_2 .

(c) $V(K/A) \cong V^2(K/A) \oplus (\mathbf{Z}/2)^{2^n-2-1} \oplus \mathbf{Z}/4$ where the third term is generated by the class of the $(+1)$ -quadratic form $(A/4A, \phi, \psi)$ where $\phi(g, g) = 1/4 \in K/A$, $\psi(g) = 1/4 \in K/2A$ and g is the generator of $A/4A$.

Proof. Since $(-1) \in K_v^{\times 2}$, nondegeneracy of the Hilbert symbol implies there is a nonsplit quaternion algebra $(-1, u/K_v)$ (see [25, 6.2.16]) for some $u \in K_v^\times$. Using (6.20) and (6.21), it is possible to assume $u = v(2 - (\zeta + \zeta^{-1}))$, $v \in A^\times$. Set $\pi = v(2 - (\zeta + \zeta^{-1}))$. The norm form of $(-1, \pi/K)$ is, by definition (see [25, p. 56]), the quaternary form $\phi = \langle 1, 1, -\pi, -\pi \rangle \in fW_0^1(K)$. Clearly $\text{dis}(\phi) \in K^{\times 2}$; and the Hasse-Witt invariant

$$h_q(\phi) = \left(\frac{-1, \pi}{K_q} \right) = 1,$$

for nondyadic finite q , by [31, 63:11a]. Hence $\phi \in I^2(K)$ (by construction), $\phi \in \text{im}\{fW_{\text{herm}}^1(A) \rightarrow fW_{\text{herm}}^1(K)\}$ (by (6.0) and (6.4)), and $h_v(\phi) = (-1, \pi/K_v) = -1$. Thus, by the commutativity of the upper right square of (6.16), $\mathcal{L}_0^1(\phi)$ is the generator of $V^2(K/A) = \mathbf{Z}/2$.

(b) Immediate from (6.13) and (6.15).

(c) Clearly, by (6.6), $\tau(A/4A, \phi, \psi) = rk(A/4A \otimes A/\mathfrak{p}) = 1$.

(d) Evidently $\mathcal{L}_{\text{herm}}^1(\langle -1, \pi \rangle)$ is the generator of $fW_{\text{herm}}^1(K_v/A_v) = fW_{\text{herm}}^1(A/\mathfrak{p})$ (see (6.4)(d)) and $2\mathcal{L}_0^1\langle -1, \pi \rangle = \mathcal{L}_0^1\langle -1, -1, \pi, \pi \rangle =$ generator of $V^2(K/A)$.

7. Arf and Dickson invariants over the maximal order \mathcal{N}_n in Γ_n . In this chapter Γ_n and \mathcal{N}_n will denote the quaternion algebra $(-1, -1)/\mathbf{Q}(\zeta + \zeta^{-1})$ and an involution-invariant maximal order in it, respectively, where $\zeta = \zeta_{2^n}$, $n \geq 2$.

Begin (as in §6) by considering the commutative diagram

$$(7.1) \quad \begin{array}{ccccccc} fW_0^{-1}(\mathcal{N}_n) & \xrightarrow{\mathcal{K}_0^{-1}} & fW_0^{-1}(\Gamma_n) & \xrightarrow{\mathcal{L}_0^{-1}} & fW_0^{-1}(\Gamma_n/\mathcal{N}_n) & \xrightarrow{\mathcal{D}_0^{-1}} & fW_1^1(\mathcal{N}_n) \\ \downarrow H & & \downarrow = & & \downarrow & & \\ fW_{\text{herm}}^{-1}(\mathcal{N}_n) & \xrightarrow{\mathcal{K}_{\text{herm}}^{-1}} & fW_{\text{herm}}^{-1}(\Gamma_n) & \xrightarrow{\mathcal{L}_{\text{herm}}^{-1}} & fW_{\text{herm}}^{-1}(\Gamma_n/\mathcal{N}_n) & & \end{array}$$

At the end of this chapter we prove

(7.2) THEOREM. $\mathcal{L}_{\text{herm}}^{-1}$ is surjective in (6.1).

This fact, together with the facts $\mathcal{K}_{\text{herm}}^{-1}$ is injective (by [6, 1.1]) and $fW_1^1(\Gamma_n) = 0$ (by [33, (4.1)]) furnish an exact sequence, derived from (7.1):

$$(7.3) \quad fW_0^{-1}(\mathcal{N}_n) \xrightarrow{H} fW_{\text{herm}}^{-1}(\mathcal{N}_n) \xrightarrow{L} V(\Gamma_n/\mathcal{N}_n) \longrightarrow fW_1^1(\mathcal{N}_n)$$

where

$$(7.4) \quad V(\Gamma_n/\mathcal{N}_n) := \ker \{fW_0^{-1}(\Gamma_n/\mathcal{N}_n) \longrightarrow fW_{\text{herm}}^{-1}(\Gamma_n/\mathcal{N}_n)\}.$$

(7.5) THEOREM.

$$V(\Gamma_n/\mathcal{N}_n) = \begin{cases} \mathbf{Z}/2 + \mathbf{Z}/2, & n = 2 \\ ((\mathbf{Z}/2)^{2^{n-2}} + \mathbf{Z}/4, & n \geq 3. \end{cases}$$

Proof. Write $\Gamma_n = \Gamma$, $\mathcal{N}_n = \mathcal{N}$. Since hermitian and quadratic forms on nondyadic torsion modules can be identified,

$$V(\Gamma/\mathcal{N}) = V(\Gamma/\mathcal{N}_{(\mathfrak{p})}) = V(\Gamma_{\mathfrak{p}}/\mathcal{N}_{\mathfrak{p}})$$

where \mathfrak{p} is the unique dyadic prime in the center $\mathbf{Z}(\zeta + \zeta^{-1})$ of \mathcal{N} .

Let $n \geq 3$. Replacing (Γ, \mathcal{N}) by $(\Gamma_{\mathfrak{p}}, \mathcal{N}_{\mathfrak{p}})$, since $(\Gamma_{\mathfrak{p}}, \mathcal{N}_{\mathfrak{p}}) \cong (M_2(\mathbf{Q}(\zeta + \zeta^{-1})_{\mathfrak{p}}), M_2(\mathbf{Z}(\zeta + \zeta^{-1})_{\mathfrak{p}}))$ with involutions described in (4.4) and (4.16), the diagram becomes by Morita theory (4.3) isomorphic to (6 0), where $(K, A) = (\mathbf{Q}(\zeta + \zeta^{-1})_{\mathfrak{p}}, \mathbf{Z}(\zeta + \zeta^{-1})_{\mathfrak{p}})$; applying (6.23) completes the proof in this case.

If $n = 2$, then $\Gamma_{\mathfrak{p}}$ is a division algebra so by [61] the discriminant induces

$$fW_0^{-1}(\Gamma_{\mathfrak{p}}) = fW_{\text{herm}}^{-1}(\Gamma_{\mathfrak{p}}) \cong \mathbf{Q}_2^{\times}/(\mathbf{Q}_2^{\times})^2 = (\mathbf{Z}/2)^3.$$

Thus, a set of generators is $\{\langle i + 2j, i \rangle, \langle i + j + k, i + 2j \rangle, \langle i, i + j \rangle\}$. Since the reduced norms of their entries lie in \mathbf{Z}_2 the first two generators are in $\text{im}(fW_{\text{herm}}^{-1}(\mathcal{N}_{\mathfrak{p}}) \rightarrow fW_{\text{herm}}^{-1}(\Gamma_{\mathfrak{p}}))$ by [42, 12.5]; $\langle i, i + j \rangle$ is not and must therefore map to the generator of $fW_{\text{herm}}^{-1}(\Gamma_{\mathfrak{p}}/\mathcal{N}_{\mathfrak{p}}) = \mathbf{Z}/2$ under $\mathcal{L}_{\text{herm}}^{-1}$. Finally, letting (m) denote the unique maximal 2-sided ideal of $\mathcal{M}_{\mathfrak{p}}$ it is generated by $(1 - \zeta)$, a direct calculation or [42, 14.3] shows

$$(7.6) \quad \mathcal{N}_{\mathfrak{p}}/(m) \cong F_4,$$

where F_4 has the nontrivial involution. By (1.13), and (2.6),

$$(7.7) \quad fW_{*}^2(\mathcal{N}_{\mathfrak{p}}) \equiv 0.$$

Putting these facts into (7.1) where (Γ, \mathcal{N}) is replaced by $(\Gamma_{\mathfrak{p}}, \mathcal{N}_{\mathfrak{p}})$ yields $V(\Gamma/\mathcal{N}) = \mathbf{Z}/2 + \mathbf{Z}/2$ as required.

(7.8) DEFINITION. Let \mathfrak{p} be a (discrete) prime in the center

$Q(\zeta + \zeta^{-1})$ of Γ_n , $\zeta = \zeta_{2^n}$, and suppose $(\Gamma_n)_v$ is split. Set $\Gamma_n = \Gamma$. If $x \in fW_0^{-1}(\Gamma) = fW_{\text{herm}}^{-1}(\Gamma)$, its \mathfrak{p} -adic Hasse-Witt invariant is the image in the composition $fW_0^{-1}(\Gamma) \rightarrow fW_0^{-1}(\Gamma_v) \xrightarrow{\cong} fW_0^{-1}(M_2(Q(\zeta + \zeta^{-1})_v)) \xrightarrow{\text{Morita}} fW_0^{-1}(Q(\zeta + \zeta^{-1})_v) \xrightarrow{h_v} \mathbf{Z}/2$, where h_v is the Hasse-Witt invariant of [60].

(7.9) COROLLARY. (a) One $\mathbf{Z}/2$ -term in (7.5) is generated by $\mathcal{L}_0^{-1}(x)$, where x has nontrivial dyadic Hasse-Witt invariant, $n \geq 3$.

(b) If $n \geq 3$, $V(\Gamma_n/\mathcal{N}_n) \xrightarrow{\mathcal{L}_0^{-1}} fW_1^{-1}(\mathcal{N}_n) \rightarrow fW_1^{-1}(\mathcal{N}_n/2\mathcal{N}_n) = \mathbf{Z}/2$ is nontrivial.

Proof. The corollary is immediate from the analogous facts, where skew-hermitian forms over $(\Gamma_n, \mathcal{N}_n)$ are replaced by symmetric forms over $(Q(\zeta + \zeta^{-1}), \mathbf{Z}(\zeta + \zeta^{-1}))$, and the fact that the proof of (7.5) used Morita theory to translate the latter context.

(7.10) THEOREM.

$$H: fW_0^{-1}(\mathcal{N}_n) \longrightarrow fW_{\text{herm}}^{-1}(\mathcal{N}_n) = \begin{cases} 0 \longrightarrow 0, & n = 2 \\ (\mathbf{Z}/2)^{2^n-2-2} \longrightarrow (\mathbf{Z}/2)^{2^n-2-2} + \mathbf{Z}/2, \\ (\text{inclusion into first term}) & n \geq 3. \end{cases}$$

The $\mathbf{Z}/2$ not in $\text{im}(H)$, $n \geq 3$, can be represented by a form with trivial discriminant and nondyadic Hasse-Witt invariants, and nontrivial dyadic Hasse-Witt invariant.

Proof. Let $\mathcal{N} = \mathcal{N}_n$, $\Gamma = \Gamma_n$. According to [23, p. 138] the kernel of the completion-induced map

$$(7.12) \quad C: fW_{\text{herm}}^{-1}(\Gamma) \longrightarrow \coprod_v fW_{\text{herm}}^{-1}(\Gamma_v),$$

(where the sum is over all valuations of the center of Γ) is trivial if and only if at most two Γ_v are nonsplit; otherwise it is an elementary 2-group of rank $|S| - 2$ where S is the set of places where Γ_v is a division algebra. Thus, if $n = 2$, C is injective, and since Γ is split at nondyadic $q \in \mathbf{Z}$ (see (4.9)), by Morita theory (4.3) (q is odd),

$$\begin{array}{ccc} fW_{\text{herm}}^{-1}(\Gamma_q) & \xrightarrow{\mathcal{L}_{\text{herm}}^{-1}} & fW_{\text{herm}}^{-1}(\Gamma_q/\mathcal{N}_q) \\ \cong \downarrow & & \cong \downarrow \\ fW_{\text{herm}}^1(\mathbf{Q}_q) & \xrightarrow{\mathcal{L}_{\text{herm}}^1} & fW_{\text{herm}}^1(\mathbf{Q}_q/\mathbf{Z}_q). \end{array}$$

Thus if $x \in \text{im}\{\mathcal{K}_{\text{herm}}^{-1}: fW_{\text{herm}}^{-1}(\mathcal{N}) \rightarrow fW_{\text{herm}}^{-1}(\Gamma)\}$ all Hasse-Witt invariants vanish at odd q (by (6.4)(b)). On the other hand, the

discriminant of x must be a positive integer and a unit, hence equal to one. Since the right side of (7.12) is detected by the discriminants and Hasse-Witt invariants if Γ_p is split (p odd) and by the discriminant if $p = 2$ (by [61]), $x = 0$. Since $\mathcal{K}_{\text{herm}}^{-1}$ is injective $fW_{\text{herm}}^{-1}(\mathcal{N}) = 0$. The argument just given, together with (7.7), (1.13) and (2.1), shows $fW_0^{-1}(\mathcal{N}) = 0$.

Now let $n \geq 3$. The argument just given also shows $\ker C \subseteq fW_{\text{herm}}^{-1}(\mathcal{N})$. By the "Existence of forms with prescribed local behavior" (7.17), there is $x \in fW_{\text{herm}}^{-1}(\Gamma)$ with $\text{dis}(x) = 0$, $h_q(x) = 0$, for $q \neq p$, the unique dyadic prime, and $h_p(x) \neq 0$. By (6.4)(a) and Morita theory, $\mathcal{L}_{\text{herm}}^{-1}(x) = 0$, so $x \in \text{im}(\mathcal{K}_{\text{herm}}^{-1})$. But $x \notin \ker C$, because $h_p(x) \neq 0$. Thus, we have

$$(7.13) \quad \ker C + \mathbf{Z}/2 \subseteq fW_{\text{herm}}^{-1}(\mathcal{N}),$$

where the $\mathbf{Z}/2$ is represented by x with $h_p(x) \neq 0$, $h_q(x) = 0$, q non-dyadic, and $\text{dis}(x) = 0$.

Conversely, if $x \in fW_{\text{herm}}^{-1}(\mathcal{N})$, then $\text{dis}(x) \in \mathbf{Z}(\zeta + \zeta^{-1})^\times$ and is positive at all real valuations (see (1.7)). Thus by (6.22) it is a square. Also, the usual Morita arguments together with (6.4)(b) show $h_q(x) = 0$, for q nondyadic. This shows $\ker C + \mathbf{Z}/2 \supseteq fW_{\text{herm}}^{-1}(\mathcal{N})$.

It will be shown in (7.15) that $fW_0^{-1}(\mathcal{N}) \rightarrow fW_0^{-1}(\mathcal{N}/2\mathcal{N})$ is trivial. This together with (2.1), shows H is injective. The arguments above show $\ker C \subseteq fW_0^{-1}(\mathcal{N})$; this is equality because by (6.4)(b), $\mathcal{L}_0^{-1}(x) \neq 0$, where x represents the $\mathbf{Z}/2$ in (7.9).

The next result is an immediate corollary of (7.5), (7.9), and (7.10). Observe how it contrasts with (5.3) and (6.18), where the Dickson invariant detected essentially all of $fW_1^\lambda(A)$.

(7.14) THEOREM. (a) For all $n \geq 3$, there is a nonsplit exact sequence

$$(\mathbf{Z}/2)^{2^{n-2}} \twoheadrightarrow fW_1^1(\mathcal{N}_n) \longrightarrow fW_1^1(\mathcal{N}_n/2\mathcal{N}_n);$$

and $fW_1^1(\mathcal{N}_n/2\mathcal{N}_n) = \mathbf{Z}/2$ if $n \geq 3$, and is trivial if $n = 2$; $fW_1^1(\mathcal{N}_2) = \mathbf{Z}/2 + \mathbf{Z}/2$.

(b) $fW_1^{-1}(\mathcal{N}_n) = 0$, for all n .

To prove (b), Morita arguments (by now routine), together with the proof of (6.18) and the splittings (1.10, show that $fW_1^1(\Gamma_n/\mathcal{N}_n) = 0$, $n \geq 3$. When $n = 2$, a theorem of [40] shows $fW_0^1((\Gamma_n)_2) = 0$, and by (7.7), $fW_1^{-1}((\mathcal{N}_n)_2) = 0$. Hence, by the localization sequence, $fW_1^1(\Gamma_n/\mathcal{N}_n) = 0$, $n = 2$. Since $fW_1^{-1}(\Gamma_n) = 0$ by [33, (4.1)], the localization sequence yields b).

(7.15) **THEOREM.** (a) $fW_0^\lambda(\mathcal{N}_n/2\mathcal{N}_n) = \mathbf{Z}/2$ if $n \geq 3$, and is trivial if $n = 2$. (b) $fW_0^\lambda(\mathcal{N}_n) \rightarrow fW_0^\lambda(\mathcal{N}_n/2\mathcal{N}_n)$ is nontrivial if and only if $n \geq 3$ and $\lambda = 1$.

Proof. When $n = 2$, (a) follows from (1.13) and (7.7). For $n \geq 3$ it follows from the fact that Γ_n is split at the dyadic prime, Morita theory, and (6.19).

For (b) consider the following diagram, where $A = \mathbf{Z}(\zeta + \zeta^{-1})$, $K = \mathbf{Q}(\zeta + \zeta^{-1})$, K_+^\times = the group of nonzero, totally positive elements of K , $\mathcal{N} = \mathcal{N}_n$, $\Gamma = \Gamma_n$ and \mathfrak{p} is the unique dyadic prime in K

$$\begin{array}{ccccc} fW_0^{-1}(\mathcal{N}) & \xrightarrow{\mathcal{K}_0^{-1}} & fW_0^{-1}(\Gamma) & \xrightarrow{\text{dis}} & K_+^\times/K^{x^2} \\ \downarrow & & \downarrow & & \downarrow \\ fW_0^{-1}(\mathcal{N}_{\mathfrak{p}}) & \xrightarrow{\quad} & fW_0^{-1}(\Gamma_{\mathfrak{p}}) & & \\ \cong \downarrow \text{Morita} & & \cong \downarrow \text{Morita} & & \\ fW_0^{-1}(A_{\mathfrak{p}}) & \xrightarrow{\quad} & fW_0^{-1}(K_{\mathfrak{p}}) & \xrightarrow{\text{dis}} & K_{\mathfrak{p}}^\times/K_{\mathfrak{p}}^{x^2}. \end{array}$$

Since $\text{im}(\text{dis} \cdot \mathcal{K}_0^{-1}) \subseteq A^\times/A^{x^2}$ consists of totally positive units, this image is trivial by (6.22). Now the proof of (6.19)(b) shows $fW_0^{-1}(A_{\mathfrak{p}})$ is detected by its discriminant; but $A_{\mathfrak{p}}^\times/A_{\mathfrak{p}}^{x^2} \rightarrow K_{\mathfrak{p}}^\times/K_{\mathfrak{p}}^{x^2}$ is injective ($A_{\mathfrak{p}}$ is integrally closed), so (b) is proved in case $\lambda = -1$.

If $\lambda = +1$, consider the diagram

$$\begin{array}{ccccc} fW_1^{-1}(\Gamma/\mathcal{N}) & \xrightarrow{\mathcal{D}_1^1} & fW_0^1(\mathcal{N}) & \xrightarrow{\mathcal{K}_0^1} & fW_0^1(\Gamma) \\ \downarrow C & & \downarrow & & \downarrow \\ fW_1^{-1}(\Gamma_{\mathfrak{p}}/\mathcal{N}_{\mathfrak{p}}) & \xrightarrow{(\mathcal{D}_1^1)_{\mathfrak{p}}} & fW_0^1(\mathcal{N}_{\mathfrak{p}}) & \xrightarrow{(\mathcal{K}_0^1)_{\mathfrak{p}}} & fW_0^1(\Gamma_{\mathfrak{p}}) \\ & & \cong \downarrow \text{Morita} & & \cong \downarrow \text{Morita} \\ & & fW_0^{-1}(\mathbf{Z}(\zeta + \zeta^{-1})_{\mathfrak{p}}) & & fW_0^{-1}(\mathbf{Q}(\zeta + \zeta^{-1})_{\mathfrak{p}}) = 0 \\ & & \cong \downarrow \text{reduction} & & \\ & & fW_0^{-1}(\mathbf{Z}(\zeta + \zeta^{-1})/\mathfrak{p}) & & \end{array}$$

The map C is surjective by (1.10). From this and the well-known fact that there is an element in $fW_0^{-1}(\mathbf{Z}(\zeta + \zeta^{-1})_{\mathfrak{p}})$ which is nontrivial after reduction mod 2, there exists $x \in fW_0^1(\mathcal{N}_{\mathfrak{p}})$ with nontrivial mod 2 reduction and $(\mathcal{K}_0^1)_{\mathfrak{p}}(x) = 0$. If $(\mathcal{D}_1^1)_{\mathfrak{p}}(\tilde{x}) = x$, $\tilde{x} \in fW_1^{-1}(\Gamma_{\mathfrak{p}}/\mathcal{N}_{\mathfrak{p}})$, then $\mathcal{D}_1^1 \cdot C^{-1}(\tilde{x}) \in fW_0^1(\mathcal{N})$ is nonvanishing in $fW_0^1(\mathcal{N}/2\mathcal{N})$.

It remains to prove Theorem (7.2): that $\mathcal{L}_{\text{herm}}^{-1}$ is surjective in Diagram (7.1). To do this, a version of the “local-to-global theorem” of [31, §72], is needed, where the number field there is replaced here by the quaternion algebra Γ_n .

(7.17) **THEOREM** (*Existence of forms with prescribed local behavior*). Let $K = \mathbb{Q}(\zeta + \zeta^{-1})$, $\zeta = \zeta_{2^n}$, $n \geq 2$, $\Gamma = (-1, -1/K)$, $S =$ the set of primes at which Γ is not split. Let $\{\delta_p\} \in \prod_{p \text{ finite}} K_p^\times / K_p^{\times^2}$ and $\{\gamma_p\} \in \prod_{p \in S} \{\pm 1\}$ be given. Then there is a skew-hermitian form $g: \Gamma^n \times \Gamma^n \rightarrow \Gamma$ such that $h_p(g) = \gamma_p$ and $\text{dis}_p(g) = \delta_p$ for all p if and only if

(a) there exists a totally positive $d \in K^\times$ satisfying $d_p = \delta_p$, p finite, and

(b) almost all $\gamma_p = 1$.

REMARK. Observe that the $\{\gamma_p\}$ are not required to satisfy the "product formula" $\prod \gamma_p = 1$, as in [31, §72].

Proof. First recall some results on the Galois cohomology of (classical) algebraic groups (for details see [23]). For an algebraic closure \bar{K} of K all skew-hermitian forms of fixed rank n say, over $\Gamma \otimes \bar{K} (= M_2(\bar{K}))$ become isometric. Let $U = U(\bar{K})$ denote the isometry group of this form. The reduced norm induces a homomorphism $U \rightarrow \mathbb{Z}/2$, whose kernel is denoted SU . There is a $\mathbb{Z}/2$ -covering of SU , denoted Spin , and there are exact sequences

$$(7.18) \quad 1 \longrightarrow SU \longrightarrow U \longrightarrow \mathbb{Z}/2 \longrightarrow 1$$

and

$$(7.19) \quad 1 \longrightarrow \mathbb{Z}/2 \longrightarrow \text{Spin} \longrightarrow SU \longrightarrow 1.$$

(7.20) **LEMMA** [23, pp. 14-15]. *There is a 1-1 correspondence*

$$H^1(K, U) \longleftrightarrow \left\{ \begin{array}{l} \text{isometry classes of rank } n \\ \text{skew-hermitian forms over } \Gamma \end{array} \right\}.$$

Using this, the map $H^1(K, U) \rightarrow H^1(K; \mathbb{Z}/2) \cong K^\times / K^{\times^2}$ induced by the second map in the sequence (6.18) can be identified with the discriminant and $H^1(K, SU)$, with forms of discriminant 1. If Γ_p is split, the connecting map in the cohomology sequence induced from (7.19), $H^1(K_p, SU) \rightarrow H^2(K_p; \mathbb{Z}/2) \cong \text{Br}_2(K_p) = \mathbb{Z}/2$ can be identified with the Hasse-Witt invariant of (7.8).

In general, $H^1(K, U)$, $H^1(K, SU)$, etc., are not groups, only pointed sets because U , SU are not abelian. However, by considering first the case where $n = 1$, so that SU becomes a torus, hence abelian, an argument of [23, p. 137] essentially identifies the map (induced by completions)

$$C: H^1(K; SU) \longrightarrow \prod_p H^1(K_p, SU)$$

with the map L in the Artin reciprocity sequence for the extension $K\sqrt{-1}/K = \mathbb{Q}\zeta_{2^n}/\mathbb{Q}(\zeta_{2^n} + \zeta_{2^n}^{-1})$ used in (5.2). Combining these remarks results in the exact commutative diagram [23, p. 136]

$$\begin{array}{ccccccc}
 & & & & \text{dis} & & \\
 & & & & \curvearrowright & & \\
 \mathbb{Z}/2 & \longrightarrow & H^1(K; SU) & \longrightarrow & H^1(K; U) & \longrightarrow & H^1(K; \mathbb{Z}/2) \cong K^\times/K^{\times 2} \\
 \downarrow \Delta & & \downarrow C & & \downarrow D & & \downarrow P \\
 \prod_{\mathfrak{p} \in S} \mathbb{Z}/2 & \longrightarrow & \prod_{\mathfrak{p}} H^1(K_{\mathfrak{p}}; SU) & \longrightarrow & \prod_{\mathfrak{p}} H^1(K_{\mathfrak{p}}; U) & \longrightarrow & \prod_{\mathfrak{p}} H^1(K_{\mathfrak{p}}; \mathbb{Z}/2) \cong K_{\mathfrak{p}}^\times/K_{\mathfrak{p}}^{\times 2} \\
 & & \downarrow r & & & & \uparrow \prod \text{dis}_{\mathfrak{p}} \\
 & & \mathbb{Z}/2 & & & &
 \end{array}$$

where $S =$ the (finite) set of primes \mathfrak{p} at which Γ is not split and Δ is the diagonal inclusion.

The interpretation given above of the sets and maps in this diagram yield Theorem (7.17). Details are left to the reader.

REMARK. Chasing in the diagram, one also gets the failure of the Hasse-principle: D has kernel consisting of $2^{|S|-2}$ elements.

Proof of (7.2). Consider the commutative diagram

$$\begin{array}{ccc}
 fW_{\text{herm}}^{-1}(\Gamma) & \xrightarrow{\mathcal{L}_{\text{herm}}^{-1}} & fW_{\text{herm}}^{-1}(\Gamma/\mathcal{N}) \\
 \downarrow & & \downarrow \cong \\
 \prod_{\mathfrak{p} \text{ finite}} fW_{\text{herm}}^{-1}(\Gamma_{\mathfrak{p}}) & \xrightarrow{(\mathcal{L}_{\text{herm}}^{-1})_{\mathfrak{p}}} & \prod_{\mathfrak{p} \text{ finite}} fW_{\text{herm}}^{-1}(\Gamma_{\mathfrak{p}}/\mathcal{N}_{\mathfrak{p}}) \\
 \downarrow \text{Morita} & & \downarrow \\
 \prod_{\mathfrak{p} \in S'} fW_{\text{herm}}^{-1}(\Gamma_{\mathfrak{p}}) + \prod_{\substack{\mathfrak{p} \notin S' \\ \text{finite}}} fW_{\text{herm}}^{-1}(K_{\mathfrak{p}}) & \longrightarrow & \prod_{\mathfrak{p} \in S'} fW_{\text{herm}}^{-1}(\Gamma_{\mathfrak{p}}/\mathcal{N}_{\mathfrak{p}}) + \prod_{\substack{\mathfrak{p} \notin S' \\ \text{finite}}} fW_{\text{herm}}^{-1}(K_{\mathfrak{p}}/A_{\mathfrak{p}})
 \end{array}$$

where S' is the set of finite primes at which Γ is not split. (When $K = \mathbb{Q}(\zeta_{2^n} + \zeta_{2^n}^{-1})$, $n \geq 3$, $S' = \emptyset$.) Consider also the commutative diagram

$$\begin{array}{ccc}
 K^\times/K^{\times 2} & = & K^\times/K^{\times 2} \\
 \downarrow & & \downarrow j \\
 \prod_{\mathfrak{p}} K_{\mathfrak{p}}^\times/K_{\mathfrak{p}}^{\times 2} & \xrightarrow{\prod \tilde{v}_{\mathfrak{p}}} & \prod_{\mathfrak{p}} (\mathbb{Z}/2) \\
 & & \downarrow \\
 & & \mathcal{C}/\mathcal{C}^2
 \end{array}$$

where $\tilde{v}_{\mathfrak{p}}$ is the $v_{\mathfrak{p}}$ -adic valuation mod 2, and \mathcal{C} is the class group.

Since \mathcal{E} has odd order [17, Satz 38'], j is an isomorphism. This, together with the interpretation of $fW_{\text{herm}}^1(K_p) \rightarrow fW_{\text{herm}}^1(K_p/A_p)$ in (6.4) and Theorem (7.17), shows $\mathcal{L}_{\text{herm}}^{-1}$ is surjective. The argument for $n = 2$ is left to the reader.

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