PACIFIC JOURNAL OF MATHEMATICS
Vol. 102, No. 1, 1982

AN AFFIRMATIVE ANSWER TO GLAUBERMAN'S CONJECTURE

MASAHIKO MIYAMOTO

Let G be a finite group and P be a Sylow p-subgroup of G for a prime p. The following question is raised by G. Glauberman.

Question 16.8. Does there exist a function f from the positive integers i to the positive integers such that

$$H^i(G, F_p) \cong H^i(N_G(K_\infty(P)), F_p) \quad \textbf{whenever} \quad p \geqq f(i)?$$

Here K_∞ denotes the section conjugacy functor constructed by G. Glauberman and F_p denotes the finite field consisting of p elements and by forgetting its multiplicative structure, we consider it as a trivial G-module.

In relation to the above conjecture, he proved the case $i = 1$ and D. F. Holt has recently proved $f(2) \leqq 11$. The purpose of this paper is to provide an affirmative answer to the question.

THEOREM C. If $p \geqq 12 \times 6^{m-2} + 3$, the $H^m(G, F_p) \cong H^m(N_G(K_\infty(P)), F_p)$ for all integers $m \geqq 2$.

This theorem is a a consequence of the following more detailed theorem since K_∞ has degree 4.

THEOREM B. *Let W be a section conjugacy functor of G of degree t. If $p \geqq 4 \times 6^{m-2} \times (t-1) + 3$, then the restriction map induces $H^m(G, V) \cong H^m(N_G(W(P)), V)$ for a trivial p-primary G-module V and all integers $m \geqq 2$.*

COROLLARY 1. *Let V be a faithful p-primary G-module, n an integer greater than 0, and r be the first integer such that $p \leqq 4 \times 6^{n-1} \times (r+1) + 2$. Set $A = \langle a \in P\colon [V, a; r+1] = 1, a^p = 1\rangle$. Then we have $H^m(G, V) \cong H^m(N_G(A), V)$ for all integers m with $1 \leqq m \leqq n$.*

2. Notations and preparations. All groups considered in this paper will be finite and we treat only finite modules by the same argument in [6]. In particular, we always assume that G is a finite group and P is a Sylow p-subgroup of G. Most of our notations are standard and taken from [1] and [2], and we adopt notations from [7] about cohomology functor and G-functors. In addition, we will use the following:

* A module V is said to be an $F_p[G]$-module if V is a G-module and an elementary Abelian p-group as group.

* For a subset $\mathscr{A}$ and a subgroup H of G, set $\mathscr{A}(H) = \mathscr{A} \cap H$.

* If T is a finite group acting on a solvable finite group S, then we denote by $ir_T(S)$ the direct product of all the composition factors of S under T.

We sometimes use the following lemma.

LEMMA 2.1. *Let H be a finite p-group acting on an $F_p[G]$-module A. If an element a acts on the semiproduct HA and satisfies the following*: $[H, a; 2] = 1$ *and* $[A, a; t] = 1$, *then we have* $[H^n(H, A), a; (n+1)t] = 1$ *for all integers* $n \geqq 0$.

Proof. Since $[H, a]$ is a normal subgroup of $H\langle a \rangle$, the assertion follows from the Lyndon-Hochschild-Serre spectral sequence.

3. Cohomological G-functors. In many parts of this paper, we will use cohomological G-functors, which are generalizations of cohomology group. The concept of G-functors were introduced by Green [3] during the study of modular representation theory and slightly changed by Yoshida [7]. We will adopt the definition from [7].

DEFINITION 3.1. A G-functor (into an Abelian category $\mathscr{C}$) is a quadruple $A = (a, \tau, \rho, \sigma)$, where a, τ, ρ, σ are families of the following kind;

$a = (a(H))$ assigns an object $a(H)$ of $\mathscr{C}$ for each subgroup H of G;

$\tau = (\tau_H^K)$ assigns a morphism $\tau_H^K = \tau^K: a(H) \to a(K)$ for each pair (H, K) such that $H \leqq K \leqq G$, we simply write $(\alpha)\tau^K = \alpha^K$;

$\rho = (\rho_H^K)$ assigns a morphism $\rho_H^K = \rho_H: a(K) \to a(H)$ for each pair (H, K) such that $H \leqq K \leqq G$, we simply write $(\alpha)\rho_H = \alpha_H$;

$\sigma = (\sigma_H^g)$ assigns a morphism $\sigma_H^g = \sigma^g: a(H) \to a(H^g)$ for each subgroup H of G and each element g of G, we write $(\alpha)\sigma^g = \alpha^g$. These families of objects and morphisms must satisfy the following:

Axioms for G-functors. (In these axioms, D, H, K, L are any subgroups of G; g, g' are elements of G.)

(G.1) $\tau_H^H = 1_{a(H)}$, $\tau_H^K \cdot \tau_K^L = \tau_H^L$ if $H \leqq K \leqq L$;

(G.2) $\rho_H^H = 1_{a(H)}$, $\rho_H^K \cdot \rho_D^H = \rho_D^K$ if $K \geqq H \geqq D$;

(G.3) $\sigma_H^g \cdot \sigma^{g'} = \sigma_H^{gg'}$, $\sigma_H^h = 1_{a(H)}$ if $h \in H$;

(G.4) $\tau_H^K \cdot \sigma^g = \sigma_H^g \cdot \tau^{K^g}$, $\rho_H^K \cdot \sigma^g = \sigma_K^g \cdot \rho_{H^g}$;

(G.5) (ackey axiom) If H and K are subgroup of L, then $\tau_H^L \cdot \rho_K^L = \sum_g \{\sigma_H^g \cdot \rho_{H^g \cap K} \cdot \tau^K : g \in H\backslash L/K$ a double coset represention$\}$.

DEFINITION 3.2. A G-functor $A = (a, \tau, \rho, \sigma)$ is called cohomological if it satisfies the following Axiom C:

(C) Whenever $H \leqq K \leqq G$, $\rho_H^K \cdot \tau_H^K = |K: H| 1_{a(K)}$.

For example, let V be a G-module and n be an integer ($\geqq 0$). For subgroups H, K of G with $H \leqq K$, set $a(H) = H^n(H, V)$, then the restrictiom map $\rho_H^K: H^n(K, V) \to H^n(H, V)$, the transfer map $\tau_H^K: H^n(H, V) H^n(K, V)$, and the conjugation map $\sigma^g: H^n(H, V) \to H^n(H^g, V)$ makes a cohomological G-functor, we call this a cohomology functor.

Although we explained the definition of G-functors, we will really use only G-functors which are induced from cohomology functors. Therefore, readers may regard all G-functor in this paper as cohomology functors. Next, we will show a construction of G-functor induced from the given G-functor. All G-functors considered in this paper will be cohomological.

3.3. A quotient G-functor. Let $A = (a, \tau, \rho, \sigma)$ be a G-functor over a field k. A G-functor $B = (a', \tau', \rho', \sigma')$ is called a sub-G-funtor of A if B satisfies the following properties:

(i) $a'(H) \subseteqq a(H)$ for all $H \leqq G$; and

(ii) $\tau' = \tau_{|a'(H)}$, $\rho' = \rho_{|a'(H)}$, and $\sigma' = \sigma_{|a'(H)}$.

We write $A \geqq B$. Then we can make a new G-functor called the quotient (or section) G-functor $A/B = (a_0, \tau_0, \rho, \sigma_0)$ as follows: Set $a_0(H) = a(H)/a'(H)$ for $H \leqq G$. Since the above inclusion are commutative with τ, ρ, and σ, we can define morphisms τ_0, ρ_0, σ_0, naturally.

In connection with the above notions, we define the following:

DEFINITION 3.4. A G-functor A is said to be irreducible if A has no nontrivial proper sub-G-functors.

DEFINITION 3.5. A chain of sub-G-functors $A = A_0 \geqq A_1 \geqq \cdots \geqq A_n = 0$ is called a composition series if each A_i/A_{i+1} is irreducible. Then the factors A_i/A_{i+1} are called its composition factors.

Then we have a proposition of Jordan-Hölder type.

PROPOSITION 3.6. *Any two composition series of a G-functor have the same length and, with respect to a suitable reodering of the composition factors, the corresponding factors are isomorphic.*

The proof is similar to that of Jordan-Hölder theorem. Thus the composition factors of a G-functor A are completely determined up to isomorphism (and ordering) by any one composition series. Especially, we have the following:

LEMMA 3.7. *The nth cohomology functor $H^n(\ ,V)$ of G with coefficient in a finite G-module V has a composition series.*

Next lemma about cohomological G-functors will be useful.

LEMMA 3.8. (*Factorization lemma*). *Assume that G has two subgroups N_1 and N_2 containing P such that $G = N_1N_2$. Let $A = (a, \tau, \rho, \sigma)$ be a cohomological G-functor. If an element α in $a(P)$ is stable in both N_1 and N_2, then α is stable in G itself.*

Proof. Suppose false, that is, there is an element g in G such that $(\alpha_F)^g \neq \alpha_{F^g}$ and $F = P^{g^{-1}} \cap P$ by the definition. Since g is an element of $G = N_1N_2$, there are elements g_1, g_2 in N_1, N_2, respectively, such that $g = g_1g_2$. We here assert that we can choose g_1 and g_2 such that $F^{g_1} \subseteqq P$. To see this, take an arbitrary representation $g = g_1g_2$. Then we get that $F^{g_1} \subseteqq N_1$ and $F^{g_1} = F^{gg_2^{-1}} \subseteqq N_2$ since $F, F^g \subseteqq P \subseteqq N_1 \cap N_2$. Combining these, F^{g_1} is contained in a Sylow p-subgroup P_0 of $N_1 \cap N_2$, which is conjugate to P in $N_1 \cap N_2$. Therefore, there is an element k in $N_1 \cap N_2$ such that $F^{g_1k} \subseteqq P$. Then we have a desired representation $g = (g_1k)(k^{-1}g_2)$. Since α is stable in both N_1 and N_2, we observe that $(\alpha_F)^{g_1} = (\alpha_{P^{g_1} \cap P} - 1)^{g_1}_{F^{g_1}} = (\alpha_{P \cap P^{g_1}})_{F^{g_1}} = \alpha_{F^{g_1}}$ and $(\alpha_F)^g = ((\alpha_F)^{g_1})^{g_2} = (\alpha_{F^{g_1}})^{g_2} = (\alpha^{-1}_{P^g \cap P})^{g_2}_{F^g} = \alpha_{F^g}$. This contradicts the choice of g.

In association with composition factors, we will use the following:

DEFINITION 3.9. Let $A = (a, \tau, \rho, \sigma)$ be a G-functor and $\{B_i : i \in I\}$ be a set of G-functors. We shall say that the G-functor A is covered by the set $\{B_i : i \in I\}$ or the set $\{B_i : i \in I\}$ is a covering of A if each composition factor of A is isomorphic to a composition factor of one of B_i.

For making research on a stability, we define the following map.

DEFINITION 3.10. Let $A = (a, \tau, \rho, \sigma)$ be a cohomological G-functor over F_p. We will define a map q_A: Image$(\rho_P^{N_G(P)} : a(N_G(P)) \to a(P)) \to a(P)$ by $q_A(\alpha) = \alpha - \beta_P^G$ where $\beta \in a(N_G(P))$ with $(\beta)_P = \alpha$.

LEMMA 3.11. (*Properties of q_A.*) *We have the following*:
(a) *If α is stable in G, then $q_A(\alpha) = 0$.*
(b) *If $\alpha^G = 0$, then $q_A(\alpha) = \alpha$.*
(c) $q_A(\alpha) \in \mathrm{Ker}(\tau_P^G) \cap \mathrm{Image}(\rho_P^{N_G(P)})$.
(d) $q_A \cdot q_A = q_A$.

(e) *If* $\alpha \in \mathrm{Image}(\rho_P^L)$ *for* $L \geqq N_G(P)$, *then* $q_A(\alpha) \in \mathrm{Image}(\rho_P^L)$.

Proof. All results are immediate consequences of the definition of the map q_A and Lemma 4.4 in [7].

Using the above map q_A, we get a few lemmas about covering.

LEMMA 3.12. *Let* $\{B_i : i \in I\}$ *be a covering of a cohomological G-functor* $A = (a, \tau, \rho, \sigma)$. *Suppose that a subgroup N of G containing* $N_G(P)$ *controls every composition factor of* B_i *for each* $i \in I$. *Then N controls A itself.*

Here, the statement that N controls A means that if an element α of $a(P)$ is stable in N then α is stable in G.

Proof. Let $A = A_0 \gneqq A_1 \gneqq \cdots \gneqq A_n = 0$ be a composition series of A. Suppose that N does not control A, then there is an element $\alpha \neq 0$ in $a(P) \cap \mathrm{Image}(\rho_P^N)\,\mathrm{Ker}(\tau_P^G)$ by Lemma 4.4 in [7]. By the choice of α and (b) of Lemma 3.11, we have $q_A(\alpha) = \alpha$. Let $A_i = (a_i, \tau_i, \rho_i, \sigma_i)$ be the quadruples of the G-functors A_i. Since $q_A((a_i(N_G(P)))\rho_P^{N_G(P)})$ is contained in $a_i(P)$, q_A defines the map $q_{\bar{A}_i}$ for each section G-functor $\bar{A}_i = A_i/A_{i+1}$ which has the same properties as q_A. Since every composition factor A_i/A_{i+1} of A is isomorphic to a composition factor of one of B_j and so is controlled by N, we have $q_{\bar{A}_0}(\alpha + a_1(P)/a_1(P)) = 0$. We thus get $\alpha \in a_1(P)$. By iteration, we finally obtain $\alpha = 0$, a contradiction.

4. Main result. In this section, we will get the result which will be useful in the next section, where we will prove theorems. Namely, we will investigate properties of the minimal counterexample on the assumption that Theorem B is false. We will divide this section into three parts. In Part 1 and 3, we will assume Hypothesis I and treat Proposition A. In Part 2, we will assume Hypothesis II and prepare results which will be used in the last part.

Part 1. At first, we will consider the following: Hypothesis I. Assume that:

(a) G a group, P a Sylow p-subgroup of G, n an integer $\geqq 1$;

(b) W a section conjugacy functor of G;

(c) W controls all composition factors of the G^*-functor $H^m(, V^*)$ in a trivial $F_p[G^*]$-module V^* in every section G^* of G for all integers m with $0 \leqq m \leqq n$;

(d) V an $F_p[G]$-module;

(e) r_n the first integer such that $p \leqq 4 \times 6^{n-1} \times (r+1) + 2$; and

(f) $\mathscr{A} = \{a \in P: a^p = 1, [ir_G(V), a; r_n + 1] = 1$, and a satisfies $\mathrm{Con}(r_n)$ in $G\}$ and set $B = \langle \mathscr{A} \rangle$.
Here we explain the notation.

DEFINITION 4.1. We shall say that an element a of G satisfies $\mathrm{Con}(s)$ in G for some integer $s (\geqq 0)$ if a satisfies the following property; whenever a normalizes a p-subgroup T of G, $[ir_{N_G(T)}(T), a; 2s + 1] = 1$.

Our final purpose in this section is to get the following:

PROPOSITION A. *Under Hypothesis I, we have that $N_G(B)$ controls all composition factors of the* nth *cohomology functor* $H^n(\ , V)$

Let us begin by listing up some properties of the hypothesis.

LEMMA 4.2. *B is weakly closed in P with respect to G.*

LEMMA 4.3. *Let N be a subgroup of G and K be a normal subgroup of N. If an element a of N satisfies* $\mathrm{Con}(s)$ *in G, then a satisfies* $\mathrm{Con}(s)$ *in N and the image aK/K of a in N/K satisfies* $\mathrm{Con}(s)$ *in N/K.*

Now we start the proof of Proposition A. Suppose that Proposition A is false and let $\mathscr{T} = \{(n, G, V)\}$ be the set of counterexamples. We introduce an order in $\mathscr{T}$ by setting $(n, G, V) \gg (n', G', V')$ if one of the following conditions holds:

(i) $n \gneqq n'$; (ii) $n = n'$, $|G| \gneqq |G'|$; (iii) $n = n'$, $|G| = |G'|$, $|V| \gneqq |V'|$. Let (n, G, V) be a minimal element in $\mathscr{T}$ with respect to the above order. Then we have the following lemmas.

LEMMA 4.4. $O_p(G) = 1$.

Proof. Suppose false and set $H = O_p(G)$. By Lemma 4.2, there is an element a in $\mathscr{A}$-H such that $[ir_G(H), a; 2r_n + 1] = 1$ by the property $\mathrm{Con}(r_n)$. By Proposition 1 in [6], we then obtain

$$[ir_G(H^i(H, V)), a; (2i + 1)r_n + 1] = 1$$

for every integer $i \geqq 0$. Especially, since $[ir_G(H^n(H, V)), a; p - 1] =$ by the choice of r_n, it follows that $X^G = X^{N_G(B)}$ $(X^G = \{x \in X: gx = x$ for all $g \in G\})$ for every composition factor X of $H^n(H, V)$ under G by Theorem A1.4 in [1], which implies that $N_G(B)$ controls all composition factors of the G-functor $H^0(\ , H^n(H, V))$. Since the G-functor $H^n(\ , V)$ is covered by the set of G-functors $H^i(\ /H, H^{n-i}(H, V))$: $i = 0, \cdots, n$ by Hochschild-Serre spectral sequence, there is an integer

$i \geqq 1$ such that $N_G(B)$ does not control a composition factor of the G-functor $H^i(\ /H, V^i)$ where V^i is a composition factor of $H^{n-i}(H, V)$ under G. Summarizing the above argument, we have got the following situation:

(a) G/H acts on an $F_p[G/H]$-module V^i;

(b) r_i is the first integer such that $p \leqq 4 \times 6^{i-1} \times (r_i + 1) + 2$; and

(c) $\mathscr{A}^i = \{\bar{a} \in P/H: \bar{a}^p = 1, [ir_{G/H}(V^i), \bar{a}; r_i + 1] = 1$, and $\bar{a}$ satisfies $\mathrm{Con}(r_i)$ in $G/H\}$ contains the image of $\mathscr{A}$ in G/H. By the minimality of G, n, $N_{G/H}(\langle \mathscr{A}^i \rangle)$ controls all composition factors of the G/H-functor $H^i(\ /H, V^i)$, which contradicts Lemma 4.2.

LEMMA 4.5. *V is an irreducible G-module.*

Proof. Suppose false and let $\{V_i: i \in I\}$ be the set of composition factors of V under G. For each i, set

$$\mathscr{A}^i = \{a \in P: a^p = 1, [V_i, a; r_n + 1] = 1 ,$$

and a satisfies $\mathrm{Con}(r_n)$ in $G\}$. We then get that $N_G(\langle \mathscr{A}^i \rangle)$ controls all composition factors of the G-functor $H^n(\ , V^i)$ for each i and $\mathscr{A}^i \supseteqq \mathscr{A}$, which is a contradiction.

LEMMA 4.6. $C_G(V) \subseteqq O_{p'}(G)$.

Proof. Suppose false and set $H = C_G(V)$ and S is a nontrivial Sylow p-subgorup of H. The Frattini argument yields that $G = N_G(S)H$. It thus follows from Lemma 4.4 that $N_G(S) \lneqq G$ and so $N_{N_G(S)}(B)$ controls all composition factors of the $N_G(S)$-functor $H^n(\ , V)$. Therefore, $N_{PH}(B)$ does not control all composition factors of the PH-functor $H^n(\ , V)$ by Lemma 3.8 (Factorization lemma) and Lemma 3.12. It thus follows from the choice of G that $G = PH$. Since H centralizes V and a p-group PH/H acts on the irreducible $F_p[G]$-module V, $G = H$. By the condition (c) in Hypothesis I, $N_G(W(P))$ controls all composition factors of the G-functor $H^n(, V)$. However, since $G \neq N_G(W(P))$, $N_{N_G(W(P))}(B)$ controls all composition factors of the $N_G(W(P))$-functor $H^n(\ , V)$, a contradiction.

LEMMA 4.7. *Let H be a finite group and X be a faithful H-module (or $C_H(X) \subseteqq O_{p'}(H)$). If an element a of H satisfies $[X, a; s + 1] = 1$ for an integer $s \geqq 1$, then a satisfies* $\mathrm{Con}(s)$ *in H.*

Proof. We get the conclusion by the same way as Theorem A2.4 and Lemma A2.3 in [1].

LEMMA 4.8. *$\mathscr{A}$ is equal to the set* $\{a \in P: a^p = 1, [V, a; r_n + 1] = 1\}$.

Because we have supposed that Proposition A is false, $N_G(B)$ does not control a composition factor $A = (\alpha, \tau, \rho, \sigma)$ of the nth cohomology functor $H^n(\ , V)$. According to the definition of control, we have $\mathrm{Image}(\rho_P^{N_G(B)}) \not\geqq \mathrm{Image}(\rho_P^G)$. By Lemma 4.4 in [7], there exists a nontrivial element α in $\mathrm{Image}(\rho_P^{N_G(B)}) \cap \mathrm{Ker}(\tau_P^G)$. The next lemma follows from the choice of α and Lemma 3.11.

LEMMA 4.9. $q_A(\alpha) = \alpha$.

4.10. Since $q_A(\alpha) = \alpha \neq 0$, we get $((\alpha_{P \cap P^g} - 1)^g)\tau^P \neq 0$ for some element $g \notin N_G(P)$ by the definition of q_A and Mackey axiom. Especially, there is a subgroup H_0 in P such that $H_0^g \subseteqq P$ and $((\alpha_{H_0})^g)\tau^P \neq 0$. From now on, let (α, g, H_0) be such a triple set.

LEMMA 4.11. $B \not\subseteqq H_0^g$.

Proof. Suppose false. By Lemma 4.2, $g \in N_G(B)$. In this case, it can be seen that $(\alpha_{H_0})^g = \alpha_{H_0^g}$ and $(\alpha_{H_0^g})\tau^p = 0$, a contradiction.

LEMMA 4.12. *B is a non-Abelian subgroup.*

Proof. Suppose false. By Lemma 4.11, we can choose an element a in $\mathscr{A}$-H_0^g. Take a subgroup K of P for which K contains H_0^g, a normalizes K, and K does not contain a. Since B is Abelian, a stabilizes $K \geqq K \cap B \geqq 1$. Furthermore, it follows from the choice of a that $[V, a; r_n + 1] = 1$. Combining these, we obtain $[H^n(K, V), a; (n + 1)(r_n + 1)] = 1$ by Lemma 2.1. Since $(n + 1)(r_n + 1) \leqq p - 1$, we get $((a(K))\tau_K^{K\langle a\rangle})_K = 0$ by the same way as Lemma A1.8 in [1]. Furthermore, since $a^p = 1$, we obtain $a(K)\tau_K^{K\langle a\rangle} = 0$, which contradicts $(a(H_0^g))\tau^P \ni ((\alpha_{H_0})^g)\tau^p \neq 0$.

Since B is not Abelian, B has a nontrivial subgroup $B_1 = [B, Z_2(B)]$.

LEMMA 4.13. $[V, a^{-1}a^k; 2r_n + 1] = 1$ for all $a \in \mathscr{A}$ and $k \in Z_2(B)$.

Proof. It follows from the definition of $\mathscr{A}$ that a^{-1}, a^k are elements of $\mathscr{A}$. Since they are commutative together, we get $(a^{-1}a^k - 1)^{2r_n+1} \cdot V \subseteqq \sum_{i+j=2r_n+1}(a^{-1} - 1)^i(a^k - 1)^j \cdot V$. Since one of i, j exceeds $r_n + 1$ in $i + j = 2r_n + 1$, we have the desired conclusion.

We now interest in the new set.

DEFINITION 4.14. Set $\mathscr{A}_1 = \{a \in P: a^p = 1, [V, a; 2r_n + 1] = 1\}$.

Then the above lemma means $\langle \mathscr{A}_1(B_1) \rangle = B_1$. Clearly, $\mathscr{A}_1$ contains $\mathscr{A}$. Thus $N_G(\langle \mathscr{A}_1 \rangle)$ does not control all composition factors of the G-functor $H^n(\ , V)$.

LEMMA 4.15. $B_1 \subseteqq H_0^g$.

Proof. Suppose false and choose a in $\mathscr{A}_1(B_1) - H_0^g$. Taking a subgroup K of P as well as Lemma 4.12, we also get a contradiction since $(n+1)(2r_n+1) \leqq p-1$ and $a^p = 1$.

In order to continue the proof, we need a few results. The remaining proofs will be completed in the last part.

Part 2. In this part, we will assume Hypothesis II but not Hypothesis I.

Hypothesis II. Assume that:

(a) L a group, S a Sylow p-subgroup of L, n as defined in Part 1;

(b) the condition (b) and (c) of Hypothesis I hold in L;

(c) X a faithful L-module (or $C_L(X) \subseteqq O_{p'}(L)$); and

(d) $\mathscr{A}^* = \{a \in S: a^p = 1, [X, a; 2r_n+1] = 1\}$ and $B^* = \langle \mathscr{A}^* \rangle$.

Furthermore, we assume the following: $G_0 = L_1/L_2$ is a section of L (that is, $L \geqq L_1 \trianglerighteq L_2$) and P_0 is a Sylow p-subgroup of G_0, and each i, $S_i = S \cap L_i$ is a Sylow p-subgroup of L_i so that $P_0 = S_1L_2/L_2$. Moreover, we denote $\mathscr{A}^*(P_0) = \{aL_2/L_2: a \in \mathscr{A}^*(S_1)\}$ and $B_0 = \langle \mathscr{A}^*(P_0) \rangle$. Then it is clear that B_0 is weakly closed in P_0 with respect to G_0.

Under the above hypotheses, the following lemmas hold.

LEMMA 4.16. *Let V_0 be an $F_p[G_0]$-module and $[V_0, a; r_n+1] = 1$ for all $a \in \mathscr{A}^*(P_0)$. Then $N_{G_0}(B_0)$ controls all composition factors of the G_0-functor $H^m(\ , V_0)$ for all m with $0 \leqq m \lneqq n$.*

Proof. Since $C_L(X) \subseteqq O_{p'}(L)$, it follows from Lemma 4.7 that all elements of $\mathscr{A}^*$ satisfy $\mathrm{Con}(2r_n)$ in L and so all elements of $\mathscr{A}^*(P_0)$ satisfy $\mathrm{Con}(2r_n)$ in G_0. Since $n \ngeqq m$, we can see $r_m \geqq 2r_n$ and we thus have that all elements of $\mathscr{A}^*(P_0)$ satisfy $\mathrm{Con}(r_m)$ in G_0. Then the minimality choice of n yields the desired assertion.

LEMMA 4.17. *Let V_0 be a trivial $F_p[G_0]$-module. Then $N_{G_0}(B_0)$ controls all composition factors of the G_0-functor $H^n(\ , V_0)$.*

Proof. Suppose false and let G_0 be a minimal counterexample section of L. By the condition (b) of Hypothesis II, $N_{G_0}(W(P_0))$ controls all composition factors of the G_0-functor $H^n(\ , V_0)$. It thus follows from the choice of G_0 that $G_0 = N_{G_0}(W(P_0))$. Set $H = O_p(G_0)$

and then the G_0-functor $H^n(\ , V_0)$ is covered by the set

$$\{H^i(\ /H, H^{n-i}(H, V_0)): i = 0, \cdots, n\}$$

by Hochschild-Serre spectral sequence. Choose an element a in $\mathscr{A}^*(P_0) - H$. An application of Proposition 1 in [6] yields that $[ir_{G_0}(H^{n-i}(H, V_0)), a; 4r_n(n-i)+1]=1$. Furthermore, since $4r_n(n-i) \leqq r_i$ for $i \lneqq n$, it follows from Lemma 4.16 that $N_{G_0}(B_0)$ controls all composition factors of the G_0-functor $H^i(\ /H, H^{n-i}(H, V_0))$ for each $i \lneqq n$. While, for $i = n$, the minimality of G_0 yields that $N_{G_0/H}(B_0H/H)$ controls all composition factors of the G_0/H-functor $H^n(\ /H, V_0)$. Summarizing, we get a contradiction.

REMARK. It should be note that we can get the same conclusion on the assumption that G_0 stabilized V_0.

Next, we assume that G_0 is a subgroup of L. Then G_0 acts on X. From now on, let $A = (a, \tau, \rho, \sigma)$ be a section of the L-functor $H^n(, X)$ and $A = A_1/A_2$ for sub-L-functors $A_i = (a_i, \tau_i, \rho_i, \sigma_i)$ of $H^n(\ , X)$ $i = 1, 2$. For each element $\alpha \in a(P_0)$, we set α^* to be an element of $a_1(P_0)$ such that $\alpha = (\alpha^* + a_2(P)/a_2(P))$. Moreover, let $\mathrm{Inf}\cdot H^n(P_0/P_1)$ denote the image of the inflation map $\mathrm{Inf}\colon H^n(P_0/P_1, X^{P_1}) \to H^n(P_0, X)$ for p-subgroups $P_0 \geqq P_1$. And $\mathrm{Inf}\cdot a(P_0/P_1)$ denotes the image of $\mathrm{Inf}\cdot H^n(P_0/P_1) \cap a_1(P_0)$ in $a(P_0)$.

LEMMA 4.18. *Assume that $H_0 = O_p(G_0) \neq 1$ and an element α of $a(P_0)$ is stable in $N_{G_0}(B_0)$. Then we have an representation $\alpha = \alpha_1 + \alpha_2$ such that $\alpha_1 \in \mathrm{Int}\cdot a(P_0/C_{P_0}(X^{\langle A^*(H_0)\rangle}))$ and α_2 is stable in G_0.*

Proof. We first note that the set of G_0-functors $\{H^i(\ /H_0, H^{n-i}(H_0, X)): i = 0, \cdots, n\}$ is a covering of the G_0-functor $H^n(\ , X)$. Suppose that Lemma 4.18 is false, especially, α is not stable in G_0, which implies that $N_{G_0}(B_0)$ does not control the section $A(G_0)$ (the restriction of A on G_0). Since all elements a of $\mathscr{A}^*(P_0)$ satisfy $[ir_{G_0}H^{n-j}(H_0, X)), a; 4r_n(n-j)+2r_n+1] = 1$ and $4r_n(n-j)+2r_n \leqq r_j$ for each $j \lneqq n$, it follows from Lemma 4.16 that $N_{G_0}(B_0)$ controls all composition factors of the G_0-functor $H^j(\ /H_0, H^{n-j}(H_0, X))$ for all $j \lneqq n$. We thus have that every composition factor of the G_0-functor $H^n(\ , X)$ which is not controlled by $N_{G_0}(B_0)$ is that of $H^n(\ /H_0, X^H0)$. Therefore, by Lemmas 3.11 and 3.12, we get $q_{A_1}(\alpha^*) \in \mathrm{Inf}\cdot H^n(P_0/H_0)+a_2(P_0)$ for $\alpha^* \in a_1(P_0)$ with $\alpha = \alpha^* + a_2(P_0)/a_2(P_0)$. Since $\alpha^* - q_{A_1}(\alpha^*)$ is stable in G_0, it is sufficient to treat $q_A(\alpha) = q_{A_1}(\alpha^*) + a_2(P_0)/a_2(P_0)$ and so we can reset $\alpha = q_A(\alpha)$ and $\alpha^* = q_{A_1}(\alpha^*)$ for the convenience of notations. If $C_{P_0}(X^{H_0}) \subseteqq H_0$, then we have already obtained the desired assertion. Set $C = C_{G_0}(X^{H_0})$ and $P_1 = P_0 \cap C(\ngeqq H_0)$. Then the Frattini argument yields $G_0 = N_{G_0}(P_1)C$.

Since P_0C stabilizes X^{H_0}, $N_{P_0C}(B_0)$ controls all composition factors of the P_0C-functor $H^n(\ /H_0, X^{H_0})$ by Lemma 4.17. Especially, α is stable in P_0C. It thus follows from Lemma 3.8 that α is not stable in $N_{G_0}(P_1)=N$. Then the N-functor $H^n(\ /H_0, X^{H_0})$ is covered by the set of N-functors $\{H^i(\ /P_1, H^{n-i}(P_1/H_0, X^{H_0}))\}$ by Lemma 4.16. Therefore, we get $q_{A_1(N)}(\alpha^*)\in \mathrm{Inf}\cdot H^n(P_0/P_1, X^{P_1})+a_2(P_0)$. Since $C\cap P_0=P_1$, it follows from the structure of G_0 that every element of $\mathrm{Inf}\cdot H^n(P_0/P_1)$ is stable in P_0C. Summarizing the above statements, we have that $\alpha - q_{A(N)}(\alpha)$ is stable in both P_0C and N, and so in G_0. We finally have a desired representation $\alpha = q_{A(N)}(\alpha)+\alpha - q_{A(N)}(\alpha)$ such that $q_{A(N)}(\alpha)\in \mathrm{Inf}\cdot a(P_0/P_1)$ and $\alpha - q_{A(N)}(\alpha)$ is stable in G_0, where $A(N)$ is the restriction of A on N, a contradiction.

Next, we consider the set $\mathscr{A}^*=\{a\in S\colon a^p=1, [X, a; 2r_n+1]=1\}$. In relation to this set, we define the following group.

DEFINITION 4.19. A subgroup H of L is called to be a C-group of depth 1 if H is generated by elements of $\mathscr{A}^*(H)(=H\cap\mathscr{A}^*)$.

LEMMA 4.20. *Let H be a C-group in S of depth 1. Assume that $\alpha\in a(S)$ is stable in $N_L(B^*)$. Then we have a representation $\alpha_{N_S(H)}=\alpha_1+\alpha_2$ such that $\alpha_1\in \mathrm{Inf}\cdot a(N_S(H)/N_S(H)\cap C(X^H))$ and α_2 is stable in $N_L(H)$.*

Proof. Suppose false and let H be a maximal counterexample. Furthermore, choose H such that $|N_S(H)|$ is maximal subject to the maximality of H. We first assert that $N_S(H)$ is a Sylow p-subgroup of $N_L(H)$. To see this, we follow the proof of [5]. Suppose false and let F be a conjugate subgroup of H contained in S such that $N_S(F)$ is a Sylow p-subgroup of $N_L(F)$. Then we have that for some element f of L, $N_S(H)^f\subsetneqq N_S(F)$ and $H^f=F$. By Alperin's theorem, there is a set of pair $\{(K_i, g_i)\colon g_i\in N_L(K_i), K_i\leqq S; i=1,\cdots,m\}$ such that it satisfies the following:

$$N_S(H)\subsetneqq K_1,\ \cdots,\ N_S(H)^{g_1\cdots g_{i-1}}\subsetneqq K_i,\ \cdots,\ N_S(H)^{g_1\cdots g_{m-1}}\subsetneqq K_m\,,\quad \text{and}\quad g_1\cdots g_m=f\,.$$

Since all conjugate subgroups of H in S are C-groups of depth 1, in order to get a contradiction, it suffices to treat only one step. We therefore assume that H^{g_1} satisfies the assertion of Lemma 4.20. It will be convenient to reset $K=K_1$, $g=g_1$. Set $K^*=\langle\mathscr{A}^*(K)\rangle$, then $K^*\supsetneqq\langle H, H^g\rangle$. By the maximality of H and the choice of H^g, we have two representations:

(i) $\alpha_{N_S(K^*)}=\beta_1+\beta_2$ such that $\beta_1\in \mathrm{Inf}\cdot a(N_S(K^*)/C_{N_S(K^*)}(X^{K^*}))$ and β_2 is stable in $N_L(K^*)$;

(ii) $\alpha_{N_S(H^g)} = \gamma_1 + \gamma_2$ such that $\gamma_1 \in \mathrm{Inf}\cdot a(N_S(H^g)/C_{N_S(H^g)}(X^{H^g}))$ and γ_2 is stable in $N_L(H^g)$. Since $N_L(K) \subseteqq N_L(K^*)$, combining the two represention and resetting $R = N_S(H)$, we get $((\beta_2)_R)^g = ((\beta_2)_K)^g R^g = ((\beta_2)_{N_S(H^g)})R^g = (\gamma_2+\gamma_1-(\beta_1)_{N_S(H^g)})_{R^g} = (\gamma_2 + (\gamma_1 - (\beta_1)_{N_S(H^g)}))_{R^g}$. We thus have $(\beta_2)_R = (\gamma_2)_R^{g^{-1}} + (\gamma_1 - (\beta_1)_{N_S(H^g)})_R^{g^{-1}}$ such that $(\gamma_1 - (\beta_1)_{N_S(H^g)})_R^{g^{-1}}$ is contained in $\mathrm{Inf}\cdot a(R/C_R(X^H))$ and $(\gamma_2)_R^{g^{-1}}$ is stable in $N_L(H)$, since $(\gamma_2)_{R^g}$ is stable in $N_L(H^g)$ and $\gamma_1 - (\beta_1)_{N_S(H^g)}$ is an element of $\mathrm{Inf}\cdot a(N_S(H^g)/C_{N_S(H^g)}(X^{H^g}))$ by the choice of H^g. Finally we obtain $\alpha_R = (\beta_1 + \beta_2)_R = (\beta_1)_R + ((\gamma_1 - (\beta_1)_{N_S(H^g)})^{g^{-1}})_R + ((\gamma_2)^{g^{-1}})_R$ such that $(\beta_1)_R + ((\gamma_1 - (\beta_1)_{N_S(H^g)})^{g^{-1}})_R \in \mathrm{Int}\cdot a(R/C_R(X^H))$ and $(\gamma_2)^{g^{-1}}$ is stable in $N_L(H)$, as desired, which is a contradiction. So we have proved that $N_S(H)$ is a Sylow p-subgroup of $N_L(H)$. Since $H \ngeqq B^*$, we easily check that $\langle \mathscr{A}^*(N_S(H))\rangle \gneqq H$. Set $H_1 = \langle \mathscr{A}^*(N_S(H))\rangle$, then H_1 is a C-group of depth 1 containing H properly. It follows from the maximality of H that we have a representation $\alpha_{N_S(H_1)} = \delta_1 + \delta_2$ such that $\delta_1 \in \mathrm{Int}\cdot a(N_{S(H_1)}/C_{N_S(H_1)}(X^{H_1}))$ and δ_2 is stable in $N_L(H_1)$. For δ_1 we get $(\delta_1)_R \in \mathrm{Int}\cdot a(R/C_R(X^H))$ where $R = N_S(H)$, while for δ_2 we have $(\delta_2)_R = \xi_1 + \xi_2$ such that $\xi_1 \in \mathrm{Inf}\cdot a(R/C_R(X^H))$ and ξ_2 is stable in $N_L(H)$ by Lemma 4.18. We therefore obtain a desired representation $\alpha_R = ((\delta_1)_R + \xi_1) + \xi_2$, a contradiction.

We next define more generalized C-groups.

DEFINITION 4.21. A subgroup H of S is called to be a C-group (or depth t) if H has a series $H = H_t \ngeqq H_{t-1} \ngeqq \cdots \ngeqq H_2 \ngeqq H_1$ such that $H_1 = \langle \mathscr{A}^*(H)\rangle$ and $H_{i+1} = \langle a \in H: a^p \in H_i, [X^{H_i}, a; 2r+1] = 1\rangle$ for $i = 1, \cdots, t-1$. Then we call H_i to be the C-subgroup of H of depth i.

LEMMA 4.22. *Let H and K be C-groups in S of depth at most t. Then $\langle H, K\rangle$ is also a C-group of depth at most t.*

Proof. This follows from the definition.

In association with C-group of depth t, we have a similar result.

LEMMA 4.23. *Let H be a C-group in S of depth t. Assume that $\alpha \in a(S)$ is stable in $N_L(B^*)$. Then we have a representation $\alpha_{N_S(H)} = \alpha_1 + \alpha_2$ such that $\alpha_1 \in \mathrm{Int}\cdot a(N_S(H)/C_{N_S(H)}(X^H))$ and α_2 is stable in $N_L(H)$.*

Proof. Suppose false and let t be a minimal counterexample. We already got $t \neq 1$ by Lemma 4.20. Furthermore let H be a maximal counterexample subject to the minimality of t. Let F be the C-subgroup of H of depth $t-1$. It follows from the same argument in Lemma 4.20 that we may assume that $N_S(F)$ is a Sylow p-subgroup of $N_L(F)$. We then have a representation $(\alpha)_{N_S(F)} = \beta_1 + \beta_2$ such that $\beta_1 \in \mathrm{Inf}\cdot a(N_S(F)/C_{N_S(F)}(X^F))$ and β_2 is stable in $N_L(F)$ by the minimality of t. Since $N_L(H)$ is contained in $N_L(F)$, $(\beta_2)_{N_S(H)}$ is stable in $N_L(H)$. Thus it is sufficient to treat β_1. Set $K = \langle a \in N_S(F)\colon a^p \in F, [X^F, a; 2r_n + 1] = 1\rangle$. Then K is a C-group of depth at most t. Suppose first $K \gneqq H$, then we have a representation $\alpha_{N_S(K)} = \gamma_1 + \gamma_2$ such that $\gamma_1 \in \mathrm{Int}\, a(N_S(K)/C_{N_S(K)}(X^K))$ and γ_2 is stable in $N_L(K)$ by the maximality of H. By combining the two representations, we can see $\alpha_{N_S(F)} = \beta_1 + \beta_2 = (\gamma_1)_{N_S(F)} + (\gamma_2)_{N_S(F)}$. Thus $\xi = (\gamma_2)_{N_S(F)} - \beta_2 = \beta_1 - (\gamma_1)_{N_S(F)}$ is stable in $N_L(K) \cap N_L(F)$ and contained in

$$\mathrm{Inf}\cdot a(N_S(F)/C_{N_S(F)}(X^F))\,.$$

Summarizing, we have the following situation:

(i) $\overline{N_L(F)} = N_L(F)/C(X^F) \cap N_L(F)$ acts on X^F faithfully;

(ii) $\bar{K}$ is generated by the set of elements $\bar{a}$ of $\overline{N_S(F)}$ with $[X^F, \bar{a}; 2r_n + 1] = 1$ and $\bar{a}^p = \bar{1}$;

(iii) $\overline{N_S(F)}$ is a Sylow p-subgroup of $\overline{N_L(F)}$; and

(iv) $\bar{\xi}$ is an inverse element of ξ in $a'(\overline{N_S(F)})$ which is stable in $\overline{N(K) \cap N(F)}$, that is, $\mathrm{Inf}(H^n(\overline{N_L(F)}, X^F) \to H^n(N_L(F), X))\colon \bar{\xi} \to \xi$. Here $(^-)$ denotes the image of the natural homomorphism: $N_L(F) \to \overline{N_L(F)}$ and $A' = (a', \tau', \rho', \sigma')$ is the section of $\overline{N(F)}$-functor $H^n(\ , X^F)$ which is isomorphic to the image of the section $N(F)$-functor $\mathrm{Inf}\cdot(H^n(\ /C_{N(F)}(X^F), X^F))$. By taking $\overline{N_L(F)}, \overline{N_S(F)}, \bar{K}$ in place of L, S, B^* of Hypothesis II, respectively, we can see that they satisfy the all conditions of Hypothesis II. In this case, since $\bar{H}$ is a C-group of $\overline{N_S(F)}$ of depth 1 and $\bar{\xi}$ is stable in $N_{\overline{N(F)}}(\bar{K})$, by Lemma 4.22 applied to $\bar{\xi}$ and $\overline{N_L(F)}$ instead of α and L, respectively, we have a representation $\xi_{N_S(H)} = \xi_1 + \xi_2$ such that $\xi_1 \in \mathrm{Inf}\cdot a(N_S(H)/C_{N_S(H)}(X^H))$ and ξ_2 is stable in $N_L(H) \subseteq N_L(F)$. Consequently, we have a representation $\alpha_R = (\gamma_1)_R + (\gamma_2)_R - (\beta_2)_R + (\beta_2)_R = (\gamma_1)_R + \xi_R + (\beta_2)_R = ((\gamma_1)_R + \xi_1) + (\xi_2 + (\beta_2)_R)$ such that $(\gamma_1)_R + \xi_1 \in \mathrm{Inf}\cdot a(R/C_R(X^H))$ and $\xi_2 + (\beta_2)_R$ is stable in $N_L(H)$, as desired, where $R = N_S(H)$, a contradiction. we therefore have that $H = K$. In this case we assert that F is the unique maximal C-group of $N_S(F)$ of depth $t-1$. To see this, let T be the unique maximal C-group of $N_S(F)$ of depth $t-1$ and $T = T_{t-1} \geqq T_{t-2} \geqq \cdots \geqq T_1$ be the chain of C-subgroups of T. Then since T_1 is a C-group of depth 1, $K \supseteq T_1$ by the definition of K. Since $K = H$ and F is the unique maximal C-subgroup of H of depth $t-1$, $F \supseteq T_1$. By iteration, we obtain $T \subseteq K$ and $T \subseteq F$, as desired. The result that F is the unique maximal C-group of $N_S(F)$ of depth $t-1$ implies that $S = N_S(F)$, which means $F \supseteq B^*$ and $N_L(H) \subseteq N_L(F) \subseteq N_L(B^*)$, a contradiction.

Part 3. Now we can return to the proof of Proposition A. In this part, we take the proof of Proposition A up again. This is a continuation of Part 1. We adopt same notations of Part 1, such as G, V, $\mathscr{A}_1$, B and the like.

LEMMA 4.24. *G, V, $\mathscr{A}_1$ satisfy the conditions of Hypothesis II.*

Proof. By taking G, V, $\mathscr{A}_1$ in place of L, X, $\mathscr{A}^*$, respectively, the assertion follows from Lemmas 4.6 and 4.8.

In the comments following Lemma 4.12, we defined the subgroup $B_1 = [B, Z_2(B)]$. We here define the following subgroups B_i.

DEFINITION 4.27. *B_i is the inverse image of $[B/B_{i-1}, Z_2(B/B_{i-1})]$ in B for $i = 1, \cdots$.*

Then clearly, the chain $B_1 \subseteqq B_2 \subseteqq \cdots$ is ascendent and there is an integer k such that $B_{k-1} \neq B_k = B_{k+1}$. It follows from the definition of B that B/B_k, B_i/B_{i-1} are all elementary Abelian. Set $B_{k+1} = B$. Moreover the following is clear.

LEMMA 4.26. *B_i is a C-group of depth at most i for $i \leqq k$.*

Now we recall the triple $(H_0 \leqq P, g \in G, \alpha \in a(P))$ defined in the statement (4.10). Since H_0, $H_0^g \subseteqq P$, there is a set of pair $\{(K_i, g_i)$: $K_i \leqq P$, $g_i \in N(K_i)$; $i = 1, \cdots, m\}$ such that H_0 is conjugate to H_0^g via g by this set, namely, this set satisfies the following:

$$H_0 \subseteqq K_1, \cdots, H_0^{g_1 \cdots g_{i-1}} \subseteqq K_i, \cdots, H_0^{g_1 \cdots g_{m-1}} \subseteqq K_m \quad \text{and} \quad g_1 \cdots g_m = g .$$

Since we choose $\{g, H_0\}$ on the only assumption that $((\alpha_{H_0})^g)^P \neq 0$, we can rechoose g and H_0 such that $((\alpha_{H_0} g_1)^{g_1^{-1} g})^P = 0$. Then we get $((((\alpha_{K_1})^{g_1} - \alpha_{K_1})_{H_0} g_1)_{g_1^{-1} g})^P \neq 0$. Set $\beta = (((\alpha_{K_1})^{g_1} - \alpha_{K_1})_{H_0} g_1)^{g_1^{-1} g}$. As we showed, $\beta^P \neq 0$, especially, $a(H_0^g)\tau^p \neq 0$. In Part 1, we already got the following:

$$B \not\subseteqq H_0^g \quad \text{and} \quad B_1 \subseteqq H_0^g . \tag{4.27}$$

We next show that $B_2 \subseteqq H_0^g$. To see this, we need some arguments. Since $B_1 \subseteqq H_0^g$ and $\langle \mathscr{A}_1(B_1) \rangle = B_1$, we have $\langle \mathscr{A}_1(K_1) \rangle \supseteqq (\mathscr{A}_1(B_1))^{g^{-1}} \neq \{1\}$. Let $L = \langle \mathscr{A}_1(K_1) \rangle$, then L is a normal C-group of K_1 of depth 1. We thus have a representation $\alpha_{N_P(L)} = \alpha_1 + \alpha_2$ such that $\alpha_1 \in \mathrm{Inf} \cdot a(N_P(L)/C_{N_P(L)}(V^L))$ and α_2 is stable in $N_G(L)$ by Lemma 4.22. It should be noted that we can choose the set $\{(K_i, g_i) \colon i \in I\}$ such that $N_P(\langle \mathscr{A}_1(K_i) \rangle)$ is a Sylow p-subgroup of $N_G(\langle \mathscr{A}_1(K_i) \rangle)$ for each $i \in I$, by the same argument in Lemma 4.20. To simplify the notation, we set $f = k_1$ and $F = K_1$. We then have $(\alpha_F)^f - \alpha_F = (((\alpha_1)_F)^f + ((\alpha_2)_F)^f) - ((\alpha_1)_F + (\alpha_2)_F) = ((\alpha_1)_F)^f - (\alpha_1)_F$ and so we get $\beta = ((\alpha_F^f - \alpha_F)_{H_0} f)^{f^{-1}g} = (((\alpha_1)_F^f - (\alpha_1)_F)_{H_0} f)^{f^{-1}g}$ is contained in $\mathrm{Inf}\cdot a(H_0^g/\langle \mathscr{A}_1(H_0^g)\rangle)$. By these arguments, the following holds.

LEMMA 4.28. $B_2 \subseteqq H_0^g$.

Proof. Suppose false and then there is an element a in $B_2 - H_0^g$ such that $[V^{B_1}, a; 2r_n + 1] = 1$ and $a^p \in B_1$ by the construction of B_2. Taking K as Lemma 4.12, we have that a stabilizes $K \geqq K \cap B_2 \geqq B_1$. We thus get $[H^n(K/B_1, V^B1), a; (n + 1)(2r_n + 1)] = 1$ by Lemma 2.1. Since $a^p \in B_1$ and $(n + 1)(2r_n + 1) \leqq p - 1$, we have

$$(\mathrm{Inf}\cdot a(K/B_1))\tau^{K\langle a\rangle} = 0 \ ,$$

which contradicts $\beta \in \mathrm{Inf}\cdot a(H_0^g/B_1)$.

Now we can get a final contradiction. Namely, we will show that $B_i \subseteqq H_0^g$ implies $B_{i+1} \subseteqq H_0^g$, and $B_k \not\subseteqq H_0^g$ which contradict together. The proof is similar to that of Lemma 4.28. We assume that $B_i \subseteqq H_0^g$. Let L be the unique maxiaml C-group in $F = K_1$. Since F contains $B_i^{g^{-1}}$, $L \supseteqq B_i^{g^{-1}}$. Since $N_G(L) \subseteqq N_G(\langle \mathscr{A}_1(F)\rangle)$, we can rechoose the set $\{(K_i, g_i) : i \in I, (K_1 = F, g_1 = f)\}$ such that $N_P(L)$ is also a Sylow p-subgroup of $N_G(L)$ by the same argument in Lemma 4.20. Then by Lemma 4.23, we have a representation $\alpha_{N_P(L)} = \alpha_1 + \alpha_2$ such that $\alpha_1 \in \mathrm{Inf}\cdot a(N_P(L)/C_{N_P(L)}(V^L))$ and α_2 is stable in $N_G(L)$. We thus have $(\alpha_F)^f - \alpha_F = ((\alpha_1)_F)^f - (\alpha_1)_F \in \mathrm{Inf}\cdot a(F/C_F(V^L))$. We therefore obtain $\beta = (((\alpha_1)_F^f - (\alpha_1)_F)_{H_0} f)^{f^{-1}g} \in \mathrm{Inf}\cdot a(H_0^g/B_i)$. By this argument, we have the following lemma which contradicts (4.27).

LEMMA 4.29. *$B_i \subseteqq H_0^g$ implies $B_{i+1} \subseteqq H_0^g$, and $B_k \not\subseteqq H_0^g$.*

Proof. Suppose that $B_i \subseteqq H_0^g$ and $B_{i+1} \not\subseteqq H_0^g$ for $i = i, \cdots, k$. Choose an element a in $B_{i+1} - H_0^g$ such that $[V^{B_i}, a; 2r_n + 1] = 1$ and $a^p \in B_i$. Taking K as Lemma 4.12, we get that a stabilizes $K \supseteqq K \cap B_{i+1} \supseteqq B_i$. We also obtain a contradiction by the same way of Lemma 4.28 since $\beta \in \mathrm{Int}\cdot a(H_0^g/B_i)$ and $(n + 1)(2r_n + 1) \leqq p - 1$.

This completes the proof of Proposition A.

5. **Proofs of theorems.** In this section, we will prove Theorem B and Glauberman's conjecture using Proposition A.

THEOREM B. *Let G be a finite group, P a Sylow p-subgroup of*

G, and W a section conjugacy functor of degree t. Furthermore, let V be a trivial $F_p[G]$-module. If $p \geqq 4 \times 6^{n-2} \times (t-1)+3$ for some integer $n \geqq 2$, then the restriction map induces an isomorphism:

$$H^n(G, V) \cong H^n(N_G(W(P)), V) .$$

Proof. Suppose false and in particular $N_G(W(P))$ does not control all composition factors of G-functor $H^n(G, V)$. Let $\{n, G\}$ be a minimal counterexample so that $N_G(W(P))$ does not controls all composition factors of the G-functor $H^n(G, V)$ in a trivial $F_p[G]$-module V. Then by the minimality of G, we have the following: (a) $O_p(G) \neq 1$; (b) $W(P) \ntrianglelefteq G$; and (c) for all normal subgroups K of P containing $O_p(G)$ properly, W controls all sections of the $N_G(K)$-functor $H^n(\ , V)$ in $N_G(K)$. Let $H = O_p(G)$. We then get that the set of the G-functors $\{H^i(\ /H, H^{n-i}(H, V)) \colon i = 0, \cdots, n\}$ is a covering of the G-functor $H^n(\ , V)$. Thus W does not control a composition factor of the G-functor $H^i(\ /H, X)$ for some i and an irreducible G/H-composition factor X of $H^{n-i}(H, V)$. On the other hand, since $W(P) \ntrianglelefteq G$, there is an element a in $P - H$ such that $[ir_G(H), a; t] = 1$ by the definition of degree. We thus obtain $[X, a; (n-i)(t-1)+1] = 1$ by Proposition 1 in [6]. By the choice of p, we especially have $[ir_G(H^n(H, V)), a; p-1] = 1$. By Theorem A1.4 in [1], there is a normal subgroup β of P containing $\langle H, a\rangle$ such that $N_G(B)$ controls all sections of the G-functor $H^0(\ , H^n(H, V))$. On the other hand, by the minimality of G, $N_{G/H}(W(P/H))$ controls all sections of the G/H-functor $H^n(\ /H, H^0(H, V))$. Thus we have $0 \lneqq i \lneqq n$, that is, we obtain the following situation:

(a) there is an integer $i(\gneqq 0)$ and G/H acts on an $F_p[G]$-module X;

(b) $N_{G^*}(W(P^*))$ controls all composition factors of the G^*-functor $H^j(\ , V^*)$ with coefficient in a trivial $F_p[G^*]$-module V^* for $0 \leqq j \leqq i$, where G^* is a section of G/H and P^* is a Sylow p-subgroup of G^*;

(c) $C_{G/H}(X) \subseteqq O_{p'}(G/H)$ by the minimality of G and the Frattini argument; and

(d) there is a nontrivial element a in P/H such that $[X, a; r_i] = 1$ and so a satisfies $\mathrm{Con}(r_i)$ in G/H since $(n-i)(t-1) \times 4 \times 6^{i-1} \leqq p-3$ by the hypothesis of p, where r_i is the first integer such that $p \leqq 4 \times 6^{i-1} \times (r_i+1)+2$. Since $G/H, i, X$ satisfy the conditions of Hypothesis I in §4 in place of G, n, V, respectively, there is a normal subgroup B of P containing $\langle H, a\rangle$ such that $N_G(B)$ controls all composition factors of the G/H-functor $H^i(/\ H, X)$, a contradiction. This completes the proof of Theorem B.

Especially, since the section conjugacy functor K_∞ in Glauberman [1] has degree 4, we have an affirmative answer to his question.

THEOREM C. *Let m be an integer with $m \geqq 2$. If $p \geqq 12 \times 6^{m-2} + 3$, then we have $H^m(G, F_p) \cong H^m(N_G(K_\infty(P)), F_p)$.*

Taking a new look at Hypothesis I, we notice that the conditions (b) and (c) always hold by Theorem C if $p \geqq 12 \times 6^{n-2} + 3$. We therefore have a new form of Proposition A.

THEOREM A. *Let V be an $F_p[G]$-module, n an integer $\geqq 1$, and r be the first integer such that $p \leqq 4 \times 6^{n-1} \times (r+1) + 2$. Set $\mathscr{A} = \{a \in P: a^p = 1, [ir_G(V), a; r+1] = 1$, and a satisfies $\mathrm{Con}(r)$ in $G\}$. If $r \geqq 1$, the restriction map induces an isomorphism:*

$$H^m(G, V) \cong H^m(N_G(\langle \mathscr{A} \rangle), V) \text{ for all integers } m \text{ with } 1 \leqq m \leqq n .$$

As corollary, we have:

COROLLARY 1. *Let V be a faithful p-primary G-module (or $C_G(V) \subseteqq O_{p'}(G)$) and n, r be as above. Set $B = \langle a \in P: a^p = 1$ and $[V, a; r+1] = 1\rangle$. Then $H^m(G, V) \cong H^m(N_G(B), V)$ for all integers m with $1 \leqq m \leqq n$.*

REMARK. It should be noted that all module in theorems are finite, but it is not necessary to be finite. For all p-primary G-modules, the same assertions hold by the same argument in [6].

REFERENCES

1. G. Glauberman, *Global and Local Properties of Finite Groups, "Finite Simple Group"* edited by M. B. Powell and G. Higman, Academic Press, London and New York, 1971.
2. D. Gorenstein, *Finite Groups*, Harper and Row, New York, 1968.
3. J. A. Green, *Axiomatic representation theory for finite groups*, J. Pure and Applied Algebra, **1** (1971), 41-77.
4. D. F. Holt, *More on the local control of Schur multipliers*, Quar. J. Math., (to appear).
5. M. Miyamoto, *On conjugation families*, Hokkaido Math. J., Vol. **VI**, No. 1, (1977).
6. ———, *A p-local control of Cohomology group*, J. Algebra, (to appear).
7. T. Yoshida, *On G-functors (I)*, Hokkaido Math. J., (1980), (to appear).

DEPARTMENT OF MATHEMATICS
HOKKAIDO UNIVERSITY
SAPPORO 060, JAPAN