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RANK$_2$ P-GROUPS, $P>3$, AND CHERN CLASSES

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In this paper, the integral cohomology ring of a Blackburn's type III rank$_2$ p-group ($p>3$) (the rank of a p-group is the rank of a maximal elementary abelian subgroup) is computed and the even dimensional generators are expressed in terms of Chern classes of certain group representations. Then this group satisfies Atiyah's conjecture on the coincidence of topological and algebraic filtrations defined on the complex representation ring of the group.

Let G be any finite group and $R(G)$ the complex representation ring of G. There is a convergent spectral sequence $\{E_r^{i,j}: 2 \leqq r \leqq \infty\}$ such that

$$E_2^{i,\text{even}} = H^i(G, Z),\ E_2^{i,\text{odd}} = 0, \text{ and } E_2^{i,j} = R_i^{\text{top}}(G)/R_{i+1}^{\text{top}}(G)$$

where

$$R(G) = R_0^{\text{top}}(G) \supseteqq R_1^{\text{top}}(G) \supseteqq \cdots R_{2k-1}^{\text{top}}(G) = R_{2k}^{\text{top}}(G) \supseteqq R_{2k+1}^{\text{top}} = \cdots$$

is a topologically defined even filtration on $R(G)$. $R(G)$ can be given an algebraic filtration by using the Grothendick operations γ^i; thus $R_{2k}^{\gamma}(G)$ is the subgroup generated by monomials $\gamma^{n_1}(\xi_1) \cdots \gamma^{n_r}(\xi_r)$, $n_1 + \cdots + n_r \geqq k$ and $\xi_1, \cdots, \xi_r$ elements of the augmentation ideal of $R(G)$. The definition is completed by $R_0^{\gamma}(G) = R(G)$ and $R_{2k-1}^{\gamma}(G) = R_{2k}^{\gamma}(G)$. $R(G)$ is a filtered ring with respect to both filtrations, $R_{2k}^{\gamma}(G) \subseteqq R_{2k}^{\text{top}}(G)$ for all k, and the equality holds for $k = 0, 1$, and 2 [2]. Atiyah conjectured that $R_{2k}^{\text{top}}(G) = R_{2k}^{\gamma}(G)$, $k \geqq 0$ and showed that a group G satisfies this conjecture if the even dimensional subring $H^{\text{even}}(G, Z)$ of the integral cohomology ring $H^*(G, Z)$ is generated by Chern classes of representations of the group G. Though the alternating group on four elements A_4 is a counter example [13], a long standing conjecture is that the two filtrations coincide when G is a finite p-group.

Rank$_2$ p-groups, $p > 3$, are classified by N. Blackburn [8, staz 14.4] as follows;

I: Metacyclic p-groups.

II: $G = \langle A, B, C\colon A^p = B^p = C^{p^{n-2}} = [A, C] = [C, B] = 1,\ [B, A] = C^{p^{n-3}}\rangle$.

III: $G = \langle A, B, C\colon A^p = B^p = C^{p^{n-2}} = [B, C] = 1,\ [A, C^{-1}] = B,\ [B, A] = C^{sp^{n-3}}\rangle$ where $n \geqq 4$ and $s = 1$ or some quadratic nonresidue mod p.

In [11] and [12], C.B. Thomas shows that $H^{\mathrm{even}}(G, \boldsymbol{Z})$ of some split metacyclic p-groups and Blackburn type II groups are generated by Chern classes, and hence they satisfy Atiyah's conjecture. He conjectured that a similar result holds for the remaining rank_2 p-groups, $p > 3$. This would be the best possible result, since there is a 4-dimensional generator of $H^*(3\boldsymbol{Z}_p, \boldsymbol{Z})$ which can not come from representations [9, Proposition 4.2]. For a metacyclic p-group in general the conjecture is proved by the author [1]. In this paper the conjecture is proved for Blackburn type III p-groups. The method used is mainly computational and the main result is given as follows:

THEOREM 9.
$H^*(G, \boldsymbol{Z}) = \boldsymbol{Z}[\alpha; \mu; \gamma_1, \cdots, \gamma_{p-1}; \chi_1, \cdots, \chi_{p-2}; \xi, \xi']$ where $\deg \alpha = 2$, $\deg \mu = 3$, $\deg \gamma_i = 2i$, $\deg \chi_i = 2i + 2$, $\deg \xi = \deg \xi' = 2p$ with the relations: $p\alpha = p\mu = sp^{n-3}\gamma_i = p\chi_i = p^{n-1}\xi = p^2\xi' = 0$, $\alpha^p = 0$, $\alpha\gamma_i = \alpha\chi_i = 0$, $\mu^2 = 0$, $\mu\gamma_i = \mu\chi_i = 0$, $\gamma_i\gamma_j = 0$, and $\chi_i\chi_j = 0$ for all i, j.

The method of computation used depends mainly on constructing a free action of the group G on a product of two spheres to determine the order of certain cohomology groups of G together with the method used by G. Lewis to compute the integral cohomology ring of a non-abelian group of order p^3 and exponent p. Lewis' method is based on the calculation of the E_2 terms of spectral sequences of two group extensions and the calculation of E_∞ terms by certain exact sequences of the restriction and corestriction maps. The reader is referred to [9] for the details of the method. $H^{\mathrm{even}}(G, \boldsymbol{Z})$ is expressed in terms of Chern classes by using a special Riemann-Roch formula [12].

Preliminaries. The group G can be given by either of the following two extensions:

$$1 \longrightarrow H \longrightarrow G \longrightarrow \boldsymbol{Z}_p\langle \bar{A}\rangle \longrightarrow 1 . \tag{1}$$

Where $H = \boldsymbol{Z}_p\langle B\rangle + \boldsymbol{Z}_{p^{n-1}}\langle C\rangle$ is a normal abelian subgroup of index p in G, and

$$1 \longrightarrow G^1 \longrightarrow G \longrightarrow \boldsymbol{Z}_p\langle \bar{A}\rangle + \boldsymbol{Z}_{p^{n-1}}\langle \bar{C}\rangle \longrightarrow 1 \tag{2}$$

where $G^1 = \boldsymbol{Z}_p\langle B\rangle + \boldsymbol{Z}_p\langle C^{sp^{n-3}}\rangle$ is the commutator subgroup of G. The group G is isomorphic to the group $G' = \langle X, Y, Z\colon X^{p^{n-2}} = Y^p = [Y, Z] = 1,\ Z^p = X^{sp},\ [X, Z] = Y,\ [X, Y] = X^{p^{n-3}}\rangle$ where $n \geqq 4$ and $s = 1$ or some quadratic non-residue mod p [3, p. 145]. The isomorphism from G' onto G is given by: $X \leftrightarrow AC$, $Y \leftrightarrow B^{-1}$, and $Z \leftrightarrow C$.

$$X^p \cong A^p C^p B^{1+2+\cdots+(p-1)} C^{p^{n-3}+2p^{n-3}+\cdots+(p-1)p^{n-3}} = C^p \cong Z^p$$
$$XZ \cong ACC = CA^{-1}BC = CACB^{-1} \cong ZXY \text{ and } XY \cong ACB^{-1}$$
$$= AB^{-1}C^{-1} = B^{-1}ACC^{p^{n-3}} \cong YX^{1+p^{n-3}} .$$

If s is a quadratic nonresidue mod p, the isomorphism can similarly be defined by: $X \leftrightarrow AC$, $Y \leftrightarrow B^{-s}$, and $Z \leftrightarrow C^s$.

PROPOSITION 1.
G' and hence G acts freely on the product of two spheres $S^{2p-1} \times S^{2p-1}$.

Proof. Let λ: $Y \mapsto e^{2\pi i/p} = a$, $Z \mapsto 1$ and λ': $Y \mapsto 1$, $Z \mapsto e^{2\pi i/p^{n-2}} = b$ be two 1-dimensional representations of the normal abelian subgroup $\langle Y, Z\rangle$ of index p in G'. The direct sum of the induced representations $i_!\lambda$ and $i_!\lambda'$ defines an action of the group G' on the product of two spheres $S^{2p-1} \times S^{2p-1}$. $1 \otimes 1$, $\bar{X} \otimes 1, \cdots, \bar{X}^{p-1} \otimes 1$ forms a basis for the induced modules associated with $i_!\lambda$ and $i_!\lambda'$. By [5, p. 75] the induced representations are explicitely given as follows:

$$i_!\lambda(X) = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \quad i_!\lambda(Y) = \begin{bmatrix} a & & & \\ & \cdot & \bigcirc & \\ & & \cdot & \\ \bigcirc & & \cdot & \\ & & & a \end{bmatrix}, \quad i_!\lambda(Z) = \begin{bmatrix} 1 & & & \\ & a^{-1} & & \\ & \cdot & \bigcirc & \\ & & \cdot & \\ \bigcirc & & \cdot & \\ & & & a^{-p+1} \end{bmatrix},$$

and

$$i_!\lambda'(X) = \begin{bmatrix} 0 & 0 & \cdots & 0 & b^p \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \quad i_!\lambda'(Y) = \begin{bmatrix} 1 & & & \\ & b^{-1} & & \\ & \cdot & \bigcirc & \\ & & \cdot & \\ \bigcirc & & \cdot & \\ & & & b^{-p+1} \end{bmatrix}, \quad i_!\lambda'(Z) = \begin{bmatrix} b & & & \\ & \cdot & \bigcirc & \\ & & \cdot & \\ \bigcirc & & \cdot & \\ & & & b \end{bmatrix}.$$

Let $g \in G'$ be any element. Then $g = Z^i Y^j X^k$ where $0 \leqq i < p^{n-2}$ and $0 \leqq j, k < p$. The action of G' on the first and second sphere is given by:

$$g(x_1, \cdots, x_p) = (a^j x_{p-k+1}, a^{j-i} x_{p-k+2}, \cdots, a^{j-(p-1)i} x_{p-k})$$

and

$$g(x_1, \cdots, x_p) = (b^{kp-i} x_{p-k+1}, a^{-j} b^{(k-1)p-i} x_{p-k+2}, \cdots, a^{-(p-1)j} b^{-i} x_{p-k})$$

respectively for every point $(x_1, \cdots, x_p) \in S^{2p-1}$. Any element $g \in G'$ which acts freely on $S^{2p-1} \times S^{2p-1}$ must equal to the identity. Thus G' and hence G acts freely on $S^{2p-1} \times S^{2p-1}$. $\square$

The group G acts on the sphere $S^{2p-1} = S^1 * \cdots * S^1$ (p-fold join) by the induced representation of $C \mapsto e^{2\pi i/p^{n-2}}$, $B \mapsto 1$. By [9. §6.2] we have the following complex $C'(S^{2p-1}) = \{C'_0 \leftarrow C'_1 \leftarrow \cdots \leftarrow C'_{p-1} \leftarrow \cdots \leftarrow C'_{2p-1}\}$ where C'_i is a G-free module except for $i = 0, 1, p-1$, and $2p-1$. $C'_0 = \boldsymbol{Z}(G/\langle B \rangle)$. $C'_1 = \boldsymbol{Z}G/\langle B \rangle \oplus F$, $C'_{p-1} = \boldsymbol{Z}G/\langle A \rangle \oplus F$, and $C'_{2p-1} = \boldsymbol{Z}G/\langle A \rangle \oplus F$ for some free G-module F. Consider $0 \leftarrow \boldsymbol{Z} \leftarrow C'_0 \leftarrow \cdots \leftarrow C'_{2p-1} \leftarrow \boldsymbol{Z} \leftarrow 0$ and let K, L, M, N, and R be the image-kernels at C'_0, C'_1, C'_{p-2}, C'_{p-1} and C'_{2p-1} respectively. Applying the Tate Cohomology to the resulting exact sequences we get the following exact sequences for i odd:

$$\begin{aligned}
0 &\longrightarrow H^i(G, M) \longrightarrow H^{i+1}(G, N) \longrightarrow H^{i+1}(\langle A \rangle, \boldsymbol{Z}) \longrightarrow H^{i+1}(G, M) \\
&\qquad\longrightarrow H^{i+2}(G, N) \longrightarrow 0 \\
0 &\longrightarrow H^i(G, R) \longrightarrow H^{i+1}(G, \boldsymbol{Z}) \longrightarrow H^{i+1}(\langle A \rangle, \boldsymbol{Z}) \longrightarrow H^{i+1}(G, M) \\
&\qquad\longrightarrow H^{i+2}(G, N) \longrightarrow 0 \\
0 &\longrightarrow H^{i+1}(G, \boldsymbol{Z}) \longrightarrow H^{i+1}(G, K) \longrightarrow H^{i+1}(\langle B \rangle, \boldsymbol{Z}) \longrightarrow H^{i+1}(G, \boldsymbol{Z}) \\
&\qquad\longrightarrow H^{i+2}(G, N) \longrightarrow 0 \\
0 &\longrightarrow H^i(G, K) \longrightarrow H^{i+1}(G, L) \longrightarrow H^{i+1}(\langle B \rangle, \boldsymbol{Z}) \longrightarrow H^{i+1}(G, K) \\
&\qquad\longrightarrow H^{i+2}(G, L) \longrightarrow 0
\end{aligned}$$

and $H^i(G, L) \cong H^{i+p-3}(G, M)$, $H^i(G, N) \cong H^{i+p-1}(G, R)$ for all i by dimensional shifting. Similarly, there are exact sequences for i even. Then

$$\begin{aligned}
|H^{i+2}(G, \boldsymbol{Z})| &\leqq |H^{i+1}(G, R)| = |H^{i-p+2}(G, N)| \leqq p\,|H^{i-2p+4}(G, K)| \\
&\leqq p\,|H^{i-2p+3}(G, \boldsymbol{Z})| \leqq p^2\,|H^{i-2p+2}(G, \boldsymbol{Z})|.
\end{aligned}$$

Thus the following lemma holds

LEMMA 2.

$$|H^{j+2p}(G, \boldsymbol{Z})| \leqq p^2\,|H^j(G, \boldsymbol{Z})| \textit{ for all } j. \qquad \square$$

Integral cohomolog rings: Consider the spectral sequence of extension (1).

$$E_2^{i,j} = H^i(\boldsymbol{Z}_p\langle \bar{A} \rangle,\ H^j(H, \boldsymbol{Z}))\ .$$

$H^*(H, Z) = P[\beta, \gamma] \otimes E[\mu]$ where $\deg \beta = \deg \gamma = 2$, $\deg \mu = 3$, and $p\beta = sp^{n-2}\gamma = p\mu = 0$ [1]. β and γ are maximal generators corresponding to $:B \mapsto 1/p$, $C \mapsto 0$ and $:C \mapsto 1/sp^{n-2}$, $B \mapsto 0$ respectively. The action of the group $\boldsymbol{Z}_p\langle \bar{A} \rangle$ on $H^*(H, \boldsymbol{Z})$ induced by A is given by:

$$\beta \longmapsto \beta + sp^{n-3}\gamma,\ \gamma \longmapsto \gamma + \beta, \text{ and } \mu \longmapsto \mu\ .$$

$$E_2^{*,0} = H^*(\boldsymbol{Z}_p\langle \bar{A} \rangle, Z) = P[\alpha]$$

where $\deg\alpha = 2$ and $p\alpha = 0$. α is a maximal generator corresponding to $\bar{A} \mapsto 1/p$. $E_2^{0,*} = H^*(H, Z)^{Z_p\langle\bar{A}\rangle}$ the invariant elements:

$$\gamma_1 = p\gamma,\ p^2\gamma,\ \cdots,\ p^{n-3}\gamma;\ \gamma_2 = p\gamma^2,\ p^2\gamma^2,\ \cdots,\ p^{n-3}\gamma^2;\ \cdots\ ;$$
$$\gamma_p = p\gamma^p,\ p^2\gamma^p,\ \cdots,\ p^{n-3}\gamma^p;\ \beta^2;\ \beta^3;\ \cdots\ ;\ \beta^p;\ \gamma^p - \gamma\beta^{p-1};\ \mu\ .$$

PROPOSITION 3. *The low dimensional cohomology groups are*

$$H^2(G, Z) \cong Z_{p^{n-3}} \times Z_p,\ H^3(G, Z) \cong Z_p,\ \text{and}\ H^4(G, Z) \cong Z_{p^{n-3}} \times Z_p \times Z_p\ .$$

Proof. $H^2(G, Z) \cong \mathrm{Hom}\,(G/G^1, Q/Z) \cong Z_{p^{n-3}} \times Z_p$ where Q is the field of rationals [4]. By spectral sequence of extension (1) $H^2(G, Z)$ is generated by α and γ_1. Let Res: $H^*(G, Z) \to H^*(H, Z)$ and Cor: $H^*(H, Z) \to H^*(G, Z)$ be the restriction and corestriction homomorphisms. Cor $(\mathrm{Res}\,(\alpha)\cdot\gamma) = \alpha$ Cor $(\gamma) = 0$ since $\mathrm{Res}_2\,(\alpha) = 0$. Res (Cor γ) $= (1 + A + \cdots + A^{p-1})\gamma = p\gamma + (1 + 2 + \cdots + p - 1)\beta + (sp^{n-3} + \cdots + sp^{n-2} - 1)\gamma = p\gamma$. Therefore $\gamma_1 = \mathrm{Cor}\,(\gamma)$ and $\alpha\gamma_1 = 0$. Similarly, $\gamma_i = \mathrm{Cor}\,(\gamma^i)$ and $\alpha^i\gamma_i = 0$ for $1 \leqq i < p$. By Res $-$ Cor sequences [9, p. 504 (5′)]

$$0 \longrightarrow H^2(H, Z)_A \longrightarrow T^3 \longrightarrow H^3(H, Z)^A \longrightarrow 0$$

is exact. $|H^2(H, Z)_A| = p^{n-3}$ and $|H^3(H, Z)| = p$. Then $|T^3| = p^{n-3} \times p$. $0 \to H^3(G, Z) \to T^3 \xrightarrow{\tau} H^2(G, Z) \xrightarrow{\cup\alpha} H^4(G, Z)$ is exact [9, p. 504 (4′)] $|I_m\tau| = |\mathrm{Ker}\cup\alpha| = p^{n-3}$ since $\alpha\gamma_1 = 0$. $|H^3(G, Z)| = |T^3|/|\mathrm{Im}\,\tau| = p$. Therefore $H^3(G, Z) \cong Z_p$ and generated by μ since

$$\mathrm{Res}_3\colon H^3(G, Z) \longrightarrow H^3(H, Z)$$

is an epimorphism. The following diagrams is commutative and the top row is exact [9, p. 504 (4)].

$$\begin{array}{ccccccccc}
H^2(H, Z) & \xrightarrow{\mathrm{Cor}} & H^2(G, Z) & \longrightarrow & H^3(G,\ K) & \xrightarrow{\theta} & H^3(H,\ Z) & \xrightarrow{\mathrm{Cor}} & H^3(G, Z) \\
\cong\big\downarrow & & \big\downarrow\cong & & & & & & \\
\mathrm{Hom}\,(H, Q/Z) & \xrightarrow{\mathrm{Cor}} & \mathrm{Hom}\,(\langle\bar{A}, \bar{C}\rangle, Q/Z) & & & & & &
\end{array}$$

where $K = \mathrm{Ker}\,\{Z_p\langle A\rangle \to Z\}$. Cor: $H^3(H, Z) \to H^3(G, Z)$ is zero since Cor μ = Cor Res $\mu = p\mu = 0$. $|\mathrm{Im\,Cor}_2| = p^{n-3}$ since Cor $\gamma = \gamma_1$ and Cor $\beta = 0$ because Cor (Res $(\alpha)\cdot\beta) = \alpha$ Cor $\beta = 0$. Then $|H^3(G, K)| = |\mathrm{Im}\,\theta|\cdot|H^2(G, Z)|/|\mathrm{Im\,Cor}_2| = p \times p$. The following sequence is exact

$$H^3(G, Z) \xrightarrow{\mathrm{Res}} H^3(H, Z) \longrightarrow H^3(G, K) \longrightarrow H^4(G, Z) \xrightarrow{\mathrm{Res}} H^4(H, Z)\ .$$

Res_3 is an epimorphism and $|\mathrm{Im\,Res}_4| = p^{n-3}$ since Res $\alpha = 0$. Then

$|H^4(G, \boldsymbol{Z})| = p^{n-3} \times p \times p$. Therefore $H^4(G, \boldsymbol{Z}) \cong \boldsymbol{Z}_{p^{n-3}}\langle\gamma_2\rangle + \boldsymbol{Z}_p\langle\alpha^2\rangle + \boldsymbol{Z}_p\langle\chi\rangle$ where χ is an additional generator. □

Consider now the spectral sequence of extension (2).

$$E_2^{i,j} = H^i(\boldsymbol{Z}_p\langle\bar{A}\rangle + \boldsymbol{Z}_{p^{n-3}}\langle\bar{C}\rangle,\ H^j(G^1, \boldsymbol{Z}))\ .$$

$E_2^{*,0} = H^*(\boldsymbol{Z}_p \times \boldsymbol{Z}_{p^{n-3}}, \boldsymbol{Z}) = P[\alpha, \gamma] \otimes E[\delta]$ where $\deg\alpha = \deg\gamma = 2$, $\deg\delta = 3$, and $p\alpha = sp^{n-3}\gamma = p\delta = 0$. $E_2^{0,*} = H^*(G^1, \boldsymbol{Z})^{\langle\bar{A},\bar{C}\rangle} = P[\beta^2, \beta^3, \cdots; p^{n-3}\gamma]$.

The odd generators in the exterior part vanished since they are trivial under the action of $\langle\bar{A}, \bar{C}\rangle$. By comparing the two spectral sequences $\gamma^i \leftrightarrow \gamma_i$ for $1 \leqq i < p$.

$$E_2^{*,2j} = H^*(\boldsymbol{Z}_p \times \boldsymbol{Z}_{p^{n-3}}, \boldsymbol{Z}_p \times \boldsymbol{Z}_p) \cong H^*(\boldsymbol{Z}_{p^{n-3}}, \boldsymbol{Z}_p \times \boldsymbol{Z}_p) \otimes H^*(\boldsymbol{Z}_p, \boldsymbol{Z}_p \times \boldsymbol{Z}_p)$$

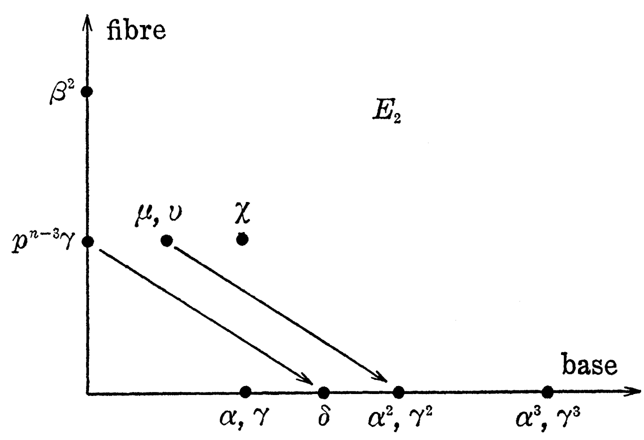
by Künneth formula. This induces a horizontal multiplication

$$\circ\colon E_2^{i,2j} \times E_2^{k,2j} \longrightarrow E_2^{i+k,2j},\ j > 0$$

and

$$\beta\colon E_2^{i,j} \longrightarrow E_2^{i,j+2}$$

is monomorphism for $j \geqq 2$ and isomorphism for $j > 0$ [4]. Let $\mu, \nu \in E_2^{1,2}$ be two independent generators. Then $\chi = \mu\circ\nu \in E_2^{2,2}$ by horizontal multiplication. Since the odd rows are zero, then $E_2 = E_3$. From the cohomology groups at the low dimensions $d_3(\alpha) = d_3(\gamma) = d_3(\mu) = d_3(\chi) = 0$. Others are easilly deduced from the E_2-diagram. Since $\gamma \leftrightarrow \gamma_1$, then $\alpha\gamma_1 = \delta\gamma_1 = \mu\gamma_1 = \nu\gamma_1 = \chi\gamma_1 = 0$. Then the additive structure of E_2 can be given are follows:



LEMMA 4.

$$E_2^{2i,0} = \boldsymbol{Z}_p\langle\alpha^i\rangle + \boldsymbol{Z}_{sp^{n-3}}\langle\gamma^i\rangle,\ E_2^{2i,4} = \boldsymbol{Z}_p\langle\chi\alpha^{i-1}\rangle + \boldsymbol{Z}_p\langle\beta^2\alpha^i\rangle\ ,$$

$$E_2^{2i,2} = \boldsymbol{Z}_p\langle \chi\alpha^{i-1}\rangle,\ E_2^{2,2} = \boldsymbol{Z}_p\langle \chi\rangle;\ E_2^{2i+1,0} = \boldsymbol{Z}_p\langle \delta\alpha^{i-1}\rangle\ ,$$
$$E_2^{2i+1,2} = \boldsymbol{Z}_p\langle \alpha^i\mu\rangle + \boldsymbol{Z}_p\langle \alpha^i\nu\rangle,\ \textit{and}\ E_2^{*,2j+1} = 0(j>0)\ .$$

The other terms are given by periodicity $E_2^{*,4} = E_2^{*,6} = \cdots$. □

LEMMA 5. *γ^p and β^p are universal cycles and hence $\beta^p\colon E_2^{i,j} \to E_2^{i,j+2p} \to$ is an isomorphism for $j>0$.*

Proof. By double cosets formula for the generalization of corestriction $\mathscr{N}$ [6, Theorem 3]

$$\begin{aligned}\operatorname{Res}_H \mathscr{N}(\gamma) &= \prod_{i=0}^{p-1}\Big(\gamma - i\beta - \frac{1}{2}i(i-1)sp^{n-3}\gamma\Big)\\ &= \prod_{i=0}^{p-1}(\gamma - i\beta) + \sum_{j=0}^{p-1}\Big(\prod_{i=0}^{p-1}(\hat{\gamma} - i\beta)\frac{1}{2}j(j-1)sp^{n-3}\gamma\Big)\\ &= \prod_{i=0}^{p-1}(\gamma - i\beta) = \gamma^p - \gamma\beta^{p-1}\end{aligned}$$

where $\hat{}$ means a deleted term. $\operatorname{Res}_H \mathscr{N}(\beta) = \prod_{i=0}^{p-1}(\beta - isp^{n-3}\gamma) = \beta^p$. Therefore γ^p and β^p are universal cycles [9, Corallary II]. □

The additive structure of E_4 can now be given as follows:

LEMMA 6.

$$\begin{aligned}&E_4^{2i,0} = \boldsymbol{Z}_p\langle\alpha^i\rangle + \boldsymbol{Z}_p\langle\gamma^i\rangle;\ E_4^{2,2j} = \boldsymbol{Z}_p\langle\chi\beta^{j-1}\rangle,\ j>0;\ E_4^{2i,2j} = 0\ ,\\ &\quad j \not\equiv 0(p),\ j>0,\ i\neq 1;\ E_4^{i,2j} = 0,\ j\not\equiv 1(p),\ j>1;\ E_4^{2i+1,2j} = 0\ ,\\ &\quad j\not\equiv 0(P) j\neq 1,\ i>0;\ E_4^{2i+1,2} = \boldsymbol{Z}_p\langle\alpha^i\mu\rangle\ ;\end{aligned}$$

and

$$E_4^{2i+1,2(p-1)} = \boldsymbol{Z}_p\langle\delta\alpha^{i-1}\beta^{p-1}\rangle\ .$$

The other terms are given by periodicity $E_4^{*,j} = E_4^{*,j+2p} = \cdots$. □

Then $E_4 = E_\infty$ in dimensions $\leqq 2p$.

LEMMA 7.

$$|H^{2p}(G, \boldsymbol{Z})| = p^{n+1}\ .$$

Proof. G acts freely on the product of the two spheres $S^{2p-1}\times S^{2p-1}$ by Proposition 1. Then by [10, Corollary 2.7] the following sequence is exact:

$$\begin{aligned}0 \longrightarrow H^{2p-1}(G, \boldsymbol{Z}) \longrightarrow H^{2p}(G, \boldsymbol{Z}) \longrightarrow \boldsymbol{Z}_{p^n}\times \boldsymbol{Z}_{p^n} \longrightarrow H^{2p}(G, \boldsymbol{Z})\\ \longrightarrow H^{2p-1}(G, \boldsymbol{Z}) \longrightarrow 0\ .\end{aligned}$$

Since $H^{2p-1}(G, \boldsymbol{Z}) = \boldsymbol{Z}_p\langle\alpha^{p-2}\mu\rangle$ by the previous spectral sequence, then $|H^{2p}(G, Z)| = p^{n+1}$. $\square$

By Res-Cor sequence

$$0 \longrightarrow Z_{p^{n-2}} \longrightarrow H^{2i}(H, Z) \longrightarrow H^{2i}(G, K) \longrightarrow H^{2i+1}(G, Z) \longrightarrow Z_p \longrightarrow 0$$

is exact where $K = \mathrm{Ker}\{Z\langle A\rangle \to Z\}$. $|H^{2i}(H, Z)| = p^{i+n-2}$ and $|H^{2i+1}(G, Z)| = p$. Therefore $|H^{2i}(G, K)| = p^i$. If $\mathrm{Cor}_{2i} = 0$, then $0 \to H^{2i-1}(G, Z) \to H^{2i}(G, K) \to H^{2i}(H, Z) \to 0$ is exact. Therefore $|H^{2i}(G, K)| = p^{i+n-1}$ which is a contradiction. Then $\mathrm{Cor}(\beta^i) \neq 0$ for $2 \leqq i < p$. Similarly, we can prove the following:

LEMMA 8. $\mathrm{Cor}(\beta^i) \neq 0$ *for* $2 \leqq i \leqq p$ *and* $\mathrm{Cor}(\gamma^p) \neq 0$. $\square$

Let $\xi = \mathscr{N}(\gamma)$ and $\xi' = \mathscr{N}(\beta)$ $\mathrm{Res}_H \mathscr{N}(\gamma) = \gamma^p - \gamma\beta^{p-1}$ and $\mathrm{Res}_H \mathscr{N}(\beta) = \beta^p$. Cor Res $\mathscr{N}(\gamma) = p\mathscr{N}(\gamma) = \mathrm{Cor}(\gamma^p) \neq 0$ and Cor Res $\mathscr{N}(\beta) = p\mathscr{N}(\beta) = \mathrm{Cor}(\beta^p) \neq 0$. Therefore $\mathscr{N}(\gamma)$ and $\mathscr{N}(\beta)$ have orders p^{n-1} and p^2 respectively and are elements in $H^{2p}(G, Z)$. Since $|H^{2p}(G, Z)| = p^{n+1}$ by Lemma 7, then $\alpha^p = 0$ in $H^*(G, Z)$.

Let $\chi_i = \mathrm{Cor}(\beta^{i+1})$, $1 \leqq i < p - 1$. χ_i is not a polynomial in α and γ since $\alpha \,\mathrm{Cor}(\beta^p) = 0$ and Res Cos $(\beta^p) = 0$. Therefore $H^{2i+2}(G, Z) = Z_p\langle \chi_i\rangle + Z_p\langle \alpha^{i+1}\rangle + Z_{sp^{n-3}}\langle \gamma^{i+1}\rangle$.

By using Cor (Res a.b) = a. Cor b, we have $\alpha\chi_i = \mu\chi_i = \chi_i\chi_j = 0$ and $\gamma_i\chi_j = 0$ since $\mathrm{Res}\,\chi_i = 0$. If $\gamma_i\gamma_j = e\,\alpha^{i+j}$, then $\alpha\gamma_i\gamma_j = e\alpha^{i+j+1} = 0$. Then $e = 0$ and hence $\gamma_i\gamma_j = 0$. Thus we have:

THEOREM 9. *The integral cohomology ring* $H^*(G, Z) = Z[\alpha; \mu; \gamma_1, \cdots, \gamma_{p-1}; \chi_1, \cdots, \chi_{p-2}, \xi, \xi']$ *where* $\deg \alpha = 2$, $\deg \mu = 3$, $\deg \gamma_i = 2i$, $\deg \chi_i = 2i + 2$, $\deg \xi = \deg \xi' = 2p$ *with the relations* $p\alpha = p\mu = sp^{n-3}\gamma_i = p\chi_i = p^{n-1}\xi = p^2\xi' = 0$, $\alpha^p = 0$, $\alpha\gamma_i = \alpha\chi_i = 0$, $\mu^2 = 0$, $\mu\gamma_i = \mu\chi_i = 0$, $\gamma_i\gamma_j = 0$, *and* $\chi_i\chi_j = 0$ *for all* i *and* j. $\square$

$H^{\mathrm{even}}(G, Z)$ is generated by $\alpha, \gamma_1, \cdots, \gamma_{p-1}, \chi_1, \cdots, \chi_{p-2}, \xi, \xi'$. $\alpha = c_1(\hat{\alpha})$ is the first Chern class of the 1-dimensional representation given by $\hat{\alpha}(A) = 1/p$. $\gamma_i = \mathrm{Cor}(\gamma^i)$ for $1 \leqq i < p$ and $\chi_i = \mathrm{Cor}(\beta^{i+1})$, $1 \leqq i \leqq p - 2$. Then by using a special Reimann-Rock formula [12, Theorem 2] we get: $\mathrm{Cor}(\gamma^i) = S_i(i_!\hat{\gamma})$, $2 \leqq i \leqq p - 2$; $\mathrm{Cor}(\gamma^{p-1}) = S_{p-1}(i_!\hat{\gamma}) + (p - 1)\alpha^{p-1}$ and $\mathrm{Cor}(\beta^i) = S_i(i_!\hat{\beta})$ $2 \leqq i \leqq p-2$; $\mathrm{Cor}(\beta^{p-1}) = S_{p-1}(i_!\hat{\beta}) + (p - 1)\alpha^{p-1}$ where α is the inflation of the generator of $H^2(\langle \bar{A}\rangle, Z)$ and $\hat{\beta}, \hat{\alpha}$ are two representations given by $\hat{\beta}: B \to 1/p$, $C \to 0$ and $\hat{\gamma}, B \to 0$: $C \to 1/sp^{n-2}$. The two generators $\xi = \mathscr{N}(\gamma) = c_p(\hat{\gamma})$ and $\xi' = \mathscr{N}(\beta) = c_p(\hat{\beta})$ are given in terms of pth Chern classes [7, Theorem 4]. By [2-Appendix] we have:

THEOREM 10. $H^{\mathrm{even}}(G, Z)$ *is generated by Chern classes and* *hence G satisfies Atiyah's Conjecture.* □

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