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Mu-THETA FUNCTIONS

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Using the technique of compact rational subgroup approximations to unitary representations on a nilmanifold, we justify the evaluation of a distribution at certain rational points of a group. This method allows us to give meaning to a distributional identity between theta-like functions at discrete points in the group. The identity itself arises from the equivalence of certain representations of the group. In attempting to compute an intertwining constant that is present, we are also able to show the existence of distributions that behave like the classical gaussians, i.e., they are eigenfunctions of the Fourier transform.

CONTENTS

I. **Introduction.** Let A be an abelian, nilpotent algebra with a nondegenerate, symmetric biliner form $B(\cdot, \cdot)$ which satisfies $B(xy, w) = B(y, xw)$ for all x, y and w in A. Let Λ be a vector lattice in A so that $\Lambda \cdot \Lambda \subset \Lambda$. Assume that $\Lambda = \Lambda^*$ where $\Lambda^* = \{x \in A \mid B(x, \Lambda) \subset Z\}$. Let $B = A \times A \times R$ be the algebra with the following multiplication (and component-wise addition):

$$(x_1, x_2, r_1)(y_1, y_2, r_2) = (x_1y_1, x_1y_2, B(x_1, y_2))$$

where x_i and y_i are in A and $r_i \in \boldsymbol{R}$, $(i = 1, 2)$. The commutant of this operation is used to generate a Lie bracket in the algebra B by $[x, y] = x \cdot y - y \cdot x$ for all x and y in B. The exponential map,

$$\exp x = \sum_{k=1}^{\infty} 1/k!\, x^k$$

applied to B yields a connected, simply connected nilpotent Lie group with multiplication, "$*$". $x*y = x + y + xy$, x, y in B. The log function (unique inverse of the $\exp(\cdot)$ function) is well defined on $(B, *)$.

Define the *scalar* log *function* on A by

$$\ell(x) = \sum (-1)^{k-1} B(x, x^{k-1})/k$$

for $k = 2$ to $r + 1$, where r is the nilpotent degree of the algebra and x is in A. This function is the third entry of the tuple, $\log(x, \sigma x, 0)$ for $(x, \sigma x, 0) \in (B, *)$ and $\sigma \in \boldsymbol{R}$. In [5], the following distributional identity is established:

$$(1) \quad \sum_{\gamma \in \Lambda} \exp 2\pi i(\sigma \mathscr{L}(\gamma) - B(\gamma, x)) = K(\sigma) \sum_{r \in \Lambda} \exp - 2\pi i \sigma \mathscr{L}(\sigma^{-1}(x + y)) .$$

$K(\sigma)$ is a (intertwining) constant depending only on $\sigma \in \boldsymbol{R}$. While this identity exhibits some promising relationships, it should be kept in mind that it is true only in a distributional sense and consequently is meaningless at distinct values of x in A. Also, unless a formula for $K(\sigma)$ exists or at least a means of computing $K(\sigma)$-independent of this identity, we are unable to attach any meaning to (1). Both of these objections will be addressed in this paper. At least for the latter, a method will be outlined for computation of $K(\sigma)$.

II. **μ-theta distributions.** We continue with group construction of the introductory section to show the derivation of (1). For details and proofs, the reader is referred to [5]. The group inverse in $(B, *)$ is denoted by $\bar{x}$ and is given by

$$\bar{x} = (\overline{x_1, x_2, r}) = \sum_{k=1}^{r} (-1)^k (x_1, x_2, r_1)^k .$$

The notation $\bar{x}$ is used to avoid confusion with multiplicative inverses in the algebra $(B, \cdot)$. Also, the log function on $(B, *)$ is given by

$$\log x = \sum_{k=2}^{r} (-1)^{k-1} (x_1, x_2, r_1)^k$$

for $x \in B$ and it is easy to show that this is the (unique) inverse of the exponential map from the Lie algebra $(B, [\cdot, \cdot])$ to $(B, *)$.

Recall that Λ is a vector lattice of A and define $\Gamma = \Lambda \times \Lambda \times \boldsymbol{R}$. Γ is a co-compact subgroup of B. The subgroup properties are easy to verify since $\Lambda \cdot \Lambda \subset \Lambda$. A character $\mu(\cdot)$ can be defined on Λ by

$$\mu(\gamma_1, \gamma_2, r) = \exp 2\pi i r .$$

The character property is verified by a straightforward computation and uses the fact that $\Lambda = \Lambda^*$. The existence of a character on Γ gives rise to a unipotent representation on $(B, *)$ and we have

PROPOSITION 1.

$$U^{\mu} = \operatorname{ind}(\Gamma, (B, *), \mu) \quad \textit{is irreducible} .$$

Another subgroup arises from $0 \times A \times \boldsymbol{R} \equiv M$. Actually M is normal in $(B, *)$ and $M \cdot M = 0$ in the algebra $(B, \cdot)$. The subgroup properties are immediate. The exp$(\cdot)$ map applied to M is the identity map. We define a character on this subgroup by

$$x(0, x, r) = \exp 2\pi r_1 .$$

Verifying that x is a character is easy since the $*$ operation in M is equivalent to adding the tuples component-wise.

PROPOSITION 2.

$U^{\infty} = \mathrm{ind}(M, (B, *), x)$ *is irreducible and equivalent to* U^{μ} .

$P_0 = A \times 0 \times \boldsymbol{R}$ *is also a subgroup of* $(B, *)$ *and has a character* $x_0(\cdot)$ *defined by*

$$\chi_0(x, 0, r) = \exp 2\pi i r .$$

In this subgroup, the $*$ *multiplication is additive in the last entry (real) of the tuple and hence* χ_0 *is a character.*

PROPOSITION 3.

$U^{0} = \mathrm{ind}(P_0, (B, *), x_0)$ *is irreducible and equivalent to* U^{μ} .

Finally, one more subgroup is to be introduced.

$$P_\sigma = \{(x, \sigma x, r) \mid x \in A,\ r, \varepsilon R,\ \sigma \in R\} .$$

Again verification of the subgroup properties is a matter of computation. P_σ, as a subset of $(B, [\ ,\ ]$ is an abelian subalgebra and the exponential map maps P_σ into itself by virtue of the closure of multiplication in P_σ. Let λ be the linear functional in the dual of the algebra $(B, [\ ,\ ])$ defined by $\lambda(x, y, r) = r$ for (x, y, r) in B. We generate a character χ_σ on $P_\sigma \subset (B, *)$ by

$$\chi(x, \sigma x, r) = \exp 2\pi i \lambda(\log(x, \sigma x, r)) .$$

PROPOSITION 4.

$U^{\sigma} = \mathrm{ind}(P_\sigma, (B, *), x_\sigma)$ *is irreducible and equivalent to* U^{μ} .

Thus far, we have constructed four subgroups, characters and corresponding induced irreducible representation—all of which are equivalent. We digress for a moment to consider these representations in terms of distributions. Let U be a unitary representation of a connected, simply connected nilpotent Lie group, G and let $C^{\infty}(U)$ be the set of vectors v in the representation space of U, $H(U)$, such that $g \to U(g)v$ is a C^{∞}, $H(V)$-valued map. Define $C^{\infty *}(U)$ as the space of continuous conjugate linear functionals on $C^\infty(U)$. Note that by virtue of the mapping $v \to (v, \cdot)$, $H(U) \subset C^{\infty *}(U)$. So for $x \in C^{\infty *}(U)$, $x(v) = (v, x) = \overline{(x, v)} = \bar{x}(v)$. $C^\infty(U)$ may be topologized so as to be a Fréchet space and V is thereby continuous on $C^\infty(U)$. Let U_∞ denote the restriction of U to $C^\infty(U)$ and let $U_{\infty *}$ be the contragredient representation to U_∞.

It is known that there is a unique, regular invariant (Haar) probability measure on the compact group Γ/G. This allows us to define a representation R of G in $L^2(\Gamma/G)$ by right translation. Let f be in $C_c^\infty(U)$ (C^∞ functions with compact support in the representation space of U) and define

$$U(f) = \int_G f(x)\, U(x^{-1})dx\ .$$

It is known that $U(f)$ maps $H(U)$ into $C^\infty(U)$ continuously. Hence there is a conjugate map $U(f)\colon C^{\infty *}(U) \to H^*(U)$, where $H^*(U)$ is the conjugate dual of $H(U)$. Actually, since $H^*(U)$ is isomorphic with $H(U)$, we may take $U(f)^*\colon C^\infty(U) \to H(U)$. Also, $U(x)U(f)^*v = U(f_x)^*v$ where $f_x = f(x^{-1}\cdot)$ and so the range of $U(f)^*$ is, in fact, contained in $C^\infty(U)$. The map $x \to f_x$ is C^∞ from N into $C_c^\infty(N)$ and $f \to U(f)^*v$ is continuous from C_c^∞ into H. Let $f^*(x) = \bar{f}(x^{-1})$ and $U(f)_\infty = U(f^*)^*$. $U(f)_\infty$ extends $U(f)$ when we consider $C^\infty \subset C^{\infty *}$. Note that this defines the action of U on a distribution acting on $C_c^\infty(U)$ by the considerations in [7]. The measure on the lattice Γ will be assumed to be the counting measure.

With these preliminaries established, we make the following definition, A Schwarz Distribution θ_σ on $(B, {}^*)$ is a *μ-theta distribution* if

(i) $\theta_\sigma(\gamma^* x) = \mu(\gamma)\theta_\sigma(x)$

and

(ii) $\theta_\sigma(x^* w) = \theta_\sigma(x)x(w)$ for all x in B, γ in Γ and w in P_σ. The real number σ is called the period of $\theta_\sigma(\cdot)$.

PROPOSITION 5. *For each σ in R, there is an essentially unique (up to a scalar multiple) μ-theta distribution.*

The existence of the μ-theta distributions is a result of the equivalence of representations established earlier. In order to show the implications of this, some consideration must be given to the intertwining operators between the four equivalent and irreducible representations. $C^\infty(U)$ is topologized by the topology of uniform convergence on compacta of the functions $g \to U(g)v$ and their derivatives of higher order. By the use of suitable semi-norms, $C^\infty(U)$

is a Fréchet space and the restriction $U_\infty(g)$ of $U(g)$ to $C^\infty(U)$ is a continuous linear operator on $C^\infty(U)$ by the closed graph theorem. From [10], we have

PROPOSITION 6. *The following are equivalent:*
(1) *U is irreducible*
(2) *U_∞ is irreducible*
(3) *The only continuous, invariant bilinear forms on $C^\infty(U) \times C^\infty(U)$ are multiples of the scalar product in H restricted to $C^\infty(U) \times C^\infty(U)$.*

Recall that $C^{-\infty}(U)$ is the space of conjugate linear functionals on $C^\infty(U)$ and let T be an intertwining operator between two continuous unitary representations U_1 and U_2 in H_1 and H_2 respectively. T has a unique continuous extension to a mapping $T^{-\infty}: C^{-\infty}(U_1) \to C^{-\infty}(U_2)$. A linear functional δ in $C^{-\infty}(U)$ is defined to be a *generalized cyclic vector* for U provided $\delta(U(g)v) = 0$ for all g in G implies that $v = 0$ for v in $C^\infty(U)$. A *generalized cyclic representation* is a pair (U, δ) consisting of a unitary representation U and a generalized cyclic vector, δ. Two such pairs (U, δ) and (V, ε) are *projectively equivalent* if there is a unitary isomorphism T which maps δ onto a multiple of ε and intertwines U and V. The pairs are *equivalent* if that scalar multiple is one. We will show that the identity (distributional) (1) is the projective equivalence of generalized cyclic representations and is equality between two μ-theta distributions. $K(\sigma)$ is the multiple of that projective equivalence.

The pair (U, δ) has a realization in the Hilbert space of distributions on $C_c^\infty(G)$ — C^∞ functions with compact support. If ϕ is in $C_c^\infty(G)$ and f is in $H(U)$, define

$$\tilde{f}(\phi) = \delta(U(\phi)f)$$

where

$$U(\phi) = \int_G \phi(g)\, U(g^{-1})dg$$

for dg (right) invariant Haar measure on G. Poulsen, [10] has shown that if ϕ is a continuous function on G vanishing at infinity, then $U(\phi)$ leaves $C^\infty(U)$ invariant and is continuous in $C^\infty(U)$. If ϕ is in $C_c^\infty(G)$, then $U(\phi)$ maps $H(U)$ continuously into $C^\infty(U)$. Furthermore, the integral defining $U(\phi)$ converges in $C^\infty(U)$ in the sense that if f is any continuous linear functional on $C^\infty(U)$ and if v is in $C^\infty(U)$, then

$$\tilde{f}(U(\phi)v) = \int_G \phi(g) f(U(g)v) dg \; .$$

The map $f \to \tilde{f}$ is injective and defines a realization of U in the space of distributions, $\tilde{H}$. $\tilde{H}$ will be referred to as the canonical space for (U, δ) and the realization of U in this space as the canonical realization. U acts as right translation in this realization, $C^\infty(U)$ is a space of locally integrable functions and δ is evaluation at the identity. In [5], Penney proves that if the map $f \to f(\phi)$ is continuous for every ϕ in $C_c^\infty(G)$ (suitably normed), if right translation defines a unitary representation U of G in H and if $C^\infty(U)$ is a space of locally integrable functions, then (U, δ) is the canonical realization of a unique equivalence class of generalized cyclic representation. We will use the following corollary to those results; namely any representation induced from a character of a closed subgroup of G is a generalized cyclic representation.

A pair (χ, H) consisting of a character χ and a subgroup H is *maximal* if there is a λ in the dual of the Lie algebra of G and a subordinate subalgebra h of maximal dimension such that $H = \exp(h)$ and $\chi(\cdot) = \exp 2\pi i \lambda(\log_G \cdot)$ restricted to H. The pair is integral if $\Gamma \cap H \backslash H$ is compact (or equivalently if ΓH is closed) and if χ restricted to $\Gamma \cap H$ is trivial. Γ, here, as elsewhere is a co-compact subgroup of G. In [1], we find

THEOREM 1. *Let (x, H) be any maximal, integral character. Then the distribution D associated with the irreducible projection onto the primary subspace of $U^x = \mathrm{ind}(H, G, x)$ is given by*

$$\langle D, f \rangle = \sum \left(\int_{\Gamma \cap H \backslash H} \bar{\chi}(h) f(\Gamma h \gamma) dh \right)$$

where the sum is over all γ in $\Gamma \cap H \backslash \Gamma$. f is in $C^\infty(\Gamma \backslash G)$, dh is normalized Haar measure on $\Gamma \cap H \backslash H$. The sum is absolutely convergent.

THEOREM 2. *Let M_1 and M_2 be closed (but not necessarily connected) subgroups of the nilpotent Lie group G. Let x_1 and x_2 be characters on M_1 and M_2 respectively such that $x_1 = x_2$ on $M_1 M_2 \neq \{e\}$. Assume $M_1 M_2$ is closed and $U_1 = \mathrm{ind}(M_1, G_1, x_1)$ and $U_2 = \mathrm{ind}(M_2, G_2, x_2)$ are irreducible representations. If T_1 and T are defined as follows then T_1 and T_2 are bounded intertwining operators between U_1 and U_2 and $T_1^* = T_2$. For f in $C^\infty(U_1)$*

$$T_1 f(g) = \int_{M_1 \cap M_2 \backslash M_2} f(mg) \bar{x}_2(m) dm$$

and for f in $C^\infty(U_2)$

$$T_2 f(g) = \int_{M_1 \cap M_2 \backslash M1} f(mg)\bar{x}_1(m)dm \ .$$

Proof. From the results in [5], $C^\infty(U_1)$ can be identified with the space of Schwarz distributions ϕ on G and in a distributional sense $\phi(m\cdot) = x_1(m)\phi(\cdot)$ for all m in H_1. If we can show that $T_1 f$ exists for all f in $C^\infty(U_1)$ and is locally integrable, then it would follow that $T_1 f$ defines a Schwartz distribution and hence an element of $C^{-\infty}(U_2)$. T_1 is then a continuous from $C^\infty(U_1)$ to $C^{-\infty}(U_2)$. From Proposition 6, T_1 takes values in $C^\infty(U_2)$, T_1 is a multiple of a unitary operator because the form $B(x_1 y) = \langle T_1 x, T_1 y\rangle$ is a U_1 invariant bilinear form and hence $B(x, y) = c\langle x_1, y\rangle$ implying that $T_1 * T_1 = cI$. The adjoint property is clear.

Let $\tilde{M}_1$ be the smallest connected subgroup containing M_1; then $M_1\backslash\tilde{M}_1$ is compact. In [5], it was proved that $C^\infty(U_1)$ is the space of C^∞ functions f which satisfy $f(mx) = x_1(m)f(x)$ for m in M_1 and f is Schwartz transverse to $\tilde{M}_1$. It is not difficult to show that $M_1 \cap M_2\backslash\tilde{M}_1 \cap M_2$ is compact. Therefore we may write

$$\int_{M_1 \cap M_2/M_2} \bar{x}_1(m)f(mx)dm = \int_{\tilde{M}_1 \cap M_2\backslash M_2}\int_{M_1 \cap M_2\backslash\tilde{M}_1 \cap M_2} \bar{x}_1(m_1 m_2)f(m_1 m_2 x)dm_1 dm_2$$

and the integrals exist by compactness and f being Schwartz transverse to $\tilde{M}$. Clearly $T_1 f$ is in fact a C^∞ function and hence locally integrable. □

Establishing this result allows us to compute explicitly the intertwining operators between the four equivalent irreducible representations.

$$\begin{tikzcd} U^\sigma \arrow[r, "T(\sigma, \infty)"] \arrow[d, "T(\sigma, 0)"'] & U^\infty \arrow[d, "T(\infty, \mu)"] \\ U^0 \arrow[r, "T(0, \mu)"'] & U^\mu \end{tikzcd}$$

FIGURE 1

More direct computations show that $P_\sigma * M = B$, $P_0 * M = B$, and $P_\sigma * P_0 = B$. For $f \in H(U^\sigma)$,

$$T(\sigma, 0)f(x) = \int_{P_\sigma \cap P_0 \backslash P_0} f(P * x)x_0(\bar{p})dp = \int_A f((p, 0, 0) * x)dp$$

since $\{(p, 0, 0) \mid p \text{ in } A\}$ is a complete set of inequivalent coset representatives of $P_\sigma \cap P_0$ in P_0. Similarly, for $f \in H(U^\sigma)$

$$T(\sigma, \infty)f(x) = \int_A f((0, p, 0)*x)dp$$

for $f \in H(U^0)$,

$$T(0, \mu)f(x) = \sum_A f((0, \gamma, 0)*x)$$

for $f \in H(U^\infty)$

$$T(\infty, \mu)f(x) = \sum_A f((\gamma, 0, 0)*x)$$

and for $f \in H(U^0)$,

$$T(0, \infty)f(x) = \int_A f((0, p, 0)*x)dp .$$

Let $w(\cdot)$ be the distribution defined by $w(m*p) = \chi(m)\chi_\sigma(p)$ for m in M and p in P_σ. Of course, this makes sense only if χ and χ_σ agree on $M \cap P_\sigma$ which is $0 \times 0 \times \boldsymbol{R}$ and if $M*P_\sigma = B$, both of which are clear. Similarly, define the distribution in $C^{-\infty}(U^0)$, $w_0(p_0*p) = \chi_0(p_0)\chi_\sigma(p)$ for p_0 in P_0 and p in P_σ. Note again that χ_0 and χ_σ agree on $P_0 \cap P_\sigma$ and $B = P_0*P_\sigma$.

PROPOSITION 7.

$$\begin{aligned} T(\sigma, 0)\delta &= w_0 \\ T(\sigma, \infty)\delta &= w . \end{aligned}$$

Proof. Let g be in $C^\infty(U^0)$ with compact support in P_0B, then

$$\begin{aligned} \langle g, T(\sigma, 0)\delta \rangle &= \langle T(0, \sigma)g, \delta \rangle \\ &= T(0, \sigma)g(0) \\ &= \int_A g(a, \sigma a, 0)\bar{\chi}_\sigma(a, \sigma a, 0)da \\ &= \langle g, w_0 \rangle . \end{aligned}$$

Similarly for f in $C^\infty(U^\infty)$ with compact support in $M \backslash B$,

$$\begin{aligned} \langle f, T(\sigma, \infty)\delta \rangle &= \langle T(\infty, \sigma)f, \delta \rangle \\ &= \int_A f(a, \sigma a, 0)\bar{\chi}_\sigma(a, \sigma a, 0)da \\ &= \langle f, w \rangle . \end{aligned}$$ □

Finally,

$$\begin{aligned} T(\infty, \mu)w(x_1, x_2, 0) &= \theta_\sigma(x_1, x_2, 0) \\ K(\sigma)T(0, \mu)w_0(x_1, x_2, 0) &= \theta_\sigma(x_1, x_2, 0) . \end{aligned}$$

PROPOSITION 8. *θ_σ is a μ-theta distribution and $K(\sigma)$ is an intertwining constant depending only on σ.*

Proof. The conjugate dual space $C^{-\infty}(U^\infty)$ may be identified with the space of Schwartz distributions Φ which satisfy $\Phi(m*x) = \chi(m)\Phi(x)$, [5]. By its very definition then $\theta_\sigma = T(\infty, \mu)w$ must be a μ-theta distribution [5, §III]. Using Schur's lemma,

$$T(\infty, \mu)T(\sigma, \infty)\delta = K(\sigma)T(0, \mu)T(\sigma, 0)\delta \tag{2}$$

for some constant, $K(\sigma)$ and both $T(\infty, \mu)w$ and $T(0, \mu)w_0$ satisfy the definition of a μ-theta distribution. $\square$

Now that we have explicit formulas for all of the intertwining operators we introduce coordinates and obtain a more concrete from for (2) — in fact we show the equivalence of (1) and (2). The following results will be useful in the computations that follow.

PROPOSITION 9.

$$\log_B(x, \sigma x, 0) = (\log_A x, \sigma \log_A x, \sigma\ell(x))$$

where $\ell(x)$ is the scalar log *function defined earlier and*
(i) $\ell(x*y) = \ell(x) + \ell(y) - B(x, y)$
(ii) $\ell(x) + \ell(\bar{x}) = B(x, \bar{x})$

Now the computations for θ_σ:

$$\begin{aligned} \theta_\sigma(x_1, x_2, 0) &= T(\infty, \mu)w(x_1, x_2, 0) \\ &= \sum w((\gamma, 0, 0)*(x_1, x_2, 0))\chi(\gamma, 0, 0) \\ &= \sum \exp 2\pi i B(\gamma, x_2)w(\gamma*x_1, x_2 + \gamma x_2, 0) \\ &= \sum \exp 2\pi i[B(\gamma, x_2) + \sigma\ell(\gamma^*x)] \,. \end{aligned}$$

where all sums are over γ in Λ. Applying Proposition 9, we have

$$\theta_\sigma(x_1, x_2, 0) = \exp 2\pi i\sigma\ell(x) \sum \exp 2\pi i[B(\gamma, x_2 - \sigma x_1) + \sigma\ell(\gamma)] \,.$$

For the other side of (2),

$$\begin{aligned} \theta_\sigma(x_1, x_2, 0) &= K(\sigma)T(0, \mu)w_0(x_1, x_2, 0) \\ &= \sum w_0(0, \gamma, 0)*(x_1, x_2, 0)\chi_0(0, \gamma, 0) \\ &= \sum \chi_0(b, 0, t_1)\chi_\sigma(a, \sigma a, 0) \end{aligned}$$

where the sum is over γ in Λ, $b = x_1 - \sigma^{-1}x_2$, $a = \bar{b}*x$ and $t_1 = \sigma - B(a, b)$.

$$\theta_\sigma(x_1, x_2, 0) = \sum \exp - 2\pi i\sigma\ell(\sigma^{-1}(\sigma x_1 - x_2 + \gamma)) \,.$$

Replacing $\sigma x_1 - x_2$ by x, we have the distributional identity (1).

THEOREM 3.

$$\sum \exp 2\pi i(\sigma \ell(\gamma) - B(\gamma, x)) = K(\sigma) \sum \exp - 2\pi i \sigma \ell \sigma^{-1}(x + \gamma))$$

The sum on both sides is over γ in Λ.

III. **μ-Theta functions.** We will in this section attempt to duplicate the results just completed in the (locally compact) group structure of the previous section. Here, however, we will be using a compact group and will obtain a function identity (as opposed to a distributional identity) analogous to (1). In a subsequent section, we will evince the connection between the two identities.

Recall that the vector lattice in A had the property that $\Lambda = \Lambda^*$. Throughout this section we will assume that the real number σ of identities (1) and (2) is rational and in lowest terms. That is, $\sigma = a/b$, $(a, b) = 1$. Define

$$A_n = \sum_{k=1}^{r} (\Lambda/n!)^k$$

where, as before, r is the nilpotent degree of the algebra A.

$$\Lambda_n = A_n^* = \{x \in A \mid B(x, A_n) \subset Z\}$$

$$A'_n = \sum_{k=1}^{r} (A_n/b)^k$$

$$\Lambda'_n = (A'_n)^*$$

PROPOSITION 1.
(a) $\Lambda'_n \subset \Lambda_n \subset \Lambda \subset A_n \subset A'_n$.
(b) *Λ_n is a two sided ideal in the ring, A_n.*
(c) *Λ'_n is a two sided ideal in the ring, A'_n.*

Proof. Actually for any sets S_1, S_2 in A such that $S_1 \subset S_2$ then $S_2^* \subset S_1^*$. Clearly $A'_n \supset A_n \supset \Lambda$ so $\Lambda^* \subset A_n^* \subset (A'_n)^*$. A_n is a ring by construction since A is nilpotent and any coefficient of the form $(1/n!)^k$ cannot appear for $k > r$. If λ is in Λ_n, x and y are in A_n, then

$$B(\lambda \cdot x, y) = B(x \cdot \lambda, y) = B(\lambda, x \cdot y) .$$

The latter form is integral since A is a ring. □

In the same method as in introduction, A_n and A'_n will be the basis for the construction of a compact ring, and subsequently, a compact group. $B_n = A_n \times A'_n \times \boldsymbol{R}$ is a ring—with multiplication defined as a subset of B and $\Gamma_n \equiv \Lambda'_n \times \Lambda_n \times z$ is a two sided ideal in B_n. Define $\tilde{B}_n = \Gamma_n \backslash B_n$ and note that it is isomorphic to $(\Lambda'_n \backslash A_n) \times (\Lambda_n \backslash A'_n) \times \pi$. $\tilde{B}_n$ is an algebra because Γ_n is an ideal. In the same fashion as before, we generate a Lie Bracket on $\tilde{B}_n$ by $[\tilde{x}, \tilde{y}] = \tilde{x}\cdot\tilde{y} - \tilde{y}\cdot\tilde{x}$ for all $\tilde{x}$ and $\tilde{y}$ in $\tilde{B}_n$. The compact group associated with $\tilde{B}_n$, denoted by $(\tilde{B}_n, {}^*)$, is $\tilde{B}_n$ with "*" multiplication defined by $\tilde{x}{}^*\tilde{y} = \tilde{x} + \tilde{y} + \tilde{x}\cdot\tilde{y}$.

In the analysis that follows, we will denote projection to $\tilde{B}_n$ by affixing "$\sim$" and form pre-images by removing "$\sim$". For instance, consider the following subset of $\tilde{B}_n$

$$\tilde{M}_n = \Gamma_n \backslash \Lambda'_n \times A'_n \times R$$

and the subset of B_n,

$$M_n = \Lambda'_n \times A'_n \times R\ .$$

PROPOSITION 2.
(a) *M_n is a subgroup of B_n*
(b) *$\tilde{M}_n$ is a subgroup of $\tilde{B}_n$.*

The proof of this straightforward. Recall that $M = 0 \times A \times \boldsymbol{R}$ and thus $M_n = M_n \cap M + \Gamma_n = (M_n \cap M)^*\Gamma_n$ by the multiplicative properties of M. The subgroup Γ_n is normal in B_n and the character x of §I is trivial on $\Gamma_n \cap (M_n \cap M)$. Therefore, there is a unique extension of χ to a character χ_n of $\Gamma_n^*(M_n \cap M) = M_n$. Since x_n is trivial on Γ_n we may also define a character $\tilde{\chi}_n$ on $\tilde{M}_n = \Gamma_n \backslash M_n$ by projection.

With the construction of §II in mind, we establish

THEOREM 1. *Let $\tilde{U}$ be any irreducible representation of $(\tilde{B}_n, {}^*)$ such that $\tilde{U}$ restricted to the center of $(\tilde{B}_n, {}^*)$ is equal to $e^{2\pi i t}I$, where I is the identity operator and t is in R. Then $\tilde{U} \cong \tilde{U}_n^\infty \equiv$* $\operatorname{ind}(\tilde{M}_n, (\tilde{B}_n, {}^*), \tilde{x}_n)$ *and $\tilde{U}_n^\infty$ is irreducible.*

Proof. Let $\tilde{U}$ restricted to $\tilde{M}_n$ be denoted by $\tilde{U}^R$. Since $\tilde{U}^R$ is finite dimensional ($\tilde{M}_n$ is compact),

$$\tilde{U}^R = \sum_{1=i}^{n} \oplus U^i$$

where the U^i's are primary and $U^i = x_i I_i$, for I_i the identity operator on the vector subspace of $H(U^i)$ operates and the x_i are distinct characters of $\tilde{M}_n$. From the results in Mackey, [3], we have that $\{\chi_i\}_{i=1}^n$ is a homogeneous $(\tilde{B}_n, {}^*)$ space under the operation $\chi_i \rightarrow \chi_i^q$. Fix a χ_0 in $\{\chi_i\}_{i=1}^n$ and let $S = \{\chi_0^q \mid q \in (\tilde{B}_n, {}^*)\}$.

LEMMA 1. *The stability subgroup for $\tilde{x}_n$ under the mapping $\tilde{x}_n \rightarrow \tilde{x}_n^q$ is $\tilde{M}_n$.*

Proof. This follows by direct computation. $\square$

LEMMA 2. *There exists a q in $\tilde{B}_n$ such that*

$$x_0 = \tilde{x}_n^q$$

where $x_0(0, 0, t) = e^{2\pi i t}$.

Proof. Since $\{x_n^{\tilde{q}}\}$ for $\tilde{q} \in \tilde{B}_n$ has M_n for its stability subgroup, there are $[\tilde{M}_n; \tilde{B}_n]$ distinct elements in $\{x_n^{\tilde{q}}\}$, $\tilde{q} \in \tilde{B}_n$. A complete set of inequivalent coset representatives for $\tilde{M}_n \backslash \tilde{B}_n$ would be $\Gamma_n \backslash A_n \times \Lambda \times z$ which is isomorphic to $\Lambda'_n \backslash A_n$. We claim:

LEMMA 3. *The number of elements in the dual of $\Lambda'_n \backslash A_n$ is $[\Lambda_n; A'_n]$.*

Proof. By the homorphism $\phi\colon A'_n \rightarrow \tilde{A}_n$ defined by $\phi(x) = \exp 2\pi i B(\cdot, x)$, we map A'_n into the dual of A_n. The kernel of this mapping is Λ_n. The annihilator of the image is precisely Λ'_n. Therefore,

$$\Lambda_n \backslash A_n \cong (\Lambda'_n \backslash A_n)^\wedge \,. \qquad \square$$

Continuing the argument in Lemma 2, there must exist a q such that $x_0 = x_n^q$ since x_0 was a character for $\tilde{M}_n$. $\square$

LEMMA 4. *If $X_0(0, 0, t) = e^{2\pi i t}$, then the stability subgroup of $\{x_0^q\}$, $q \in \tilde{B}_n$ is $\tilde{M}_n$.*

Proof. From Lemma 2, $x_0 = \tilde{x}_n^{\tilde{q}'}$ for some $\tilde{q}' \in \tilde{B}_n$. Applying Lemma 1, we see that the stability subgroup of x_0 is $\tilde{M}_n$. $\square$

We now apply the Mackey machinery, [3] to prove that $\tilde{U} \cong \operatorname{ind}(\tilde{M}_n, (\tilde{B}_n, {}^*), x_0)$. This will establish that $\tilde{U} \cong \tilde{U}_n^\infty$. We isolate this result in

LEMMA 5.

$$\tilde{U} \cong \operatorname{ind}(\tilde{M}_n, (\tilde{B}_n, {}^*), x^0).$$

Proof. As the stability subgroup of x_0 is $\tilde{M}_n$, it follows from the Mackey machinery that $\operatorname{ind}(\tilde{M}_n, (\tilde{B}_n, {}^*), x_0)$ is irreducible and equivalent with $\tilde{U}_n^\infty$. In particular, $\tilde{U}_n^\infty$ is irreducible. $\square$

Let $\sigma = a/b$ be a rational number such that $(a, b) = 1$. Recall that the definition of A'_n depended on b and we choose n (used in the definition of A_n) so that $n \geqq b$ and $n \geqq a$.

In $(B_n, *)$ define the subsets $P_{\sigma,n} = (P_\sigma \cap B_n)*\Gamma n$ and $P_{\sigma,n,0} = P_\sigma \cap P_{\sigma,n}$.

PROPOSITION 3.

$$P_{\sigma,n} \text{ is a subgroup of } (B_n, *) .$$

Proof. This is clear since $P_\sigma \cap B_n$ is a subgroup and Γ_n is normal in B_n. By the properties of Λ_n and Λ'_n, the latter is contained in Γ_n and thus

$$P_{\sigma,n} \cong P_\sigma \cap B_n + \Gamma_n . \qquad \square$$

As a result of this proposition, we have

COROLLARY 4.

$$\tilde{P}_{\sigma,n} \text{ is a subgroup of } (\tilde{B}_n, *) .$$

PROPOSITION 5.

$$P_{\sigma,n} = P_{\sigma,n,0}*(0 \times \Lambda_n \times Z)$$

Proof. From the previous result, elements of $P_{\sigma,n}$ are of the form $(x + \lambda, \sigma x + \lambda, r + z)$ for x in A_n, λ in Λ_n, λ' in Λ'_n, r in R and z in Z.

However, $(x+\lambda', \sigma x+\lambda, r+Z)=((x+\lambda'), (x+\lambda'), r)+(0, \lambda-\sigma\lambda', Z))$ and the latter element is in $0\times\Lambda_n\times Z$. $P_{\sigma,n,0}*(0\times\Lambda_n\times Z) = P_{\sigma,n,0} + (0 \times \Lambda_n \times z)$ by a previous argument. Elements in $P_{\sigma,n,0}$ are of the form $(x, \sigma x, r) + (0, \lambda, Z)$ for x in A_n. It is clear that this is in $P_\sigma \cap B_n + \Gamma_n = P_{\sigma,n}$. $\square$

The subgroup $\Gamma_n = \Lambda'_n \times \Lambda_n \times Z$ is normal in B_n and χ_σ is trivial on $\Gamma_n \cap P_{\sigma,n,0}$. There is a unique, well-defined extension of χ_σ to a character $x_{\sigma,n}$ of $P_{\sigma,n,0}{}^*\Gamma_n = P_\sigma, n$.

Since $x_{\sigma,n}$ is trivial on Γ_n, we may define a character $x_{\sigma,n}$ on $\tilde{P}_{\sigma,n} = \Gamma_n \backslash P_{\sigma,n}$ by projection.

PROPOSITION 6.

$$\tilde{U}^\sigma_n = \operatorname{ind}(\tilde{P}_{\sigma,n}, (\tilde{B}_n, *), \tilde{x}_{\sigma,n}) \quad \text{is irreducible}.$$

Proof. By Theorem 1, $\tilde{U}^\sigma_n$ is primary and quasi-equivalent with $\tilde{U}^\infty_n$. $\square$

LEMMA 6.

$$[\tilde{P}_{\sigma,n}: \tilde{B}_n] = [\Lambda_n: A'_n]$$

Proof. Let $\Delta_n = \Lambda'_n \times \Lambda_n \times R$. Then $\tilde{P}_{\sigma,n} \supset \tilde{\Delta}_n$ and $P_{\sigma,n} = \Delta_n^* P_{\sigma,n,0}$. Hence, $\Delta_n \backslash P_{\sigma,n} = \Delta_n \cap P_{\sigma,n,0} \backslash P_{\sigma,n,0}$. Now $P_{\sigma,n} = \{(x, \sigma x, t) \mid x \in A_n\}$ and $\Delta_n \cap P_{\sigma,n} = \{(\lambda, \sigma\lambda, t) \mid \lambda \in \Lambda'_n\}$. It follows that $\Delta_n \backslash P_{\sigma,n} \cong \Lambda_n \backslash A_n$. Clearly, $\tilde{P}_{\sigma,n} \backslash \tilde{B}_n \cong (\Delta_n \backslash P_{\sigma,n}) \cap (\Delta_n \backslash B_n)$. But $\Delta_n \backslash B_n \cong \Lambda'_n \backslash A_n \times \Lambda_n \backslash A'_n$. Our lemma follows. □

The dimension of $\tilde{U}_n^\sigma$ is $[\tilde{P}_{\sigma,n}; \tilde{B}_n]$ which is $[\Lambda_n; A'_n]$ by the lemma.

Establishing the equality $[\Lambda_n; A'_n] = [\Lambda'_n; A_n]$ will prove that $\tilde{U}_n^\sigma$ and $\tilde{U}_n^\infty$ have the same dimension; by primarity of $\tilde{U}_n^\sigma$ and irreducibility of $\tilde{U}_n^\infty$, $\tilde{U}_n^\sigma$ contains only one copy of $\tilde{U}_n^\infty$. We isolate this equality in

LEMMA 7.

$$[\Lambda_n; A'_n] = [\Lambda'_n; A_n] .$$

Proof. In Lemma 3, an isomorphism was established between the dual of $\Lambda'_n \backslash A_n$ and $\Lambda_n \backslash A'_n$. That same argument can be used to show an isomorphism between the dual of $\Lambda_n \backslash A'_n$ and $\Lambda'_n \backslash A_n$. Thus the number of elements in $\Lambda'_n \backslash A_n$ is equal to the number of elements in $(\Lambda_n \backslash A'_n)^\wedge$ which is less than or equal to the number of elements in $\Lambda_n \backslash A'_n$. Applying the equality suggested by the first isomorphism proves the lemma. □

From Proposition 1, $\Lambda'_n \times \Lambda_n \times \boldsymbol{Z} \subset \Lambda \times \Lambda \times \boldsymbol{R} \subset A_n \times A'_n \times \boldsymbol{R}$. Therefore, $\tilde{\Gamma}_n = \Gamma_n \backslash \Lambda \times \Lambda \times R$ is a nontrivial subgroup of $\tilde{B}_n$. Define $\tilde{\mu}$ on $\tilde{\Gamma}_n$ by

$$\tilde{\mu}((\gamma_1, \gamma_2, t)^\sim) = \exp 2\pi i t$$

for (γ_1, γ_2, t), a coset representative of Γ_n in $\Lambda \times \Lambda \times R$.

PROPOSITION 7.

$$\tilde{\mu} \quad \textit{is a character on } \tilde{\Gamma}_n .$$

Proof. Recall from Chapter I, the character μ defined on Γ by

$$\mu(\gamma_1, \gamma_2, r) = e^{2\pi i r}$$

for (γ_1, γ_2, r) in Γ. μ is trivial on Γ_n and $\tilde{\mu}$ is a projection of μ onto cosets of $\Lambda'_n \times \Lambda_n \times Z$ in Γ. Thus $\tilde{\mu}$ is well defined. □

In the following proposition, we establish the irreducibility of

$\widetilde{U}_n^{\mu}$ and continue the analogy with Chapter I:

PROPOSITION 8.

$\widetilde{U}_n^{\mu} = \mathrm{ind}(\widetilde{\Gamma}_n, (\widetilde{B}_n, {}^*), \widetilde{\mu})$ *is irreducible and equivalent to* $\widetilde{U}_n^{\infty}$.

Proof. In $\widetilde{B}_n$, the commutativity of the *-multiplication for $\widetilde{X} = (X_1, X_2, r_1)^{\sim}$ and $Y^{\sim} = (Y_1, Y_2, r_2)^{\sim}$ is equivalent to

$$\begin{aligned} &\widetilde{X} * \widetilde{Y} * \widetilde{X}^{-} = \widetilde{Y} \\ &\widetilde{Y} + [\widetilde{X}, \widetilde{Y}] = \widetilde{Y} \\ &[\widetilde{X}, \widetilde{Y}] = \widetilde{O} \ . \end{aligned}$$

In B_n, this implies that $[X, Y]$ is in Γ_n.

$$[X, Y] = (0, X_1Y_2 - Y_1X_2, B(X_1, Y_2) - B(X_2, Y_1)) \ .$$

This is, in $\Gamma_n = \Lambda'_n \times \Lambda_n \times \boldsymbol{Z}$ for all such X_1 in A_n and X_2 in A'_n if $B(x_1, y_2) - B(x_2, y_1)$ is integral and $x_2y_1 - x_1y_2$ is in Λ_n. In particular, this must hold if $x_2 = \sigma x_1$ for $\sigma = a/b$. Then $(\sigma - 1)B(x_1, y_2)$ is in $\boldsymbol{Z}$ for all x_1 in A_n. This implies that y_2 is in Λ_n by definition. Therefore, $B(x_2, y_1)$ is in $\boldsymbol{Z}$ for all x_2 in A'_n which implies that $y_1 \in \Lambda'_n$. It is clear that $\Gamma_n \backslash \Lambda'_n \times \Lambda \times \boldsymbol{R}$ is contained in the center of $\widetilde{B}_n$. We have just proven:

LEMMA 8. *The center of* $\widetilde{B}_n$ *is* $\Gamma_n \backslash \Lambda'_n \times \Lambda_n \times \boldsymbol{R}$.

$\widetilde{\mu}$ restricted to the center is of the form $e^{2\pi i t}$ for $(0, 0, t)^{\sim}$ in $\widetilde{B}_n$. Once we establish irreducibility of $\widetilde{U}_n^{\mu}$, equivalence with $\widetilde{U}_n^{\infty}$ will follow from Theorem 1. $\widetilde{U}_n^{\mu}$ is primary and its dimension $[\widetilde{\Gamma}_n; \widetilde{B}_n]$.

$$\begin{aligned} \widetilde{\Gamma}_n \backslash \widetilde{B}_n &\cong (\Gamma_n \backslash \Gamma) \backslash (\Gamma_n \backslash B_n) \\ &\cong \Gamma \backslash B_n \\ &\cong (\Lambda \backslash A_n) \times (\Lambda \backslash A'_n) \times \{0\} \ . \end{aligned}$$

The number of cosets is given by $[\Lambda; A'_n] \cdot [\Lambda; A_N]$. By an argument similar to the proof of Lemma 3, we can prove:

LEMMA 9.

$$[\Lambda'_n; \Lambda] = [\Lambda; A'_N]$$

Proof. Map Λ into the dual of A'_n by $\phi(\lambda)(x) = \exp 2\pi i(B(\lambda, x))$ for λ in Λ and x in A'_n. The kernel of this map and the annihilator of the range are Λ'_n and Λ respectively. $\square$

Therefore,

$$\begin{aligned}[\Gamma_n;\, B_n] &= [\Lambda'_n;\, \Lambda]\cdot[\Lambda;\, A_N] \\ &= [\Lambda'_n;\, A_n] \\ &= [\widetilde{M}_n;\, \widetilde{B}_n] \\ &= \dim \widetilde{U}_n^{\infty}\,.\end{aligned}$$

As $\widetilde{U}_n^{\mu}$ is primary and $\widetilde{U}_n^{\infty}$ is irreducible, this completes the proof of Proposition 8. □

There is one last subgroup to consider. Let

$$\widetilde{P}_{0,n} = \Gamma_n \backslash A_n \times \Lambda_n \times R$$

and

$$X_{0,n}^{\sim}((x,\, 0,\, r)) = \exp 2\pi i r$$

for $(x,\, 0,\, r)^{\sim}$ in $P_{0,n}^{\sim}$.

PROPOSITION 9. *$\widetilde{P}_{0,n}$ is a subgroup of $(\widetilde{B}_n,\, {}^*)$ and $\widetilde{X}_{0,n}$ is a charactes on $\widetilde{P}_{0,n}$.*

Proof. This follows the same line of reasoning as in our previous arguments. That is, the normality of Γ_n in B_n and the subgroup property of $P_0 \cap B_n$ provide the basis for proof. □

PROPOSITION 10. *$\widetilde{U}_n^{0} = \operatorname{ind}(\widetilde{P}_{0,n}),\, \widetilde{B}_n,\, {}^*),\, \widetilde{X}_{0,n})$ is irreducible and equivalent to $\widetilde{U}_n^{\infty}$.*

Proof. Again, by a counting argument, $[\widetilde{P}_{0,n};\, \widetilde{B}_n] = [\Lambda_n;\, A'_n]$ and so $\dim \widetilde{U}_n^{0} = \dim \widetilde{U}_n^{\infty}$. It is clear that $\widetilde{U}_n^{0}$ restricted to the center is of the form $\exp 2\pi i t$ for $(0,\, 0,\, t)^{\sim}$ in the center of $\widetilde{B}_n$. Therefore, by Theorem 1, $\widetilde{U}_n^{0} \cong \widetilde{U}_n^{\infty}$. □

The compact group construction is now complete and we illustrate the construction by means of the following diagram of equivalent, irreducible representations:

$$\begin{tikzcd} \widetilde{U}_n^{\sigma} \arrow[r] \arrow[d] & \widetilde{U}_n^{\infty} \arrow[d] \\ \widetilde{U}_n^{0} \arrow[r] & \widetilde{U}_n^{\mu} \end{tikzcd}$$

FIGURE 2

By Schur's lemma, this diagram commutes up to a multiplicative constant. However, we defer the definition of any intertwining operators or the computation of that multiplicative constant until certain normalization constants are computed. We will establish a relationship between Figure (1) and Figure (2) by proving that the multiplicative constant referred to above is $K(\sigma)$.

A function θ_σ on $(B_n, {}^*)$ will be called a *μ-theta function* if

(i) $\theta_\sigma(\tilde{\gamma}^*\tilde{X}) = \tilde{\mu}_n(\tilde{\gamma})\theta_\sigma(\tilde{X})$

and

(ii) $\theta_\sigma(\tilde{X}^*\tilde{p}) = \tilde{X}_{\sigma,n}(\tilde{p})\theta_\sigma(\tilde{X})$

for all $\tilde{\gamma}$ in $\tilde{\Gamma}_n$, $\tilde{p}$ in $\tilde{P}_{\sigma,n}$, and $\tilde{X}$ in $\tilde{B}_n$.

PROPOSITION 11. *There is a unique, up to a multiplicative constant, μ-theta funtion on $(\tilde{B}_n, *)$.*

Proof. It follows from the Frobenius reciprocity theorem that the subspace of vectors V in $H(\tilde{U}_n^\mu)$ such that $\tilde{U}_n^\mu(p)V = \tilde{X}_{\sigma,n}(\tilde{p})V$ for $\tilde{p} \in \tilde{P}_{\sigma,n}$ has dimension one. □

We compute the μ-theta functions more explicitly. Let W_n be the function on $(B_n, {}^*)$ defined by

$$W_n(\tilde{m}^*\tilde{p}) = \tilde{X}_n(\tilde{m})\tilde{X}_{\sigma,n}(\tilde{p})$$

for $\tilde{m}$ in $\tilde{M}_n$ and $\tilde{p}$ in $\tilde{P}_{\sigma,n}$.

LEMMA 10.

$$\tilde{M}_n^*\tilde{P}_{\sigma,n} = \tilde{B}_n$$

Proof. Let (x, y, t) be in B_n. Then,

$$(x, y, t)^\sim = (0, b, t_1)^{\sim *}(a, \sigma a, 0)^\sim$$

where $(a, b, t_1)^\sim = (x, y - \sigma x, t)^\sim$. □

$(\Gamma_n\backslash\Lambda'_n \times A_n \times R)\backslash(\Gamma_n\backslash A_n \times \Lambda_n \times R)$ is isomorphic to $\Gamma_n\backslash\Lambda'_n \times \Lambda_n \times R$. By Lemma 8, $\tilde{M}_n \cap \tilde{P}_{\sigma,n}$ is the center of $\tilde{B}_n$. Also, for $p_1^\sim$, $p_2^\sim$ in $P_{\sigma,n}^\sim$ and $\tilde{m}_1$, $\tilde{m}_2$ in $M_1^\sim$, $M_n^{\sim *}P_1^\sim = M_2^* P_2^\sim$ implies that $W_n(m_1^{\sim *}p_1^\sim) = W_n(m_2^{\sim *}p_2^\sim)$. Therefore, $W_n(\cdot)$ is a well-defined function on $\tilde{B}_n$.

Define

$$\theta_\sigma(\tilde{X}) = \sum_{\tilde{\Gamma}_n \cap \tilde{M}_n \backslash \tilde{\Gamma}_n} W_n(\gamma^*\tilde{X})$$

where $X^\sim$ is in $B_n^\sim$ and γ is a coset representative of $\tilde{\Gamma}_n \cap \tilde{M}_n$ in $\tilde{\Gamma}_n$. In terms of coordinates,

$$\begin{aligned}\theta_\sigma((x_1, x_2, t)^\sim) &= \theta_\sigma((0, 0, t_1)^{\sim *}(x_1, x_2, 0)^\sim) \\ &= e^{2\pi i t_1}\theta_\sigma((x_1, x_2, 0)^\sim)\ .\end{aligned}$$

We compute $\theta_\sigma((x_1, x_2, 0)^\sim)$ more explicitly.

$$(x_1, x_2, 0)^\sim = (0, x_2 - \sigma x_1, 0)^\sim * (x_1, \sigma x_1, 0)^\sim .$$

Thus,

$$\begin{aligned} W_n(x_1, x_2, 0) &= \tilde{X}_n((0_2, x_2 - \sigma x_1, 0)^\sim)\tilde{X}_{\sigma,n}((x_1, \sigma x_1, 0)^\sim) \\ &= \exp 2\pi i \sigma \ell(x_1) . \end{aligned}$$

The set $\tilde{C}_n = \{(\gamma, 0, 0)^\sim$ in $\Gamma_n \backslash \Lambda \times \Lambda_n \times R\}$ is a complete set of coset representatives of $\tilde{\Gamma}_n \cap \tilde{M}_n$ in $\tilde{\Gamma}_n$.

$$W_n((\gamma, 0, 0)^\sim * (x_1, x_2, 0)^\sim) = \exp 2\pi i [B(x_2, \gamma) + \sigma l(\gamma^* x_1)] \quad \text{and}$$
$$\tilde{C}_n \cong C_n = \Lambda_n' \backslash \Lambda .$$

As a consequence,

$$\begin{aligned} \theta_\sigma((x_1, x_2, 0)^\sim) &= \sum W[(\gamma, 0, 0)^\sim * (x_1, x_2, 0)^\sim] , \quad (\gamma \in C_n) \\ &= \exp 2\pi i \sigma \ell(x_1) \sum \exp 2\pi i [B(\gamma, x_2 - \sigma x_1) + \sigma \ell(\gamma)] , \quad (\gamma \in C_n) . \end{aligned}$$

The proof that θ_σ is a nontrivial function will be deferred until an explicit connection can be made between these functions and the distributions constructed in § II.

PROPOSITION 12. *$P_{0,n}^* P_{\sigma,n} = A_n \times \sigma A_n \times \boldsymbol{R} + 0 \times \Lambda_n \times 0$ is a subgroup of $(B_n, {}^*)$. Furthermore, $X_{0,n} = X_{\sigma,n}$ on $P_{0,n} \cap P_{\sigma,n} = (A_n \cap \sigma^{-1}\Lambda_n) \times \Lambda_n \times \boldsymbol{R}$.*

Proof. By previous considerations,

$$P_{\sigma,n} = \{(x, \sigma x + \lambda, t_1) | x \text{ in } A_n, \lambda \text{ in } \Lambda_n, t_1 \text{ in } \boldsymbol{R}\}$$
$$P_{0,n} = \{(y, \lambda', t_2) | y \text{ in } A_n, \lambda' \text{ in } \Lambda_n, t_2 \text{ in } \boldsymbol{R}\} .$$
$$(y, \lambda', t_2)^*(x, \sigma x + \lambda, t_1) = (y^* x, \sigma(y^* x - y) + \lambda'', t_3) \quad \text{for}$$
$$\lambda'' = \lambda Y + \lambda + \lambda'$$

and $t_3 = t_1 + t_2 + B(x, \lambda')$. Note that λ'' is in Λ_n since Λ_n is an ideal in A_n. Conversely, given an element of the form $(a, \sigma b + \lambda, t_3)$ with a, b, in A_n and t_3 in $\boldsymbol{R}$, it is easy to show that

$$(a, \sigma b + \lambda, t_3) = (y, \lambda', t_2)^*(x, \sigma x + \lambda'', t_1)$$

for x and y in A_n; λ', λ'' in Λ_n and t_1, t_2 in $\boldsymbol{R}$. So $P_{0,n}^* P_{\sigma,n}$ is set theoretically equal to $A_n \times \sigma A_n \times R + 0 \times \Lambda_n \times 0$. Multiplication can be easily shown to be well-defined. Therefore, $P_{0,n}^* P_{\sigma,n}$ is a subgroup of $(B_n, {}^*)$.

In order for $(x, \sigma x + \lambda, t_1)$ in $P_{\sigma,n}$ to also be in $P_{0,n}$, $\sigma x + \lambda \in \Lambda_n$. Thus, $x \in \sigma^{-1}\Lambda_n$. Therefore,

$$P_{0,n} \cap P_{\sigma,n} \subset \{(\sigma^{-1}\lambda_1, \lambda_2, t) \mid \lambda_1, \lambda_2 \in \Lambda_n, t \in R , \quad \text{and} \quad \sigma^{-1}\lambda_1 \in A_n\} .$$

The opposite inclusion is obvious. Finally, $X_{0,n}(\sigma^{-1}\lambda_1, \lambda_2, t) = e^{2\pi i t}$

$$\begin{aligned} X_{\sigma,n}(\sigma^{-1}\lambda_1, \lambda_2, t) &= X_{\sigma,n}(\sigma^{-1}\lambda_1, \lambda_1 + (\lambda_2 - \lambda_1), t) \\ &= \exp 2\pi i(t + \sigma\ell(\sigma^{-1}\lambda_1)) . \end{aligned}$$

However, $\sigma\ell(\cdot)$ is integral on $(\sigma^{-1}\Lambda_n)\Lambda_n$ because $n \geqq \max\{r, a, b\}$ and so $X_{0,n} \equiv X_{\sigma,n}$ on $P_{0,n} \cap P_{\sigma,n}$. $\square$

Define the function $W_{0,n}(\cdot)$ on the subgroup $\tilde{P}^*_{0,n}\tilde{P}_{\sigma,n}$ by

$$W_{0,n}(\tilde{p}_0^*\tilde{p}) = \tilde{X}_{0,n}(\tilde{p}_0)\tilde{X}_{\sigma,n}(\tilde{p})$$

for $\tilde{p}_0$ in $\tilde{P}_{0,n}$ and $\tilde{p}$ in $\tilde{P}_{\sigma,n}$. By the preceding proposition, $W_{0,n}$ is well-defined on $\tilde{P}^*_{0,n}\tilde{P}_{\sigma,n}$. Define $W_{0,n}(\cdot)$ on the entire group $(\tilde{B}_n, *)$ by

$$W_{0,n}(\tilde{x}) = \begin{cases} \tilde{X}_{0,n}(\tilde{p}_0)\tilde{X}_{\sigma,n}(\tilde{p}) & \tilde{X} \in \tilde{p}^*_{0,n}\tilde{P}_{\sigma,n} \\ 0 \quad \text{otherwise} . \end{cases}$$

For $(x_1, x_2, 0)^\sim$ in $(\tilde{B}_n, *)$, define

$$\Psi_\sigma((x_1, x_2, 0)^\sim) = \sum_{\tilde{P}_{0,n} \cap \tilde{\Gamma}_n \backslash \tilde{\Gamma}_n} W_{0,n}(\gamma^{\sim *}(x_1, x_2, 0)^\sim) .$$

If $(x_1, x_2, 0)^\sim$ is in $\tilde{P}_{0,n} * \tilde{P}_{\sigma,n}$, $(x_1, x_2, 0)^\sim = (u, 0, 0)^{\sim *}(v, \sigma v, t_1)^\sim$ where $U = x_1 - \sigma^{-1}x_2$, $v = \bar{u}^* x_1$ and $t_1 = -B(u, v)$.

Then,

$$\begin{aligned} W_0((u, 0, 0)^{\sim *}(v, \sigma v, t_1)^\sim) &= \exp(-2\pi i B(u, \sigma v)) \cdot \tilde{X}_{\sigma,n}((v, \sigma v, 0)^\sim) \\ &= \exp 2\pi i \sigma(\ell(v) - B(u, v)) . \end{aligned}$$

If $(x_1, x_2, 0)$ is not in $P^*_{0,n}P_{0,n}$ then $W_0(x_1, x_2, 0) = 0$. A direct adaptation of these arguments show that $(U', 0, 0)^{\sim *}(V, \sigma V, t_1)^\sim = (U', 0, 0)^{\sim *}(V', V', t_i')^\sim$ implies that

$$W_0((U, 0, 0)^{\sim *}(V, \sigma V, t_1)^\sim) = W_0(U', 0, 0)^{\sim *}(V', \sigma V', t_i)^\sim) .$$

It is easy to show that

$$\begin{aligned} (1) \qquad \ell(V) &= \ell(\bar{U}^* X_1) \\ &= \ell(\bar{U}) + \ell(X_1) - B(X_1, \bar{U}) \end{aligned}$$

$$(2) \qquad B(U, V) = -B(\bar{U}, X_1) + B(U, \bar{U})$$

and

$$(3) \qquad \ell(U) + \ell(\bar{U}) = B(U, \bar{U}) .$$

Therefore,

$$W_0(X_1, X_2, 0) = \exp 2\pi i \sigma(\ell(X_1) - \ell(X_1 - o^{-1}X_2)) .$$

The set $D_n^\sim = \{\gamma \mid (0, \gamma, 0)^\sim$ in $\Lambda'_n \times \Lambda_n \times \boldsymbol{Z} \backslash \Lambda'_n \times \Lambda \times \boldsymbol{Z}\}$ is a complete set of inequivalent coset representatives for $\widetilde{P}_{0,n} \cap \widetilde{\Gamma}_n$ in $\widetilde{\Gamma}_n$. Note that $D_n^\sim \subset \widetilde{P}^*_{0,n}\widetilde{P}_{\sigma,n}$.

Finally,

$$\Psi_\sigma[(x_1, x_2, 0)^\sim] = \exp 2\pi i\sigma\ell(x_1) \sum \exp - 2\pi i\sigma\ell(\sigma^{-1}(x_1 - x_2 + \gamma)) .$$

Replace $x_2 - \sigma x_1$ by x in the expressions for θ_σ and Ψ_σ and

$$\Psi_\sigma[(x_1, x_2, 0)^\sim] = \exp 2\pi i\sigma\ell(x_1) \sum \exp - 2\pi i\sigma\ell(\sigma^{-1}(x + \gamma)) , \quad (\gamma \in \Lambda_n\backslash\Lambda)$$

$$\theta_\sigma[(X_1, X_2, 0)^\sim] = \exp 2\pi i\sigma\ell(X_1) \sum \exp + 2\pi i[-B(\gamma, x) + \sigma\ell(\gamma)] , \quad (\gamma \in \Lambda'_n\backslash\Lambda) .$$

By virtue of Proposition 11 and the construction of θ_σ and Ψ_σ, we have

THEOREM 2. *For $\widetilde{x}$ in $\widetilde{B}_n$, $\theta_\sigma(\widetilde{x}) = c\Psi_\sigma(\widetilde{x})$ for some nonzero scalar c.*

This establishes a similar relationship between these μ-theta functions as in §II. However, it will be shown later that c is an intertwining constant $K_n(\sigma)$ which is equal to $K(\sigma)$.

IV. The relationship between μ-theta functions and μ-theta distributions. We will now relate the results of §§II and III. Our intent is to derive, in a more meaningful manner, the constant multiple of Theorem 2 in §III and establish equality between that constant and $K(\sigma)$.

First, we establish an equivalency that is the key of our investigation. Let $\rho\colon B_n \to \widetilde{B}_n$ be the canonical map onto $\Lambda'_n \times \Lambda_n \times \boldsymbol{Z}$ cosets in $A_n \times A'_n \times \boldsymbol{R}$. Let $\widetilde{U}^\mu_n = \operatorname{ind}(\widetilde{\Gamma}_n, (\widetilde{B}_n, {}^*), \widetilde{\mu}_n)$ and $U_n = \operatorname{ind}(\Gamma, (B_n, {}^*), \lambda_n)$.

PROPOSITION 1.

$$U_n \cong \widetilde{U}^\mu_n \cdot \rho$$

Proof. $U^\mu = \operatorname{ind}(B_n, B, \operatorname{ind}(\Gamma, B_n, \mu))$ by induction in stages. Since U^μ is irreducible, U_n must be irreducible. The following diagram is commutative:

$$\begin{array}{ccc} B_n & \xrightarrow{\ \rho\ } & \widetilde{B}_n \\ {\scriptstyle U_n}\big\downarrow & & \big\downarrow{\scriptstyle \widetilde{U}_n} \\ L^2(\Gamma\backslash B_n) & \xrightarrow{\ \sim\ } & L^2(\widetilde{\Gamma}_n\backslash\widetilde{B}_n) \end{array}$$

FIGURE 3

That is, for f in $L^2(\tilde{\Gamma}_n\backslash\tilde{B}_n)$,

$$\tilde{U}^{\mu}_n(\rho g)f(\tilde{X}) = U_n(g)f(\rho x) .$$

This follows from the definition of $\tilde{B}_n$ and from $\Lambda'_n \times \Lambda_n \times \boldsymbol{Z}$ being a two sided ideal in $A_n \times A'_n \times \boldsymbol{R}$. Therefore $\tilde{U}^{\mu}_n$ and U_n have the same invariant subspaces. Since both representations are irreducible, they must be equivalent. □

We will now improve upon Figure 2.

$$\begin{tikzcd} \tilde{U}^{\sigma}_n \arrow[r, "\tilde{T}_n(\sigma, \infty)"] \arrow[d, "\tilde{T}_n(\sigma, 0)"'] & \tilde{U}^{\infty}_n \arrow[d, "\tilde{T}_n(\infty, \mu)"] \\ \tilde{U}^{0}_n \arrow[r, "T_n(0, \mu)"'] & \tilde{U}^{\mu}_n \end{tikzcd}$$

FIGURE 4

Each of the mappings $T_n(\cdot, \cdot)$ is an intertwining operator between equivalent irreducible representations and hence this diagram commutes up to a constant, $K_n(\sigma)$ by Schur's lemma. We lift this diagram up to the entire group B by means of the inducing map and the pull back, ρ. Each representation is lifted to B by the following:

$$\begin{aligned} U^{\sigma}_n &= \mathrm{ind}(B_n, B, \tilde{U}^{\sigma}_n \circ \rho) \\ U^{0}_n &= \mathrm{ind}(B_n, B, \tilde{U}^{0}_n \circ \rho) \\ U^{\infty}_n &= \mathrm{ind}(B_n, B, \tilde{U}^{\infty}_n \circ \rho) \\ U^{\mu}_n &= \mathrm{ind}(B_n, B, \tilde{U}^{\mu}_n \circ \rho) . \end{aligned}$$

We may, without ambiguity also make the additional identifications

$$\begin{aligned} U^{\sigma}_n &= \mathrm{ind}(P_{\sigma,n}, B, X_{\sigma,n}) \\ U^{0}_n &= \mathrm{ind}(P_{0,n}, B, X_{0,n}) \\ U^{\infty}_n &= \mathrm{ind}(M_n, B, X_n) \\ U^{\mu}_n &= \mathrm{ind}(\Gamma_n, B, \mu_n) . \end{aligned}$$

By equivalence of the inducing representations we have the following diagram of irreducible, equivalent representations of $(B, *)$.

$$\begin{tikzcd} U^{\sigma}_n \arrow[r, "T_n(\sigma, \infty)"] \arrow[d, "T_n(\sigma, 0)"'] & U^{\infty}_n \arrow[d, "T_n(\infty, \mu)"] \\ U^{0}_n \arrow[r, "T_n(0, \mu)"] & U^{\mu}_n \end{tikzcd}$$

FIGURE 5

For f in the appropriate representation space, we have

$$
\begin{aligned}
[T_n(\sigma, \infty) U_n^0 f](0, 0, 0) &= [\tilde{T}_n(\sigma, \infty) \tilde{U}_n^\sigma \circ \rho f](0, 0, 0) \\
[T_n(\infty, \mu) U_n^\infty f](0, 0, 0) &= [\tilde{T}_n(\infty, \mu) \tilde{U}_n^\infty \circ \rho f](0, 0, 0) \\
[T_n(\sigma, 0) U_n^\sigma f](0, 0, 0) &= [\tilde{T}_n(\sigma, 0) \tilde{U}_n^\sigma \circ \rho f](0, 0, 0) .
\end{aligned}
$$

Consequently, the diagram of Figure 5 also commutes up to the constant, $K_n(\sigma)$.

That is,

$$T_n(\infty, \mu) T_n(\sigma, \infty) = K_n(\sigma) T_n(0, \mu) T_n(\sigma, 0) .$$

Through the equivalence of the representation U_n and $\tilde{U}_n^\mu \circ \rho$, we have established an equivalency between U^μ and U_n^μ. Since all these representations are irreducible, consider the diagram of Figure 6.

$$
\begin{tikzcd}
& U_n^\sigma \arrow[dl, "T_n(\sigma, \infty)"'] \arrow[dd, "T_n(\sigma, n)"] & & U^\sigma \arrow[ll, "A(\sigma, n)"'] \arrow[dl, "T(\sigma, \infty)"'] \arrow[dd, "T(\sigma, 0)"] \\
U_n^\infty \arrow[dd, "T_n(\infty, \mu)"] & & U^\infty \arrow[ll, "A(\infty, n)"'] \arrow[dd, "T(\infty, \mu)"] & \\
& U_n^0 \arrow[dl, "T_n(0, \mu)"'] & & U^0 \arrow[ll, "A(0, \mu)"'] \arrow[dl, "T(0, \mu)"] \\
U_n^\mu & & U^\mu \arrow[ll, "A(\mu, n)"] &
\end{tikzcd}
$$

FIGURE 6

It has been shown that the right and left sides of this diagram commute up to the constants $K(\sigma)$ and $K_n(\sigma)$ respectively. It is our intent to show that $K_n(\sigma) = K(\sigma)$.

However, before pursuing that line of reasoning, we exhibit the intertwining operators of Figure 5. For sake of definition, let f be an appropriate function in each instance below.

$$
\begin{aligned}
A(\sigma, n) f(x, y, t) &= \int_{P_{\sigma,n} \cap P_\sigma \backslash P_{\sigma,n}} f(p^*(x, y, t)) \bar{X}_{\sigma,n}(p) dp \\
A(\mu, n) f(x, y, t) &= f(x, y, t) \\
A(0, n) f(x, y, t) &= [\Lambda_n; \Lambda]^{-1} \int_{P_{0,n} \cap P_0 \backslash P_{0,n}} f(p^*(x, y, t)) \bar{X}_{0,n}(p) dp \\
A(\infty, n) f(x, y, t) &= [\Lambda_n'; \Lambda]^{-1} \int_{M_n \cap M \backslash M_n} f(p^*(x, y, t)) \bar{X}_n(p) dp
\end{aligned}
$$

$$T_n(\sigma, 0)f(x, y, t) = \int_{P_{0,n}\cap P_{0,n}\backslash P_{0,n}} f(p^*(x, y, t))\bar{X}_{0,n}(p)dp$$

$$T_n(\sigma, \infty)f(x, y, t) = \int_{P_{\sigma,n}\cap M_n\backslash M_n} f(p^*(x, y, t))\bar{X}_n(p)dp$$

$$T_n(0, \mu)f(x, y, t) = [\Lambda_n; \Lambda]\int_{P_{0,n}\cap\Gamma\backslash\Gamma} f(p^*(x, y, t)\bar{\mu}(p)dp$$

$$T_n(\infty, \mu)f(x, y, t) = [\Lambda'_n; \Lambda]\int_{M_n\cap\Gamma\backslash\Gamma} f(p^*(x, y, t)\bar{\mu}(p)dp$$

In each definition dp is Haar measure. These definitions are (formally) the intertwining operators between the irreducible, equivalent representations. The following proposition is necessary for the latter group of operators defined above.

PROPOSITION 2.
(a) *The characters* $X_{\sigma,n}$ *and* $X_{0,n}$ *agree on* $P_{\sigma,n}\cap P_{0,n}$.
(b) *The characters* $X_{\sigma,n}$ *and* X_n *agree on* $P_{\sigma,n}\cap M_n$.
(c) *The characters* $X_{0,n}$ *and* μ *agree on* $P_{0,n}\cap\Gamma$.
(d) *The characters* X_n *and* μ *agree on* $M_n\cap\Gamma$.

Proof. (a) This was proven in Proposition 12 in Chapter II.

(b) $P_{\sigma,n}\cap M_n = \Lambda'_n\times\Lambda_n\times R$ and both $X_{\sigma,n}(\lambda', \lambda, t)$ and $X_{0,n}(\lambda', \lambda, t)$ are $e^{2\pi it}$ for (λ', λ, t) in $P_{\sigma,n}\cap M_n$.

(c) $M_n\cap\Gamma = \Lambda'_n\times\Lambda\times \boldsymbol{Z}$ and both characters are trival on this subgroup.

(d) $P_{0,n}\cap\Gamma = \Lambda\times\Lambda_n\times Z$ and both characters are trivial on this subgroup. □

Before justifying the existence of such operators, some consideration will be given to normalization constants associated with the invariant measures.

Let G by a locally compact abelian group K, a subgroup of G. Let f and $\hat{f}$ be integrable functions on G and $\hat{G}$. Let $K^\perp$ = {characters of the dual group of G that are trivial on K}.

The Poisson summation formula is given by

$$\int_K f(k)dk = \int_{K^\perp}\hat{f}(m)dm$$

if K and $K^\perp$ are given compatible Haar measures, i.e.,$K^\perp$ has the measure dual with the measure on $K\backslash G$ which is determined by the measure on G and the measure on K. We want to choose that measure so that $\hat{f}^\vee = f$. If G is compact, then $\hat{G}$ is discrete and Haar measure on $\hat{G}$ must be counting measure. If the measure of the compact group is C, then the Haar measure on $\hat{G}$ is C^{-1} times counting measure.

We shall say the measures on G, KG and K are normalized if

$$\int_G = \int_{K\backslash G}\int_K .$$

Hence, we shall always assume that our measures are so normalized. Furthermore, we will choose the measure on $K^{\perp}$ to equal that on $(K\backslash G)^{\wedge}$ so that the Poisson summation formula works. As a consequence, we give discrete sets counting measure. If G and K are both discrete, the triple G, K, $K\backslash G$ is then normalized.

Set	*Measure*
$A_n\backslash A$	$[\Lambda_n; \Lambda]^{-1}$
$A'_n\backslash A$	$[\Lambda_n; \Lambda]^{-1}$
$\Lambda_n\backslash A$	$[\Lambda_n; \Lambda]$
$\Lambda'_n\backslash A$	$[\Lambda_n; \Lambda]$.

With these considerations completed, we now investigate the intertwining operators.

PROPOSITION 3. *For f appropriately chosen in the representation space of each operator,*

$$T_n(\infty, \mu)f(x, y, t) = [\Lambda_n; \Lambda]\sum_{\Lambda'_n\backslash\Lambda} f((\gamma, 0, 0)^*(x, y, t))$$
$$T_n(0, \mu)f(x, y, t) = [\Lambda_n; \Lambda]\sum_{\Lambda_n\backslash\Lambda} f((0, \gamma, 0)^*(x, y, t))$$
$$T_n(\sigma, \infty)f(x, y, t) = \sum_{\Lambda_n\backslash\Lambda'_n} f((0, \gamma, 0)^*(x, y, t))$$
$$T_n(\sigma, 0)f(x, y, t) = \int_{P_{\sigma,n}\cap P_{0,n}\backslash P_{0,n}} f(P^*(x, y, t))\bar{X}_{0,n}(p)dp$$

where dp is normalized Haar measure.

Proof. The last equality is stated only for completeness and reference, as no attempt will be made to simplify it.

By definition, for f in $S(A \times 0 \times 0) \cap H(U_n^{\infty})$,

$$T_n(\infty, \mu)f(x, y, t) = [\Lambda'_n; \Lambda]\int_{M_n\cap\Gamma\backslash\Gamma} f(p^*(x, y, t))\bar{\mu}(p)dp .$$

It is easy to show that $\{(\gamma, 0, 0) \mid \gamma \text{ in } \Lambda'_n\backslash\Lambda\}$ is a complete set of inequivalent coset representatives of $M_n \cap \Gamma$ in Γ and $\bar{\mu}(\gamma, 0, 0) = 1$. The other identities follow similarly. $\square$

Before attempting any simplification of $A(\mu, n)$, $A(\infty, n)$, $A(\sigma, n)$ and $A(0, n)$ we state the following proposition whose proof may be found in [8].

PROPOSITION 4. *If H and K are rational subgroups of N (a nilpotent Lie group) and K normalizes H then HK is rational and, in particular, is closed.*

By the Noether isomorphism theorem.

$$P_{\sigma,n} \cap P_\sigma \backslash P_{\sigma,n} \cong P_\sigma \backslash P_{\sigma,n}^* P_\sigma$$
$$M_n \cap M \backslash M_n \cong M \backslash M^* M_n$$
$$P_{0,n} \cap P_0 \backslash P_{0,n} \cong P_0 \backslash P_{0,n}^* P_0$$

and the latter portion of each isomorphism is closed by the previous remarks. $\{(k, 0, 0) \mid k \text{ in } \Lambda_n'\}$ is a complete set of inequivalent coset representatives of $M_n \cap M$ in M_n as is $\{(0, k, 0) \mid k \text{ in } \Lambda_n\}$ for $P_{0,n} \cap P_0$ in $P_{0,n}$. Also, $\{(0, k, 0) \mid k \text{ in } \Lambda_n\}$ is a complete set of inequivalent coset representatives for $P_{0,n} \cap P_\sigma$ in $P_{\sigma,n}$. As a consequence we have,

$$A(\mu, n) f(X, Y, t) = f(X, Y, t)$$
$$A(\sigma, n) f(X, Y, t) = \sum_{\Lambda_n} f((0, k, 0)^*(X, Y, t))$$
$$A(0, n) f(X, Y, t) = [\Lambda_n; \Lambda]^{-1} \sum_{\Lambda_n} f((0, k, 0)^*(X, Y, t))$$
$$A(\infty, n) f(X, Y, t) = [\Lambda_n'; \Lambda]^{-1} \sum_{\Lambda_n'} f((k, 0, 0)^*(X, Y, t)) .$$

It is left to the reader to show the triviality of each character on the coset representatives. With all of these considerations of operators completed, we present the first theorem of this chapter.

THEOREM 1. *The diagram of Figure 6 is commutative except for the right and left sides, which have been shown to commute up to the constants $K_n(\sigma)$ and $K(\sigma)$ respectively.*

Proof. The really meaningful part of this proof is the commutativity of the back portion of this diagram and we defer justification of that to the end of this proof. Recall $S(G)$ is the Schwarz space of the group G. It is sufficient to compute the constant of commutativity for two intertwining operators between equivalent and irreducible representations by application of them on an arbitrary nonzero function evaluated at the group identity.

First consider the front of the diagram.

$$\begin{tikzcd} U_n^\infty \arrow[d, "T_n(\infty, \mu)"'] & U^\infty \arrow[l, "A(\infty, n)"'] \arrow[d, "T(\infty, \mu)"] \\ U_n^\mu & U^\mu \arrow[l, "A(\mu, n)"'] \end{tikzcd}$$

Let f be in $H(U^\infty) \cap S(A \times 0 \times 0)$. Now,

$$\begin{aligned} A(\mu, n)T(\infty, \mu)f(0, 0, 0) &= T(\infty, \mu)f(0, 0, 0) \\ &= \sum_{\Lambda} f(\gamma, 0, 0), \quad (\gamma \text{ in } \Lambda) . \end{aligned}$$

In the other direction,

$$\begin{aligned} T_n(\infty, \mu)A(\infty, n)f(0, 0, 0) &= [\Lambda_n'; \Lambda] \sum_{\Lambda_n' \backslash \Lambda} A(\infty, n)f(\gamma, 0, 0) \\ &= [\Lambda_n'; \Lambda] \sum_{\Lambda_n' \backslash \Lambda} [\Lambda_n'; \Lambda]^{-1} \sum_{\Lambda_n'} f(k^*(\gamma, 0, 0)) \\ &= \sum_{\Lambda} f(k', 0, 0) . \end{aligned}$$

Thus the diagram commutes since the sums converge.

The bottom of (6) is

$$\begin{tikzcd} U_n^0 \arrow[d, "T_n(0, \mu)"'] & U^0 \arrow[l, "A(0, n)"'] \arrow[d, "T(0, n)"] \\ U_n^\mu & U^\mu \arrow[l, "A(\mu, n)"'] \end{tikzcd} .$$

Let f be in $H(U^0) \cap S(0 \times A \times 0)$, then

$$\begin{aligned} A(\mu, n)T(0, \mu) &= T(0, \mu)f(0, 0, 0) \\ &= \sum_{\Lambda} f(0, \gamma, 0) . \end{aligned}$$

In the other direction,

$$\begin{aligned} T_n(0, \mu)A(0, n)f(0, 0, 0) &= [\Lambda_n; \Lambda] \sum_{\Lambda_n \backslash \Lambda} A(0, n)f(0, \gamma, 0) \\ &= [\Lambda_n; \Lambda] \sum_{\Lambda_n \backslash \Lambda} [\Lambda_n; \Lambda]^{-1} \sum_{\Lambda_n} f((0, k, 0)^*(0, \gamma, 0)) \\ &= \sum_{\Lambda} f(0, \gamma', 0) . \end{aligned}$$

Therefore, the bottom portion commutes.

The top of the diagram is

$$\begin{tikzcd} U_n^\sigma \arrow[d, "T_n(\sigma, \infty)"'] & U^\sigma \arrow[l, "A(\sigma, n)"'] \arrow[d, "T(\sigma, \infty)"] \\ U_n^\infty & U^\infty \arrow[l, "A(\infty, n)"'] \end{tikzcd} .$$

Let f be in $H(U^\sigma) \cap S(0 \times A \times 0)$.

$$\begin{aligned} A(\infty, n)T(\sigma, \infty)f(0, 0, 0) &= [\Lambda_n; \Lambda]^{-1} \sum_{\Lambda_n'} T(\sigma, \infty)f(k, 0, 0) \\ &= [\Lambda_n'; \Lambda]^{-1} \sum_{\Lambda_n'} \int_{P_\sigma \cap M \backslash M} f(p^*(k, 0, 0))\bar{X}(p)dp \\ &= [\Lambda_n'; \Lambda]^{-1} \sum_{\Lambda_n'} \int_{A} f(k, p, 0)dp \end{aligned}$$

$$(k, p, 0) = (a, \sigma a, t)^*(0, b, 0)$$

where $a = k$, $b = p + \bar{k}p + \sigma\bar{k}$ and $t = -B(a, b)$.

Note that $db = dp$. Thus, since f is in $H(U^\sigma)$,

$$\begin{aligned} A(\infty, n)T(\sigma, \infty)f(0, 0, 0) &= [\Lambda'_n; \Lambda]^{-1} \sum_{\Lambda'_n} \int_A \exp - 2\pi i(B(k, b) \\ &\quad + \sigma\ell(k_1))\cdot f(0, b, 0)db \\ &= [\Lambda'_n; \Lambda]^{-1} \sum_{\Lambda'_n} \exp 2\pi i\sigma\ell(k)\hat{q}(x_k) , \end{aligned}$$

where $q(x) = f(0, x, 0)$ and X_k is the character $e^{-2\pi iB(k,x)}$. Also by construction, $\sigma\ell(k)$ is integral for k in Λ'_n. In the other direction,

$$\begin{aligned} T_n(\sigma, \infty)A(\sigma, n)f(0, 0, 0) &= \sum_{\Lambda_n\backslash\Lambda'_n} A(\sigma, n)f(0, \gamma, 0) \\ &= \sum_{\Lambda_n\backslash\Lambda'_n} \sum_{\Lambda_n} f((0, k, 0)^*(0, \gamma, 0)) \\ &= \sum_{\Lambda'_n} f(0, k', 0) . \end{aligned}$$

By the Poisson summation formula,

$$[\Lambda'_n; \Lambda]^{-1} \sum_{\Lambda'_n} \hat{q}(X_k) = \sum_{\Lambda'_n} f(0, k', 0) ,$$

i.e., this portion commutes also.

Finally, this portion of the diagram is the most difficult for proving commutativity

$$\begin{array}{ccc} U^\sigma_n & \xleftarrow{A(\sigma, n)} & U^\sigma \\ {\scriptstyle T_n(\sigma, 0)}\big\downarrow & & \big\downarrow{\scriptstyle T(\sigma, 0)} \\ U^0_n & \xleftarrow{A(0, n)} & U^0 . \end{array}$$

Commutativity will, in fact, be demonstrated by showing that

$$A(0, n)^{-1}T_n(\sigma, 0)A(\sigma, n) = T(\sigma, 0) .$$

For f in $H(U^\sigma) \cap S(A \times 0 \times 0)$, we have

$$\begin{aligned} T_n(\sigma, 0)A(\sigma, n)f(0, 0, 0) &= \int_{P_{\sigma,n}\cap P_{0,n}\backslash P_{0,n}} A(\sigma, n)f(p_1)\bar{X}_{0,n}(p_1)dp_1 \\ &= \iint_{P_{\sigma,n}\cap P_{0,n}\backslash P_{0,n}\ P_{\sigma,n}\cap P_\sigma\backslash P_{\sigma,n}} f(p_2^* p_1)\bar{X}_{0,n}(p_1)\bar{X}_{\sigma,n}(p_2)dp_2dp_1 \end{aligned}$$

where dp_1 and dp_2 are normalized Haar measures. As before, define the function W_0 on $A_N \times \sigma A_n \times R$ by $W_0(p_2^* p_1) = X_{0,n}(p_1)X_{\sigma,n}(p_2)$ for p_1 in $P_{0,n}$ and p_2 in $P_{\sigma,n}$. This is well-defined since $X_{0,n} = X_{\sigma,n}$ on $P_{0,n} \cap P_{\sigma,n}$ and $P^*_{\sigma,n}P_{0,n} = A_N \times \sigma A_n \times R$. Also, by the Noether isomorphism theorem, $P_{0,n} \cap P_{\sigma,n}\backslash P_{0,n} \cong P_{\sigma,n}\backslash P^*_{0,n}P_{\sigma,n}$.

$$T_n(\sigma, 0)A(\sigma, n)f(0, 0, 0)$$
$$= \iint_{P_{\sigma,n}\backslash P^*_{0,n}P_{\sigma,n}\; P_{\sigma,n}\cap P_\sigma\backslash P_{\sigma,n}} f(p_2^* p_1)\, W(p_2^* p_1) dp_2 dp_1$$
$$T_n(\sigma, 0)A(\sigma, n)f(0, 0, 0) = \int_{P_{\sigma,n}\cap P_\sigma\backslash P^*_{0,n}P_{\sigma,n}} f(p)\, W(p) dp$$

where $p = p_2^* p_1$ and dp is normalized Haar measure. $P_{\sigma,n} \cap P_\sigma = \{(a, \sigma a) | a \text{ in } A_n\} \times \boldsymbol{R}$ and $\{(a, 0, 0) | a \text{ in } A_n\}$ is a complete set of inequivalent coset representatives of $P_{\sigma,n} \cap P_\sigma$ in the subgroup $P^*_{0,n} P_{\sigma,n}$.

Thus,

$$T_n(\sigma, 0)A(\sigma, n)f(0, 0, 0) = \int_{A_n} f(a, 0, 0)\bar{W}_0(a, 0, 0) da\ .$$

$W_0(a, 0, 0) = 1$ because $(a, 0, 0)$ is in $P_{0,n}$ and hence by normalization of Haar measure, we have $T_n(\sigma, 0)A(\sigma, n)f(0, 0, 0) = \sum_{A_n} f(p, 0, 0)$. In order to prove commutativity, we will show that

$$A^*(0, n)T_n(\sigma, 0)A(\sigma, n)f(0, 0, 0) = T(\sigma, 0)f(0, 0, 0)$$

where

$$A^*(0, n)f(X, Y, t) = \int_{A_n\backslash A} f((p, 0, 0)^*(X, Y, t))dt$$

and $A^*(0, n)A(0, n) = I$, the identity operator on $H(U^0)$. That is, $A^*(0, n) = A^{-1}(0, n)$.

LEMMA 1.

$$A^*(0, n)A(0, n) = I\ .$$

Proof.

$$A^*(0, n)A(0, n)f(0, 0, 0) = \int_{An\backslash A} A(0, n)f(p, 0, 0)dp$$
$$= \int_{A_n\backslash A} [A_n; A]^{-1} \sum_{A_n} f(p, k, 0)dp$$
$$(p, k, 0) = (p, 0, -B(p, k + \bar{p}k))^*(0, k + \bar{p}\cdot k, 0)$$
$$A^*(0, n)A(0, n)f(0, 0, 0) = \int_{A_n\backslash A} [A_n; A]^{-1} \sum_{A_n} \exp$$
$$-2\pi i B(p, k + \bar{p}k)f(0, k + \bar{p}k, 0)dp\ .$$

Let $h_p(k) = \exp - 2\pi i(B(p, k + \bar{p}k)f(0, k + \bar{p}k, 0))$.
$[A_n, A]^{-1} \sum_{A_n} h_p(k) = \sum_{A_n} \hat{h}_p(X_k)$ by the Poisson summation formula.

$$\begin{aligned} A^*(0, n)A(0, n)f(0, 0, 0) &= \int_{A_n\backslash A} \sum_{A_n} \hat{h}_p(X_k)dp \\ &= \int_{A_n\backslash A} \sum_{A_n} \int_A h_p(a)\bar{X}_k(a)dadp \\ &= \int_{A_n\backslash A} \sum_{A_n} \int_A \exp \\ &\quad -2\pi iB(p, a+\bar{p}a)e^{-2\pi iB(a,k)}f(0, a+\bar{p}a, 0)dadp . \end{aligned}$$

However, the sum over k in A_n is zero except for $a = 0$ in which case the right hand side becomes $f(0, 0, 0)$. □

Next,

$$A^*(0, n)T_n(\sigma, 0)A(\sigma, n)f(0, 0, 0) = \int_{A_n\backslash A} T_n(\sigma, 0)A(\sigma, n)f(p, 0, 0)dp$$

and

$$\begin{aligned} T_n(\sigma, 0)A(\sigma, n)f(p, 0, 0) &= [U_n^0(p, 0, 0)T_n(\sigma, 0)A(\sigma, n)f](0, 0, 0) \\ &= [T_n(\sigma, 0)A(\sigma, n)U(p, 0, 0)f](0, 0, 0) \\ &= \sum_{A_n} [U_n^0(p, 0, 0)f](a, 0, 0) \\ &= \sum_{A_n} f((a, 0, 0)^*(p, 0, 0)) . \end{aligned}$$

So,

$$\begin{aligned} A^*(0, n)T_n(\sigma, 0)A(\sigma, n)f(0, 0, 0) &= \int_{A_n\backslash A} \sum_{A_n} f((a, 0, 0)^*(p, 0, 0))dp \\ &= \int_A f(p', 0, 0)dp' \end{aligned}$$

where $p' = a^*p$ and dp' is Haar measure of A. The latter expression is precisely $T(\sigma, 0)f(0, 0, 0)$. We have shown that

$$A^*(0, n)T_n(\sigma, 0)A(\sigma, n) = T(\sigma, 0)$$

and by Lemma 1,

$$\begin{aligned} T_n(\sigma, 0)A(\sigma, n) &= A^{*-1}(0, n)T(\sigma, 0) \\ T_n(\sigma, 0)A(\sigma, n) &= A(0, n)T(\sigma, 0) . \end{aligned}$$

The back of the diagram (D4) commutes and this completes the proof of Theorem I. □

THEOREM 2. *For n appropriately chosen, that is sufficiently large, $K_n(\sigma) = K(\sigma)$ if σ is a rational number.*

Proof. By diagram of Figure 6 and f chosen in $H(U^\sigma)$, we have

$$T_n(\infty, \mu)T_n(\sigma, \infty)A(\sigma, n)f = T_n(\infty, \mu)A(\infty, n)T(\sigma, \infty)f$$

and hence

$$\begin{aligned} K_n(\sigma)T_n(0, \mu)T_n(\sigma, 0)A(\sigma, n)f &= A(\mu, n)T(\infty, \mu)T(\sigma, \infty)f \\ K_n(\sigma)T_n(0, \mu)A(0, n)T(\sigma, 0)f &= K(\sigma)A(\mu, n)T(0, \mu)T(\sigma, 0)f \\ K_n(\sigma)A(\mu, n)T(0, \mu)T(\sigma, 0)f &= K(\sigma)A(\mu, n)T(0, \mu)T(\sigma, 0)f\ . \end{aligned}$$

Therefore, $K_n(\sigma) = K(\sigma)$. □

Consider the diagonal plane of the diagram:

$$\begin{tikzcd} U_n^\infty \arrow[d, "T_n(\infty, 0)"'] & U^\infty \arrow[l, "A(\infty, n)"'] \arrow[d, "T(\infty, 0)"] \\ U_n^0 & U^0\ . \arrow[l, "A(0, n)"'] \end{tikzcd}$$

This diagram is commutative by the considerations in §II. Therefore we have the following:

PROPOSITION 4.
(1) $T(0, \mu)T(\infty, 0) = T(\infty, \mu)$
(2) $T_n(0, \mu)T_n(\infty, 0) = T_n(\infty, \mu)$.

Proof. By Schur's lemma,

$$T(\infty, \mu) = CT(0, \mu)T(\infty, 0)\ .$$

We will show that $C = 1$. For f in $H(U^\infty) \cap S(A \times 0 \times 0)$,

$$\begin{aligned} T(\infty, \mu)f(0, 0, 0) &= \sum_\Lambda f(\gamma, 0, 0) \\ CT(0, \mu)T(\infty, 0)f(0, 0, 0) &= C\sum_\Lambda T(\infty, 0)f(0, \gamma, 0) \\ &= C\sum_\Lambda \int_A f(p, p\gamma + \gamma, B(p, \gamma))dp\ . \end{aligned}$$

Noting that $(p, p\gamma + \gamma, B(p, \gamma)) = (0, p\gamma + \gamma, 0)^*(p, 0, B(p, \gamma))$ then

$$\begin{aligned} CT(0, \mu)T(\infty, 0) &= C\sum_\Lambda \int_A e^{2\pi i B(p,\gamma)} f(p, 0, 0)dp \\ &= C\sum_\Lambda \hat{f}(r, 0, 0)\ . \end{aligned}$$

By the Poisson summation formula, $C = 1$ (recall that $\Lambda = \Lambda^\perp$). Also, $T_n(\infty, \mu) = T_n(0, \mu)T_n(\infty, 0)$ follows by the commutativity of diagrams. □

COROLLARY 1. $T(\infty, 0)T(\sigma, \infty) = K(\sigma)T(\sigma, 0)$.

Proof.

$$\begin{aligned} T(\infty, 0)T(\sigma, \infty) &= T(0, \mu)^{-1}T(\infty, \mu)T(\sigma, \infty) \\ &= T(0, \mu)^{-1}K(\sigma)T(0, \mu)T(\sigma, 0) \\ &= K(\sigma)T(\sigma, 0) . \end{aligned} \qquad \square$$

The following results answer some open questions from Chapter II. It has been shown that

$$\begin{aligned} T(0, \mu)T(\sigma, 0)\delta &= T(0, \mu)W_0 \\ &= K(\sigma)\theta_\sigma \end{aligned}$$

and

$$\begin{aligned} T(\infty, \mu)T(\sigma, \infty)\delta &= T(\infty, \mu)W \\ &= \theta_\sigma . \end{aligned}$$

Those same arguments apply to the operators in §III

$$\begin{aligned} \widetilde{T}_n(0, \mu)\widetilde{T}_n(\sigma, 0)\delta &= \psi_\sigma \\ \widetilde{T}_n(\infty, \mu)\widetilde{T}_n(\sigma, \infty)\delta &= \theta_\sigma \end{aligned}$$

and we have

COROLLARY 2. *$\theta_\sigma(\tilde{x}) = K(\sigma)\psi_\sigma(\tilde{x})$ for $\tilde{x}$ in $\widetilde{B}_n$ and θ_σ and ψ_σ are nonzero functions on $(\widetilde{B}_n, *)$.*

Proof.

$$\theta_\sigma(\tilde{x}) = K_n(\sigma)\psi_\sigma(\tilde{x}) .$$

$K_n(\sigma)$ is the intertwining constant for the operators of the compact group and by Theorem 2, of this section $K_n(\sigma) = K(\sigma)$. ψ_σ is a nonzero function on $(B_n, *)$—it is the image of a nontrivial function under an intertwining operator between two irreducible, equivalent representations. Also, $K_n(\sigma) = K(\sigma) \neq 0$. Therefore θ_σ is also nontrivial. $\square$

V. $K(\sigma)$. In this section, we will attempt to compute $K(\sigma)$ more explicitly. At the very least, we will derive an interesting identity involving $K(\sigma)$ and present some examples to illustrate the depth of these results.

From Corollary 1, §IV, we have $T(\infty, 0)T(\sigma, \infty) = K(\sigma)T(\sigma, 0)$. This result is the basis for a remarkable identity. Let $\Phi_\sigma(X) = \exp 2\pi i\sigma\ell(x)$ for x in the nilpotent algebra, A and $l(\cdot)$ the scalar log function as defined earlier. As usual, denote the Fourier Transform of a function, g, over the algebra A by $\hat{g}$.

THEOREM 1. *If n is the vector space dimension of A, then $K(\sigma)$ is a factor of the proportionality constant in the following distributional Fourier transform identity*:

$$\hat{\Phi}_\sigma(\cdot) = \sigma^{-n}K(\sigma)\Phi_{-\sigma}(\sigma^{-1}\cdot) .$$

Proof. Let f be in $C^\infty(U^\sigma)$ and have compact support in $P_\sigma\backslash B$.

$$T(\infty, 0)T(\sigma, \infty)f(0, 0, 0) = K(\sigma)T(\sigma, 0)f(0, 0, 0)$$
$$\int_A T(\sigma, \infty)f(p, 0, 0)dp = K(\sigma)\int_A f(p, 0, 0)dp$$
$$\int_A\int_A f((0, p', 0)^*(p, 0, 0))dp = K(\sigma)\int_A f(p, 0, 0)dp .$$

Now

$$(0, p', 0)^*(p, 0, 0) = (p, p', 0)$$
$$(p, p', 0) = (a, \sigma a, 0)^*(b, 0, 0)$$

where $a = \sigma^{-1}p'$, $b = \overline{\sigma^{-1}p'^*p}$.

From this $(p^*\bar{b}) = p'$. Since this transformation of coordinates is unipotent, we have $\sigma^n db = dp'$, where n is the vector space dimension of the algebra, A. Thus the left hand side of the equality is, after a change of coordinates,

$$\sigma^n\int_A\int_A \exp 2\pi i\sigma\ell(p^*\bar{b})f(b, 0, 0)dbdp$$
$$= \sigma^n\int_A\int_A e^{-2\pi iB(p,\sigma\bar{b})}e^{2\pi i\sigma\ell(\bar{b})}f(b, 0, 0)dbe^{2\pi i\sigma\ell(p)}dp .$$

With the change $b \to \bar{b}$, $db = d\bar{b}$ by the invariance of Haar measure and the inner integrand becomes $\exp(-2\pi iB(p, \sigma b))\exp 2\pi i\sigma\ell(b)f(\bar{b}, 0, 0)$. However, since f is in $H(U^\sigma)$,

$$e^{2\pi i\sigma\ell(b)}f(\bar{b}, 0, 0) = X_\sigma(b, \sigma b, 0)f(\bar{b}, 0, 0)$$
$$= f(0, \sigma b, 0) . \tag{1}$$

Let $g(b) = f(0, b, 0)$ and the previous integral is now

$$\sigma^n\int_A\int_A e^{-2\pi iB(p,\sigma b)}g(\sigma b)dbe^{2\pi i\sigma\ell(p)}dp .$$

Again change coordinates by $b \to \sigma^{-1}b$ and we have

$$\int\hat{g}(p)\exp^{2\pi i\ell(p)}dp = \int\hat{g}(p)\Phi_\sigma(p)dp .$$

On the other side of our operator identify involving $K(\sigma)$, we have

$$K(\sigma)\int_{A} f(p, 0, 0)dp \ .$$

Invoking the algebraic properties of (1),

$$f(p, 0, 0) = \exp(-2\pi i \measuredangle(\bar{p}))f(0, \sigma\bar{p}, 0)$$

and this changes the latter integral (after the coordinate transformation $p \rightarrow \sigma^{-1}\bar{p}$)

$$\sigma^{-n}K(\sigma)\int_{A} \exp(-2\pi i\sigma\measuredangle(\sigma^{-1}p)) = \sigma^{-n}K(\sigma)\int_{A} \Phi_{-\sigma}(\sigma^{-1}p)g(p)dp \ . \qquad \square$$

While this distributional formula does not provide us with an explicit formula for $K(\sigma)$, it will allow us to compute $K(\sigma)$ in some examples that follow. The examples, at least the first, indicate the complexity of the integrals involved. This complexity does give some promise for future research. First, however, the examples to illustrate the application of our results.

EXAMPLE 1. Let $N^1 = \mathrm{span}_R\{e\}$, where $B(e, e) = 1$ and $e^i = 0$ for $i > 1$. Assume $\sigma = a/b$, $(a, b) = 1$. Let $X = t\cdot e$ for t in $\boldsymbol{R}$ and $\gamma = k\cdot e$ for k in $\boldsymbol{Z}$. By definition, $\measuredangle(X) = -(1/2)B(te, te) = -(1/2)t^2$. As a consequence, $\Phi_\sigma(X) = \exp -\pi i\sigma t^2$. The Fourier transform identity

$$\hat{\Phi}_\sigma(X) = e^{-\pi i/4}\Phi_{-\sigma}(\sigma^{-1}X)/\sqrt{\sigma}$$

is well-known. Thus, $K(\sigma) = \sqrt{\sigma}\, e^{-\pi i/4}$. From the definitions in §III, Λ_n is isomorphic to $b!Z$ and Λ_n' is isomorphic to $bn!\,Z$. Thus $\Lambda_n'\backslash\Lambda$ is isomorphic to the integers modulo $bn!$ and $\Lambda_n\backslash\Lambda$ to the integers modulo $n!$. The μ-theta function identity of §II (with $C = K(\sigma)$) is

$$(2) \qquad \sum_{k=0}^{n!-1} \exp\pi i\left(\frac{b}{a}\right)(k + t)^2 = K(\sigma)\sum_{k=0}^{bn!-1} \exp\pi i\left(-2kt - \frac{a}{b}k^2\right) .$$

Let $t = 0$ and after cancellation, we have

$$\sum_{k=0}^{a-1} \exp\pi i\,\frac{b}{a}\,k^2 = \sqrt{\frac{a}{b}}e^{-\pi i/4}\sum_{k=0}^{b-1} \exp - \pi i\frac{a}{b}k^2 \ .$$

This is, of course, the familiar Gaussian Summation Formula. Formula (2) may be intepreted a transformation of the classical sums (with some restrictions on t).

EXAMPLE 2. Let N^2 be the nilpotent algebra with basis $\{e_1, e_2\}$

where $e_1^2 = e_2$, $e_1^3 = 0$ and $B(e_i, e_j) = 1$ if and only if $i + j = 3$ and zero else. For $X = t_1e_1 + t_2e_2$ and t_1, t_2 in $\boldsymbol{R}$,

$$\measuredangle(X) = \measuredangle(t_1e_1 + t_2e_2) = \frac{1}{3}t_1^3 - t_1t_2 .$$

Let g be in $C_C^\infty(R^2)$, then

$$\begin{aligned}&\int_A \Phi_\sigma(X)\hat{g}(X)dX \\ &\quad = \int_{R^4} \exp 2\pi i\sigma\frac{1}{3}t_1^3 \exp(-2\pi i(\sigma t_1 + s_1)t_2 + t_1s_2)g(s_1, s_2)ds_1ds_2dt_1dt_2 .\end{aligned}$$

Note that the integral, $\int_R \cdot dt_2$ is zero except when $s_1 = -\sigma t_1$. This integral then reduces to:

$$\begin{aligned}&\int_{R^2} \exp 2\pi i\sigma t_1^3/3 \exp - 2\pi i t_1 s_2 g(-\sigma t_1, s_2)ds_2dt_1 \\ &\quad = \sigma^{-1}\int_{R^2} \exp - 2\pi i\sigma\left(\frac{1}{3}(\sigma^{-1}t_1)^3 - \sigma^{-2}t_1s_2\right)g(t_1, s_2)ds_2dt_1 \\ &\quad = \sigma^{-1}\int_A \Phi_{-\sigma}(\sigma^{-1}x)g(x)dx .\end{aligned}$$

Therefore, $K(\sigma) = \sigma$.

Let $X = t_1e_1 + t_2e_2$ as before and $\gamma = k_1e_1 + k_2e_2$ for k_1 and k_2 in $\boldsymbol{Z}$. The μ-theta function identity becomes

$$\begin{aligned}&\sum_{A'_n\setminus A} \exp 2\pi i - \sigma^{-1}\left[\frac{1}{3}\sigma^{-1}(t_1^3 + k_1^3) + (t_1 - k_1)(t_2 - k_2)\right] \\ &\quad = K(\sigma)\sum_{A_n\setminus A} \exp 2\pi i\left(\frac{1}{3}k_1^3 - k_1k_2 + k_1t_2 + k_2t_1\right) .\end{aligned}$$

It is not difficult to show that A_n is the span over $\boldsymbol{Z}$ of $(n!)^2e_1$ and $n!e_2$ and A'_n is span over $\boldsymbol{Z}$ of $b^2(n!)^2e_1$ and $bn!e_2$. For $t_1 = 0$,

$$\begin{aligned}&\sum_{k_1=0}^{b^2(n!)^2-1}\sum_{k_2=0}^{bn!-1} \exp 2\pi i - \frac{b}{a}\left(\frac{1}{3}\frac{b}{a}k_1^3 - k_1k_2\right) \\ &\quad = \frac{a}{b}\sum_{k_1=0}^{(n!)^2-1}\sum_{k_2=0}^{n!-1} \exp 2\pi i\frac{a}{b}\left(\frac{1}{3}k_1^3 - k_1k_2\right) .\end{aligned}$$

Since $n \geqq a$, $n \geqq b$ these sums reduce by cancellations to

$$\sum_{k_1=0}^{3a-1}\sum_{k_2=0}^{a-1} \exp 2i - \frac{b}{a}\left(\frac{1}{3}\frac{b}{a}k_1^3 - k_1k_2\right) = \frac{a}{b}\sum_{k_1=0}^{3b=1}\sum_{k_2=0}^{b-1} \exp 2i\frac{a}{b}\left(\frac{1}{3}k_1^3 - k_1k_2\right) .$$

However, the sum over k_2 in both instances is trivial except when $k_1 = 0$, the identity is then: $a = a/b \cdot b$.

The results from Example 2 are somewhat disappointing, as we had hoped to derive a formula for the cubic Gauss Sums. The difficulty might be a peculiarity of the algebra. It could be, however, the tendency of nilpotent algebras to behave in a general fashion akin to the quadratic case. Nonetheless, it serves as a basis for future research. It should also be mentioned that Example 1 was not just coincidental—it is the motivation for this group construction, [5].

In the final analysis, there are several problems that come immediately to mind. The first is the possibility of using the adele group constructions and thereby obtaining a product formula for the μ-theta functions and distributions. Secondly, there seems to be a lot more structure to $K(\sigma)$ as indicated by the examples and the proof of the first theorem of this section. Finally, the analogy of these functions with classical theta functions certainly begs some consideration to establish formulas comparable to these of the classical case.

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