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THE LEFSCHETZ NUMBER AND BORSUK-ULAM THEOREMS

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Let M be a manifold, with or without boundary, which is dominated by a finite complex. Let G be a finite group which acts faithfully and freely on M. Let $f: M \to M$ be a G-map. Let Λ_f denote the Lefschetz number of f and let $o(G)$ denote the order of G. The main result states, under the conditions above, that $o(G)$ divides Λ_f. Even in the case of compact M this result was not widely known. We use Wall's finiteness obstruction theory to extend the result from compact M to finitely dominated M.

The remainder of the paper is devoted to various easy applications of the result. In Theorem 5 we assume that $\pi_i(M)$ is finitely generated for all $i > 1$. Then we show that if $\pi_1(M)$ has torsion, $\pi_*(M)$ cannot be only torsion.

In Theorem 6, we have a connected Lie group L acting on M and f is an L-map. We show that the orbit map $\omega: L \to M$ induces the trivial homomorphism on fundamental groups if $\Lambda_f \neq 0$. This implies that the action of L on M can be lifted to any regular covering space.

We show that any linear transformation $T: R^n \to R^n$ which commutes with the based free action of a finite group G of order greater than 2 must have a non-negative determinant (Theorem 8).

Then we come to the Borsuk-Ulam type results. We consider maps $f: (C^{n+1} - 0) \to C^n$. A primitive k-root of unity ξ gives rise to a free Z_k-action on C^n. We show that the equation $\sum_{i=0}^{k-1} \bar{\xi}^i f(\xi^i x) = 0$ always has a solution $x \in C^{n+1} - 0$. This result gives various conditions on the degeneracy of the images of the orbit of the Z_k action in C^n. In particular, we show that if $f: S^n \to R^r$ and if $n \geqq r(p-1)$, then some orbit of the Z_p-action must be mapped into a point. The proof uses the equation above and Vandermonde determinants.

2. Free actions and the Lefschetz number.

A manifold M (or space) is *dominated* by a finite complex K if there exists maps $f: M \to K$ and $g: K \to M$ such that $g \cdot f$ is homotopic to the identity of M. We will need various facts about finitely dominated spaces in order to prove the result that $o(G)$ divides Λ_f for noncompact M. It is easily shown that this is true for compact M. We use the theory of C.T.C. Wall, [8], to extend to the noncompact case.

LEMMA 1. *Let M be a finitely dominated manifold. The orbit* *space M/G is a finitely dominated manifold.*

Proof. Consider the universal principal G-bundle $G \to E_G \to B_G$. Since G acts on M, we may replace the fibre to obtain the bundle $M \to M_G \to B_G$. Since G acts freely on M, we know that M_G is a homotopy equivalent to M/G. Since G is finite and acts freely on M, we see that $M \to M/G$ is a covering, so M/G is a manifold, and hence it is finite dimensional.

We know from Wall's work [8] that a space is finitely dominated if and only if it is homotopy equivalent to a finite dimensional CW-complex and also homotopy equivalent to a possibly different CW-complex of finite type (i.e., a complex whose n-skeletons are finite complexes for all n). Thus M is homotopy equivalent to a complex of finite type. Since M_G is the total space of a fibration whose base and fibre are of finite type, M_G is homotopy equivalent to a complex of finite type [4; Lemma 1.1]. Also, by the first paragraph, M_G is homotopy equivalent to M/G which is a finite dimensional complex. So M/G is finitely dominated.

REMARK. The above lemma follows from Lemma 5.5 in [2].

Now we come to the main result.

THEOREM 2. *Let M be a finitely dominated manifold, and let G act freely and faithfully on M and $f: M \to M$ be a G-map. Then $o(G)$ divides Λ_f.*

Proof. First we show the result for a compact manifold M with a finite group G acting freely on M and a G-map $f: M \to M$. We obtain a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & M \\ \downarrow & & \downarrow \\ M/G & \xrightarrow{\bar{f}} & M/G \end{array}$$

where the vertical maps are covering projections and $\bar{f}$ is induced by f. Now M/G is a finite CW-complex and we may adjust $\bar{f}$ by a homotopy so that it has finitely many isolated fixed points. Now f restricted to a fibre over a fixed point maps the fibre onto itself and it either leaves all the points fixed or leaves no points fixed. Thus if x is a fixed point of f then all the points in the fibre, which is an orbit of G, are fixed, and they all must have the same fixed point index. The fibre contains $o(G)$ points, so $o(G)$ must divide Λ_f.

Now suppose that M is a finitely dominated manifold. It follows from Wall's theory [8] that the cartesian product of M with a finite complex, whose Euler-Poincare number is zero, must be a homotopy equivalent to a finite complex. Thus we have a compact manifold K and homotopy equivalences $h: K \to S^3 \times (M/G)$ and $h^{-1}: S^3 \times (M/G) \to K$. We let G act on $S^3 \times M$ by letting $g(s, m) = (s, g(m))$. Then G acts freely on $S^3 \times M$ and gives rise to a diagram

$$\begin{array}{ccc} \tilde{K} & \xrightarrow{\tilde{h}} & S^3 \times M \\ \downarrow{\scriptstyle p} & & \downarrow \\ K & \xrightarrow{h} & S^3 \times (M/G) \end{array}$$

where $\tilde{K} \to K$ is in the pull back of the G-bundle $S^3 \times M \to S^3 \times (M/G)$. So $\tilde{K} \to K$ is a covering space and $\tilde{h}$ is a G-map. Also there is some lifting $\tilde{h}^{-1}$ of h^{-1} which is a G-map and $\tilde{h} \circ \tilde{h}^{-1}$ is homotopic to the identity by equivariant homotopies.

Now let $F: S^3 \times M \to S^3 \times M$ be given by $F(s, m) = (s_0, f(m))$ where $s_0 \in S^3$ is a base point. F is a G-map and so the composition $\tilde{h}^{-1}F\tilde{h}$: $\tilde{K} \to K$ is a G-map. Now $\Lambda_{\tilde{h}^{-1}F\tilde{h}} = \Lambda_F = \Lambda_f$ and by the theorem for compact complexes $o(G)$ divides $\Lambda_{\tilde{h}^{-1}F\tilde{h}}$ and hence it divides Λ_f.

REMARK. An earlier version of this theorem was proved in [1; Theorem 4] by transfer methods for G a cyclic group and M homotopy equivalent to a finite complex. Nakaoka in [7] developed a transfer based on the coincidence number $\Lambda_{f,g}$ for maps f and g from M to itself where M is a *compact* manifold. Using a result related to these transfers he succeeded in showing that $o(G)$ divides $\Lambda_{f,g}$ where f and g are G-maps. This result gives Theorem 2 in the compact case. The compact case was originally proved by G. Hirsch [3a].

COROLLARY 3. *$o(G)$ divides $\chi(M)$, the Euler-Poincare number.*

COROLLARY 4. *If $f: M \to M$ is a G-map where G acts freely and is finite, then f cannot be homotopic to a constant map.*

3. The fundamental group. At this time, despite our great knowledge of homotopy groups, we do not possess a single example of a finite complex, with nontrivial higher homotopy groups, whose homotopy groups are completely known. The following result illustrates a difficulty. It generalizes the famous result that if π has torsion, then $K(\pi, 1)$ cannot be a finite complex.

THEOREM 5. *Assume that M is a manifold such that $\pi_i(M)$ is finitely generated for all $i > 1$. Then if some nontrivial element of $\pi_1(M)$ has finite order, some element of $\pi_i(M)$ does not have finite order for some $i > 1$. (Here we need not assume that M has finite type.)*

Proof. Consider the universal cover $\tilde{M}$ of M. Now $\tilde{M}$ has finite dimension and $\pi_i(\tilde{M}) \cong \pi_i(M)$ is finitely generated for $i > 1$, and $\pi_1(\tilde{M}) = 0$. Hence it follows from Wall's theory that $\tilde{M}$ is homotopy equivalent to a finite complex. Now $\pi_1(M)$ acts freely on $\tilde{M}$ and if it contains an element of finite order, then a finite cyclic group G acts freely on $\tilde{M}$ and $o(G)$ divides $\chi(\tilde{M})$ by Corollary 3.

Now if $\pi_i(M) \cong \pi_i(\tilde{M})$ were all torsion groups for $i > 1$, then by the Hurewicz theorem mod finite groups, the homology groups of $H_i(\tilde{M})$ would be finite for all i. Thus $\chi(\tilde{M})=1$ so $o(G)=1$ which contradicts the hypothesis that G is not trivial.

Now we turn to the action of a Lie group L on a compact manifold M with or without boundary. There is the orbit map or evaluation map $\omega: L \to M$ given by evaluating $l \in L$ at a base point $m_0 \in M$. Thus $\omega: l \mapsto l(m_0)$. We are concerned with the induced homomorphism $\omega_*: \pi_1(L) \to \pi_1(M)$.

THEOREM 6. *Let L be a connected Lie group acting on a compact manifold M with or without boundary. Suppose there is an equivariant map $f: M \to M$ such that $\Lambda_f \neq 0$. Then $\omega_*: \pi_1(L) \to \pi_1(M)$ is trivial.*

Proof. Every element $\alpha \in \pi_1(L)$ can be represented by a homomorphism $S^1 \to L$. Thus we need only prove the theorem in the case of $L = S^1$.

We shall show that the S^1 action must have a fixed point and this will give us $\omega_* = 0$. There is an element $t \in S^1$ such that $\{t^n \mid \text{all } n \in \mathbf{Z}\}$ is dense in S^1. There is a sequence $s_p \in S^1$ for p a prime such that s_p converges to t and s_p has order p. Since only a finite number of primes divide Λ_f, all but a finite number of s_p must have a fixed point x_p by Theorem 2. Now the sequence $\{x_p\}$ must have a cluster point x since M is compact, thus x must be a fixed point for t and hence a fixed point for all of S^1.

COROLLARY 7. *With the hypotheses above, the action of L on M can be lifted to any regular covering $\tilde{M}$ of M.*

Proof. By Corollary 11 of [3], $\omega_* = 0$ is sufficient.

REMARK. Theorem 6 is related to Theorem 5 of [1]. Using the first paragraph of the proof of Theorem 6 and Theorem 5 of [1] we can eliminate the compactness condition on M. Theorem 6 and the result from [1] that $\omega_*: \tilde{H}_*(L) \to \tilde{H}_*(M)$ is zero for integral homology under the hypothesis of Theorem 6 are interesting to compare.

4. Borsuk-Ulam theorems. In this section we study spheres and complex and real vector spaces. The classical Borsuk-Ulam theorem involves a continuous map $f: (R^{n+1} - 0) \to R^n$ and states that some orbit of the action $x \mapsto -x$ on $R^{n+1} - 0$ is mapped onto a single point of R^n. We investigate the situation of a map $(C^{n+1} - 0) \xrightarrow{f} C^n$ and ask if an orbit of the action $x \mapsto \xi x$ is mapped onto a single point.

But before we begin with the Borsuk-Ulam theorems we prove a different result on the determinant of a linear transformation. We say that a finite group G acts based freely on R^n if it acts faithfully on R^n and freely on $R^n - 0$. We say a linear transformation $T: R^n \to R^n$ *commutes* with the action of G if T commutes with every element $g \in G$.

THEOREM 8. *If G acts based freely on R^n and if T is a linear transformation which commutes with the action of G, then* $\det T \geqq 0$ *or* $o(G) = 1$ *or* 2.

Proof. If $\det T \neq 0$, then T is an isomorphism and we can regard $T: R^n - 0 \to R^n - 0$. Now $\Lambda_T = 1 + (-1)^{n-1} \deg T$ and $\deg T = \det T / |\det T|$. Assume $o(G) > 2$. Then n must be even since otherwise $\chi(R^n - 0) = 2$ and $o(G)$ divides $\chi(R^n - 0)$.

Thus $\Lambda_T = 1 - (\det T / |\det T|)$ and so Λ_T can either be 2 or 0. But it cannot be 2 since $o(G) > 2$ and divides Λ_T. Then $\Lambda_T = 0$ and hence $\det T > 0$.

Now we come to the main underlying result for the Borsuk-Ulam theorem.

THEOREM 9. *Let G be a finite group acting based freely on a vector space V and let W be a proper invariant subspace. Then any G-map $f: (V - 0) \to W$ must contain 0 in its image.*

Proof. Suppose 0 is not in the image of f. Then the composition $(V - 0) \to W - 0 \hookrightarrow V - 0$ is a G-map and it must be homotopic to a constant, so by Corollary 4 either G is trivial or $f(x) = 0$ for some $x \in V - 0$.

THEOREM 10. *Let ξ be a primitive kth root of unity. Let $f\colon C^{n+1} - 0 \to C^n$ be any continuous map. Then there is an $x \in C^n - 0$ such that $\sum_{i=0}^{k-1} \bar{\xi}^i f(\xi^i x) = 0$.*

Proof. This follows from Theorem 9 after noting that $x \mapsto \xi x$ leads to a based free action of Z_k on $C^{n+1} - 0$ and that C^n can be regarded as an invariant subspace and $F(x) = \sum_{i=1}^{k} \bar{\xi} f(\xi^i x)$ is an equivariant map.

Observe that we may let S^{2n+1} be the unit sphere in C^{n+1} and Theorem 10 guarantees that $\sum_{i=1}^{k} \bar{\xi}^i f(\xi^i x) = 0$ for some $x \in S^{2n+1}$ and $f\colon S^{2n+1} \to C^n$.

We will call the set $\{x, \xi x, \cdots, \xi^{k-1} x\}$ a k-orbit on S^{2n+1} where ξ is a primative kth root of unity.

COROLLARY 11. *Given k and $f\colon S^{2n+1} \to C^n$, there is a k-orbit whose image lies in a $k - 2$ dimensional complex hyperplane.*

Proof. Choose x so that $\sum \bar{\xi}^i f(\xi^i x) = 0$. Let $x_i = f(\xi^i x)$. Consider the set of $k - 1$ vectors. $\{(x_1 - x_k), \cdots, (x_{k-1} - x_k)\}$. These lie in a $k - 2$ dimensional subspace since they are linearly dependent since they satisfy $\sum_{i=1}^{k} \bar{\xi}^i (x_i - x_k) = 0$. Hence the translation by x_k of this subspace is a $k - 2$ dimensional hyperplane which contains the vectors $x_1, \cdots, x_k$.

COROLLARY 12. *Given k and $f\colon S^{2n+1} \to R^n$, there is a k-orbit whose image under f lies in a $k - 3$ dimensional real hyperplane.*

Proof. The equation $\sum_{i=1}^{k} \bar{\xi}^i f(\xi^i x) = 0$ gives two equations, one for the real part and one for the imaginary part.

COROLLARY 13. *A map $f\colon S^{2n+1} \to R^n$ carries some 3-orbit to a point.*

COROLLARY 14. *A map $f\colon S^{2n+1} \to R^n$ carries some 4-orbit into one or two points so the two pairs of antipodal points are each carried into a point.*

Proof. Let $x_j = f(\xi^j x)$. Then

$$-ix_1 - x_2 + ix_3 + x_4 = 0 .$$

So $x_1 = x_3$ and $x_2 = x_4$.

THEOREM 15. *Suppose p is a prime and $f\colon S^n \to R^r$. If $n \geqq r(p - 1)$, then there is a p-orbit whose image is a single point.*

Proof. For $p = 2$ this is the classical Borsuk-Ulam theorem. For $p > 2$ we see that n must be odd and $r(p-1)$ must be even so we may assume that $n > r(p-1)$.

Let $n = 2k-1$. Then we extend $f: S^{2k-1} \to R^r$ radially to a map $g: \boldsymbol{C}^k \to R^r$ by letting $g(z) = \|z\| f(z/\|z\|)$. Then we double the dimension of the range and domain and define a map $G: \boldsymbol{C}^{2k} \to \boldsymbol{C}^r$ by setting $G(z_1, z_2) = g(z_1) + ig(z_2)$ where $(z_1, z_2) \in \boldsymbol{C}^k \oplus \boldsymbol{C}^k \cong \boldsymbol{C}^{2k}$. Now we define a map $F: (\boldsymbol{C}^{2k} - 0) \to \boldsymbol{C}^{r(p-1)}$ by letting $F(z) = (G(z), G(z)^2, \cdots, G(z)^{p-1})$ where we understand that a vector $v = (v_1, \cdots, v_r)$ raised to the jth power is the vector $v^j = (v_1^j, \cdots, v_r^j)$.

Now since $2k > r(p-1)$ (because $n \geqq r(p-1)$ by hypothesis), we can apply Theorem 10 which guarantees a solution to the equation

$$\sum_{j=0}^{p-1} \bar{\xi}^j F(\xi^j x) = 0 .$$

Let $x_j = G(\xi^j x)$ and $x_j^k = (G(\xi^j x))^k$. Then we obtain the vector equations

$$\sum_{j=0}^{p-1} \bar{\xi}^j x_j = 0$$
$$\sum_{j=0}^{p-1} \bar{\xi}^j x_j^2 = 0$$
$$\vdots$$
$$\sum_{j=0}^{p-1} \bar{\xi}^j x_j^{p-1} = 0 .$$

These equations are vector equations since x_j^k is a vector. If we look at, say, the first coordinates of the vectors $\{x_j^k\}$ and use x_j^k to denote its own first coordinate then the set of equations can be written in matrix form as

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_{p-1} \\ & & \vdots & \\ x_1^{p-1} & & \cdots & x_{p-1}^{p-1} \end{pmatrix} \begin{pmatrix} 1 \\ \bar{\xi} \\ \vdots \\ \bar{\xi}^{p-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} .$$

Now the matrix gives the Vandermonde matrix and

$$\begin{vmatrix} 1 & 1 \cdots & 1 \\ x_1 & \cdots\cdots & x_{p-1} \\ & \vdots & \\ x_1^{p-1} & \cdots\cdots & x_{p-1}^{p-1} \end{vmatrix} = \prod_{i<j} (x_j - x_i) .$$

Then an easy induction argument shows that the row echelon form of the Vandermonde matrix consists of rows composed of entries which are 0's or 1's. We have the equation

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ & & & \vdots & & \\ & & & \vdots & & \end{pmatrix} \begin{pmatrix} 1 \\ \bar{\xi} \\ \vdots \\ \bar{\xi}^{p-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since p is a prime and ξ is a primitive pth root of unity the only sum of these roots which equals zero is the sum of them all, $1 + \xi + \cdots + \xi^{p-1} = 0$. Thus the row echelon form must be

$$\begin{pmatrix} 1 & 1 \cdots 1 \\ 0 & 0 \cdots 0 \\ & \vdots \\ 0 & \cdots 0 \end{pmatrix}$$

which can only happen if $x_1 = x_2 = \cdots = x_{p-1}$. Thus all the 1st coordinates of the vectors $x_1, \cdots, x_{p-1}$ agree, and similarly all the jth coordinates agree. So

$$G(x) = G(\xi x) = \cdots = G(\xi^{p-1}x) \text{ for some } x \in \boldsymbol{C}^{2k} - 0 .$$

This implies that

$$g(z) = g(\xi z) = \cdots = g(\xi^{p-1}z)$$

for some $z \in \boldsymbol{C}^k - 0$ and since g is a radial extension of f we see that f maps an orbit to a single point.

REMARK. Theorem 15 was proved by Munkholm [5] and Nakaoka [6]. In both cases their methods apply in a more general situation. On the other hand Corollary 14 is new and is different from Mumkholm's and Nakaoka's work in that it considers an action of a cyclic group of order 4 and 4 is not a prime.

REFERENCES

1. A. Casson and D. H. Gottlieb, *Fibrations with compact fibres*, Amer. J. Math., **99**, (1977), 159-189.

2. A. L. Edmonds, *Equivariant regular neighborhoods in transformation groups*, Proc. of the Conference at the University of Newcastle upon Tyne, August 1976, Editor Czes Kosniowski, Cambridge University Press, 51-69.

3. D. H. Gottlieb, *Lifting actions in fibrations*, Geometric applications of homotopy theory I, (Proceedings of the Evanston Conference, 1977.) Lecture Notes in Mathematics #657, Springer-Verlag, 217-254.

3a. G. Hirsch, Ann. Sci. Ecole. Sup. (3), **60** (1943), 113-142.

4. G. Mislin, *Conditions for finite domination for certain complexes*, Proceedings of the Vancouver Conference in Algebraic topology, 219-224, Lecture Notes in Mathematics #673, Springer-Verlag.

5. H. J. Munkholm, *Borsuk-Ulam type theorems for proper Z_p-actions on (mod p homology) n-spheres*, Math. Scand., **24** (1969), 167–185.
6. M. Nakaoka, *Generalizations of the Borsuk-Ulam theorem*, Osaka J. Math., **7** (1970), 423–441.
7. ———, *Coincidence Lefschetz numbers for a pair of fibre preserving maps*, J. Math. Soc. Japan, **32** (1980), 751–779.
8. C.T.C. Wall, *Finiteness conditions for CW-complexes*, Ann. of Math., **81** (1965), 56–69.

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