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CONVERGENCE AND APPROXIMATION THEOREMS FOR VECTOR-VALUED DISTRIBUTIONS

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We prove here that under adequate restrictions, convergence of a sequence of vector-valued distributions $\{P_n\}$ and boundedness of the sequence of their convolution inverses $\{S_n\}$ implies convergence of $\{S_n\}$; boundedness and convergence are formulated with respect to "fractional derivative norms" which include ordinary boundedness and convergence as a particular case. The results include diverse results for convergence of solutions of differential, difference and functional equations proved by Trotter, Kato, Goldstein, Ujishima, Ponomarev and others.

Table of Contents

1. Introduction. Let $\{S_n(\cdot)\}$ be a sequence of strongly continuous semigroups in a Banach space E. Trotter proved in [**32**] that if the S_n are uniformly bounded in $t \geq 0$, then $S_n(\cdot)$ converges in the strong topology to a strongly continuous semigroup $S(\cdot)$ if and only if $R(\lambda; A_n) = (\lambda I - A_n)^{-1}$ converges strongly to $R(\lambda; A)$, where A_n (resp. A) denotes the infinitesimal generator of S_n (resp. S); an addition of Kato [**12**] deals with the case where S is not assumed to exist at the outset but is obtained from A, in turn defined from the strong limit of the $R(\lambda; A_n)$. Trotter also proves in [**32**] convergence results for the $\{S_n\}$ based directly on the convergence of the $\{A_n\}$ in certain sets, as well as results on approximation of S by discrete semigroups, corresponding to finite difference approximations of abstract differential equations. A parallel (and somewhat earlier) treatment of the discrete case was originated by Lax (see [**18**], [**26**]). Its main theoretical result is the *Lax equivalence theorem*, where, as in the Trotter-Kato theorem, convergence (of a sequence of discrete semigroups to a strongly continuous semigroup) is deduced from uniform boundedness (*stability* of the difference scheme in [**18**]) plus convergence of the infinitesimal generators (*consistency* in [**18**]).

Since the Trotter-Kato and Lax theorems, numerous variants and generalizations have appeared both for semigroups and for other operator-valued solutions of abstract differential equations, dealing with continuous and discrete approximations (see [**1**], [**10**], [**13**], [**14**], [**15**], [**16**], [**17**], [**22**], [**23**], [**24**], [**25**], [**26**], [**31**], [**33**]). In all of these results (named A-B-C theorems by Ujishima [**33**]), convergence (C) is deduced from uniform boundedness (B) and convergence of infinitesimal generators or of their resolvents (A). Theorems of the same type have been obtained for nonlinear semigroups but we restrict ourselves to the linear case here.

We present in this paper a general scheme including most of the known results for the linear case and based on the following observation. Let $S_n(\cdot)$, A_n be as above; call X_n the domain $D(A_n)$ of A_n endowed with its graph norm. Then each S_n can be thought of as a distribution with values in the space $(E; X_n)$ of linear bounded operators from E into X_n (through the assignation $S_n(\varphi) = \int S_n(t)\varphi(t)dt$ for any Schwartz test function in $\mathcal{D}$). Consider the $(X_n; E)$-valued distribution $P_n = \delta' \otimes I - \delta \otimes A_n$, I the identity operator, δ the Dirac delta. We easily check that $P_n * S_n = \delta \otimes I$, $S_n * P_n = \delta \otimes I_n$ (I_n the restriction of I to X_n), thus S_n is the convolution inverse of P_n (in symbols, $S_n = P_n^{*-1}$). Accordingly, the Trotter-Kato and similar theorems can be formulated as particular cases of the following result: if $\{P_n\}$ is a sequence of operator-valued distributions such that the inverses $\{S_n\}$ are bounded in a function space $\mathcal{F}$, convergence of the P_n (or convergence of the inverse of its Laplace transform) will be equivalent to convergence of the S_n in $\mathcal{F}$. Obviously, a general theorem of this type will only hold under definite restrictions on the form of the P_n (see §4) but even under those restrictions the result (Theorem 4.7) includes most, if not all, instances of A-B-C theorems hitherto known, both in the continuous and discrete cases in substantially generalized versions; apart from the fact that P_n is a distribution of a fairly general form, boundedness is postulated (and convergence is obtained) in a whole gamut of norms, roughly corresponding to the supremum of fractional derivatives of order $\leq \eta$ for arbitrary η, $-\infty < \eta < \infty$; $\eta = 0$ is the uniform convergence case considered in most of the existing results. The theorem applies equally well not only to abstract differential equations but to hereditary equations describing systems where the whole past (rather than only the present) must be brought into play to predict the future. The idea of the proof of Theorem 4.7 is to base the argument on the "inverse Laplace transform" formula (2.6) for inverses of vector-valued distributions (as done by Piskarev in [**23**]) rather than on special properties of S, S_n only valid for very particular distributions (such as the semigroup equation associated with $P_n = \delta' \otimes I - \delta \otimes A$).

One final bonus of this approach is that it lends itself fairly well to a unified treatment of the continuous and the discrete case; although Theorem 4.7 only refers to the first, it can be easily twisted to accomodate the second (§§9 and 10) and provides ample generalizations, in the sense pointed out above, of the Lax equivalence theorem.

A parallel thread runs through several sections of the present paper, and is that of obtaining results on convergence of the S_n not in specific fractional derivative norms but in the (weaker) sense of distributions, in terms of convergence of the inverse Laplace transform (also called resolvent) of the P_n. As might be expected from the very general nature of distributional convergence the results become simpler and all restrictions on the form of the P_n are blown away. (Theorem 4.1.) The applications of this line of thought are not without interest; in particular we obtain extensions (or possibilities of extension) of the diverse exponential formulas in **[11]** to the ambit of regular distribution semigroups (Theorem 6.3). These applications, together with the more interesting and important applications of Theorems 4.7. and 9.1 are found in §§6, 7, 8, 11, 12, and 13. We discuss in Section 14, by way of conclusion, some extensions and generalizations.

2. Distribution inverses. We denote by E, F, $X,\ldots$ complex Banach spaces: $(E;\, F)$ is the space of all linear bounded operators from E into F equipped with its usual uniform operator norm (we usually write (E) as a shorthand for $(E;\, E)$). The symbol $\mathfrak{D}'(F)$ indicates the space of all distributions U with values in F defined in $-\infty < t < \infty$ and $\mathfrak{D}_0'(F)$ is the subspace thereof consisting of distributions with support in $t \geq 0$. When F coincides with $\mathbf{C}$ (i.e., when $\dim F = 1$) we write simply $\mathfrak{D}'$, $\mathfrak{D}_0'$ as customary. The space of all tempered, F-valued distributions and its subspace consisting of distributions with support in $t \geq 0$ will be denoted by $\mathcal{S}'(F)$, $\mathcal{S}_0'(F)$ respectively, abbreviated to $\mathcal{S}'$, $\mathcal{S}_0'$ when $F = \mathbf{C}$. Both $\mathfrak{D}'(F)$ and $\mathcal{S}'(F)$ will be equipped with their usual topology. The symbol $\mathcal{S}_{0,\omega}'(F)$ indicates the space of all $U \in \mathfrak{D}_0'(F)$ such that $e^{-\omega t}U \in \mathcal{S}_0'(F)$, convergence of U_n in $\mathcal{S}_{0,\omega}'(F)$ meaning convergence of $e^{-\omega t}U_n$ in $\mathcal{S}_0'(F)$. We shall use the convolution $U * V$ of vector-valued distributions only in the case where $U \in \mathfrak{D}_0'((X;\, F))$ and $V \in \mathfrak{D}_0'(X)$ or $V \in \mathfrak{D}_0'((E;\, X))$, the convolution being defined with respect to the natural bilinear maps from $(X;\, F) \times X$ into F and from $(X;\, F) \times (E;\, X)$ into $(E;\, F)$. (All facts on vector-valued distributions here and below can be found in enormously general versions in **[27]** and **[28]**; see also **[20]** for a more elementary exposition.) A distribution $P \in \mathfrak{D}_0'((X;\, E))$ is said to belong to the class $\mathfrak{D}_0'((X;\, E))^{-1}$ (or $\mathfrak{D}_0'^{-1}$ for short) if it has a convolution inverse $S = P^{*-1}$

with support in $t \geq 0$, that is, if there exists $S \in \mathfrak{D}'_0((E; X))$ such that

$$P * S = \delta \otimes I, \qquad S * P = \delta \otimes J \tag{2.1}$$

where I (resp. J) denotes the identity operator in E (resp. X); see [**8**] for a proof of the fact that $P \in \mathfrak{D}_0'^{-1}$ is equivalent to the well posedness of the problem of solving the equation $P * U = V$ with $V \in \mathfrak{D}'_0(E)$ in a sense made explicit there. Distributions in $\mathfrak{D}_0'^{-1}$ have been characterized in [**4**] under a compact support assumption (but in spaces of distributions in several time variables). The identification in the general case was given in [**8**]. This result is reproduced below, restricted to the case (amply sufficient for applications) where $P \in \mathfrak{S}'_0((X; E))$.

We indicate by $\varphi(\hat{t})$ a function $t \to \varphi(t)$ or the distribution it defines, as opposed to $\varphi(t)$, which is the value of the function at t. The same "functional" notation will be used for distributions; for instance $\delta(\hat{t} - 1)$ indicates the Dirac measure at $t = 1$. Given a distribution $U \in \mathfrak{S}'_0(E)$ we denote by $\mathfrak{L}U$ the Laplace transform of U defined by $\mathfrak{L}U(\lambda) = U(\exp(-\lambda\hat{t}))$; $\mathfrak{L}U$ is analytic in $\operatorname{Re}\lambda > 0$ and

$$\|\mathfrak{L}U(\lambda)\| \leq C(1 + |\lambda|)^m \qquad (\operatorname{Re}\lambda > 0)$$

for suitable constants C, m.

Let P be a distribution in $\mathfrak{S}'_0((X; E))$, and let $\mathfrak{P}(\lambda) = \mathfrak{L}P(\lambda)$. The $(X; E)$-valued function P is defined and analytic in a (maximal) domain $\pi(P)$ containing the half-plane $\operatorname{Re}\lambda > 0$. We denote by $\rho(P)$ the *resolvent set* of P consisting of all $\lambda \in \pi(P)$ such that $\mathfrak{P}(\lambda)$ has a bounded inverse $\mathfrak{R}(\lambda) \in (E; X)$ called the *resolvent* of P. Obviously, $\mathfrak{R}$ is analytic in $\rho(P)$. The complement $\sigma(P)$ of $\rho(P)$ is the *spectrum* of P.

2.1. THEOREM. *A distribution $P \in \mathfrak{S}'_0((X; E))$ belongs to $\mathfrak{D}'_0((E; X))^{-1}$ if and only if $\rho(P)$ contains a region $\Lambda = \Lambda(\alpha, \beta, \omega)$ defined by an inequality of the form*

$$\operatorname{Re}\lambda \geq \max\{\alpha \log|\operatorname{Im}\lambda| + \beta, \omega\} \tag{2.2}$$

($\alpha \geq 0$) *and*

$$\|\mathfrak{R}(\lambda)\|_{(E;X)} \leq C(1 + |\lambda|)^m \qquad (\lambda \in \Lambda) \tag{2.3}$$

for adequate constants C, m. The distribution $S = P^{*-1}$ belongs to $\mathfrak{S}'_0((E; X))$ if and only if the conditions above are satisfied for a half-plane $\Xi = \Xi(\omega) = \{\lambda; \operatorname{Re}\lambda \geq \omega\}$.

For a proof see [**8**], Theorem 2.5. Regions defined by inequalities of the type of (2.2) are called *logarithmic regions*. We may and will assume without loss of generality that $\omega > 0$. A *reverse logarithmic region*

$\Omega(\alpha, \beta, \omega)$ is defined by the inequality

$$\text{(2.4)} \qquad \operatorname{Re} \lambda \geq \min\{\beta - \alpha \log|\operatorname{Im} \lambda|, \omega\}$$

for $\alpha \geq 0$, $\omega > 0$. Reverse logarithmic regions are basic in the identification of abstract parabolic distributions $P \in \mathcal{S}_0'((X; E))$. A distribution P is *abstract parabolic* if it belongs to $\mathcal{D}_0'((E; X))^{-1}$ and if $S = P^{*-1}$ coincides with an $(E; X)$-valued infinitely differentiable function in $t > 0$ (note that singular behavior is allowed at $t = 0$). Abstract parabolic distributions were characterized in [8] under a compact support assumption. To simplify the statement of the result we introduce the space $\mathcal{E}_0'(F)$, consisting of all distributions in $\mathcal{D}_0'(F)$ which have compact support. Clearly $\mathcal{E}_0'(F) \subseteq \mathcal{S}_0'(F) \subseteq \mathcal{D}_0'(F)$.

2.2 THEOREM. *Let $P \in \mathcal{E}_0'((X; E))$. Then P is abstract parabolic if and only if for every $\alpha > 0$ there exists $\beta = \beta(\alpha)$, $\omega = \omega(\alpha)$ such that the reverse logarithmic region $\Omega = \Omega(\alpha, \beta, \omega)$ contains $\rho(P)$ and*

$$\text{(2.5)} \qquad \|\mathfrak{R}(\lambda)\| \leq C(1 + |\lambda|)^m \qquad (\lambda \in \Omega)$$

where C (but not m) may depend on α.

For a proof see [8], Theorem 6.1.

It will be essential later to have direct integral representations of S in terms of $\mathfrak{P}$ for the classes identified by Theorems 2.1 and 2.2. To this end we introduce the "fractional differentiation" distributions Y_ζ defined as follows. For $\operatorname{Re} \zeta > 0$, $Y_\zeta(\hat{t}) = h(\hat{t})\hat{t}^{\zeta-1}/\Gamma(\zeta)$, where h is the Heaviside function ($h(t) = 1$ for $t \geq 0$, $h(t) = 0$ for $t < 0$). The function $\zeta \to Y_\zeta$ can be analytically extended to all complex ζ. (See [29]). The family of distributions $\{Y_\zeta; \zeta \in \mathbf{C}\}$ so defined satisfies the following conditions: (a) $Y_\zeta \in \mathcal{S}_0'$, (b) $Y_z * Y_\zeta = Y_{z+\zeta}$, (c) $Y_\zeta' = Y_{\zeta-1}$, (d) $\mathfrak{L}Y_\zeta(\lambda) = \lambda^{-\zeta}$ ($\operatorname{Re} \lambda > 0$), these four equalities holding for all complex ζ; (e) $Y_{-m} = \delta^{(m)}$ for $m = 1, 2, \ldots$. Note that convolution by Y_m is the operator of integration (from 0 to t) iterated m times.

Let $P \in \mathcal{S}_0'((X; E)) \cap \mathcal{D}_0'((X; E))^{-1}$, $T > 0$, and let $\gamma > \alpha T + m + 1$ (α the constant in (2.2). Then $Y_\gamma * S$ coincides with an $(E; X)$-valued continuous function in $t \leq T$ and admits the representation

$$\text{(2.6)} \qquad (Y_\gamma * S)(t) = \frac{1}{2\pi i} \int_\Gamma \lambda^{-\gamma} e^{\lambda t} \mathfrak{R}(\lambda)\, d\lambda$$

where Γ is the boundary of the logarithmic region Λ in Theorem 2.1 oriented upwards. This representation can be made global when $P \in \mathcal{S}_0'((E; X))^{-1}$, that is, when $S \in \mathcal{S}_0'((E; X))$; in this case Λ reduces to a half-plane $\operatorname{Re} \lambda \geq \omega$ and (2.6) holds for all t provided that $\gamma > m + 1$ and that the contour of integration is taken to be the line $\operatorname{Re} \lambda = \omega$. Further improvements are possible when P is abstract parabolic: here Γ is deformed to the boundary of the reverse logarithmic region Ω in Theorem 2.2 and formula (2.6) holds again when $\gamma > m + 1$ but is now valid in $t > 0$ for all γ, in particular for $\gamma = -1, -2, \ldots$ thus providing representations for all derivatives of S. These representations imply that for each $\varepsilon > 0$ and each γ there exists $C = C(\varepsilon, \gamma)$ such that

$$(2.7) \qquad (Y_\gamma * S)(t) \le Ce^{\omega t} \qquad (t \ge \varepsilon)$$

We shall indicate by $P \in \mathcal{C}_+^\infty((E;\ X))^{-1}$ or simply $P \in (\mathcal{C}_+^\infty)^{-1}$ the fact that P is abstract parabolic. An important subclass of $\mathcal{C}_+^\infty((E;\ X))^{-1}$ is $\mathcal{Q}(\varphi;\ (E;\ X))^{-1}$ $(0 < \varphi < \pi/2)$ consisting of all abstract parabolic P such that (a) $S = P^{*-1}$ can be extended to an $(E;\ X)$-valued function $S(\zeta)$ analytic in the sector $\Sigma_+(\varphi) = \{\zeta;\ |\arg \zeta| \le \varphi,\ \zeta \neq 0\}$ and satisfies

$$(2.8) \qquad \|S(\zeta)\|_{(E;X)} \le Ce^{\omega|\zeta|} \qquad (\zeta \in \Sigma_+(\varphi),\ |\zeta| \ge 1).$$

(b) For every ψ, $|\psi| \le \varphi$ the function that equals $S(te^{i\psi})$ for $t > 0$ and vanishes in $t < 0$ defines a distribution $S_\psi \in \mathcal{D}_0'((E;\ X))$. (c) There exists a real γ such that $Y_\gamma * S_\psi$ coincides with a jointly continuous $(E;\ X)$-valued function of t, ψ for $-\infty < t < \infty$, $|\psi| \le \varphi$. These distributions have been characterized in **[9]**, Theorem 3.8, which we reproduce here.

2.3 THEOREM. *Let $P \in \mathcal{S}_0'((X;\ E))$, and assume $P \in \mathcal{Q}(\varphi;\ (E;\ X))^{-1}$. Then $\rho(P)$ contains a sector $\Sigma = \Sigma(\varphi + \pi/2, \gamma) = \{\lambda;\ |\arg(\lambda - \gamma)| \le \varphi + \pi/2\}$ and*

$$(2.9) \qquad \|\mathfrak{R}(\lambda)\| \le C(1 + |\lambda|)^m \qquad (\lambda \in \Sigma).$$

Conversely, assume $\mathfrak{R}(\lambda)$ exists in $\lambda \in \Sigma$ and satisfies (2.9). *Then $P \in \mathcal{Q}(\varphi';\ (X;\ E))^{-1}$ for any φ', $0 < \varphi' < \varphi$.*

3. The approximation scheme. Boundedness assumptions. Let E, E_n $(n \ge 1)$ be complex Banach spaces, and let $\mathfrak{A}_n$: $E \to E_n$ be a bounded operator for each n. Following Trotter **[32]** we say that the sequence $\{(E_n, \mathfrak{A}_n);\ n \ge 1\}$ *approximates* E if and only if

$$(3.1) \qquad \lim_{n \to \infty} \|\mathfrak{A}_n u\|_{E_n} = \|u\|_E$$

for all $u \in E$. It follows easily from the closed graph theorem that (3.1) implies

$$(3.2) \qquad \|\mathfrak{A}_n\|_{(E;E_n)} \le C \qquad (n \ge 1).$$

The sequence $\{u_n; u_n \in E_n\}$ *converges to* $u \in E$ (in symbols $u_n \Rightarrow u$) if and only if

$$\|u_n - \mathfrak{A}_n u\|_{E_n} \to 0 \qquad (n \to \infty). \tag{3.3}$$

Often (but not always) E is a function space in some domain Ω of Euclidean space and E_n is finite dimensional: for instance, if $E = \mathcal{C}(\Omega)$ (continuous bounded functions in $\mathrm{Cl}(\Omega)$ with supremum norm) $\mathfrak{A}_n$ might be the set of values of u at a set of grid points depending on n, or if $E = L^p(\Omega)$ we may define $\mathfrak{A}_n u$ as the totality of means of u in the sets of a finite subdivision of Ω, also depending on n. We refer the reader to **[13]**, **[16]**, **[32]** for examples and additional details. To unburden the notation we indicate from now on by $\|\cdot\|$ the norms of E or of $(E; E)$; the symbol $\|\cdot\|_n$ stands for the norms of E_n, $(E; E_n)$ or $(E_n; E_n)$ (precise identification will follow from the context). The basic ingredient of the results below will be a sequence $\{P_n\}$ of vector-valued distributions, each P_n in $\mathcal{S}_0'((X_n; E_n)) \cap \mathfrak{D}_0'((E_n; X_n))^{-1}$ or subspaces thereof and a distribution P in $\mathcal{S}_0'((X; E)) \cap \mathfrak{D}_0'((E; X))^{-1}$ which will be the limit (in a sense to be made precise later) of the P_n. The assumptions on the complex Banach spaces X, X_n are that $X \subseteq E$, $X_n \subseteq E_n$, the injection being bounded in each case, that is,

$$\begin{aligned} \|u\| &\le C\|u\|_X \qquad (u \in X), \\ \|u_n\|_n &\le C\|u_n\|_{X_n} \quad (u \in X_n, n \ge 1). \end{aligned} \tag{3.4}$$

The norms of X, X_n will be of little further use since all "measurements" shall be made in the norms of E and E_n.

We shall consider incessantly in what follows sequences $\{G_n\}$ of operator-valued distributions; G_n will belong to $\mathfrak{D}_0'((E_n))$ or to subspaces thereof. Given two real numbers η, T ($T > 0$) we shall say that $\{G_n\}$ is *η-uniformly bounded* in $0 \le t \le T$ if

(a) *For each $u_n \in E_n$ the distribution $Y_{-\eta} * G_n u_n$ coincides in $t < T$ with an E_n-valued function $u(\cdot)$ continuous in $t \le T$.*

(b) *There exists a constant C independent of n such that*

$$\|(Y_{-\eta} * G_n u_n)(t)\|_n \le C\|u_n\|_n \qquad (u_n \in E_n, t \le T). \tag{3.5}$$

Note that (b) implies that $(Y_{-\eta} * G_n)(t)$ is a bounded operator in E_n for each $t \le T$ (this also results from (a) and the closed graph theorem). It does not follow, however, that $t \to Y_{-\eta} * G_n$ is a continuous (E_n)-valued function, although $Y_{-\eta+1} * G_n = Y_1 * (Y_{-\eta} * G_n)$ will of course be continuous in the norm of (E_n). The sequence $\{G_n\}$ is *η-strongly uniformly convergent to $G \in \mathfrak{D}_0'(E)$* if (a) holds for $\{G_n\}$, G and $(Y_{-\eta} * G_n \mathfrak{A}_n u)(t) \Rightarrow (Y_{-\eta} * Gu)(t)$ uniformly in $t \le T$ for every $u \in E$; recall this means that

$$\|(Y_{-\eta} * G_n \mathfrak{A}_n u)(t) - \mathfrak{A}_n(Y_{-\eta} * Gu)(t)\|_n \to 0 \tag{3.6}$$

as $n \to \infty$, uniformly for $0 \leq t \leq T$. Similar definitions will be used in $t \geq 0$; if ω is an additional real parameter, the sequence $\{G_n\}$ is declared to be (η, ω)*-uniformly bounded in* $t \geq 0$ if (a) holds in $t \geq 0$ with $u(\cdot)$ continuous there and

$$(3.7) \qquad \|(Y_{-\eta} * G_n u_n)(t)\|_n \leq Ce^{\omega t}\|u_n\| \qquad (u_n \in E_n, t \geq 0).$$

The sequence $\{G_n\}$ is (η, ω)*-strongly uniformly convergent in* $t \geq 0$ if (a) holds for $\{G_n\}$, G in $-\infty < t < \infty$ and if $e^{-\omega t}(Y_{-\eta} * G_n \mathfrak{A}_n u)(t) \Rightarrow e^{-\omega t}(Y_{-\eta} * Gu)(t)$ uniformly in $t \geq 0$.

We note that the diverse notions of boundedness may be formulated in spaces other than (E_n), for instance in $(E_n; X_n)$; it suffices to replace the norm of (E_n) by that of (E_n, X_n) in (3.5) or (3.7) (see a use of this notion in Lemma 3.2 below). The extension of the convergence definitions is not so immediate since we have not introduced "convergence in the norm of X_n" (but see Lemma 4.5).

We introduce one last convergence notion. The sequence G_n is η-*uniformly convergent* in $0 \leq t \leq T$ if the $Y_{-\eta} * G_n$ (resp. $Y_{-\eta} * G$) are (E_n)-continuous functions (resp. an (E)-continuous function) in $0 \leq t \leq T$ and if (3.6) holds uniformly for $0 \leq t \leq T$ and $u \in E$, $\|u\| \leq 1$. The notion of (η, ω)-uniform convergence is correspondingly formulated.

We go back to the distributions $\{P_n\}$, P at the beginning of this section. The sequence $\{P_n\}$ is called *equi-invertible* in a region Δ (closure of an open connected set) of the complex plane if Δ is contained in $\rho(P_n)$ for all n and there exist C, m independent of n such that $\mathfrak{R}_n(\lambda) = \mathfrak{P}_n(\lambda)^{-1} = \mathfrak{L}P_n(\lambda)^{-1}$ satisfies

$$(3.8) \qquad \|\mathfrak{R}_n(\lambda)\|_{(E_n)} \leq C(1 + |\lambda|)^m \qquad (\lambda \in \Delta, n \geq 1).$$

In subsequent uses of the definition the region Δ will be a logarithmic region Λ, a half-plane Ξ, a reverse logarithmic region Ω or a sector Σ.

In the case $E = E_n$, $X = X_n$, $\mathfrak{A}_n = I$ the definitions above are related to familiar concepts. A sequence $\{G_n\}$ in $\mathfrak{D}_0'((E))$ is η-uniformly bounded in $0 \leq t \leq T$ for all $T > 0$ (with η depending in general on T) if and only if it is bounded in $\mathfrak{D}_0'((E))$. On the other hand, $\eta(T)$-uniform convergence of $\{G_n\}$ in $0 \leq t \leq T$ for all T is equivalent to convergence of $\{G_n\}$ in $\mathfrak{D}_0'((E))$, the strong version corresponding to convergence of each $G_n u$ in $\mathfrak{D}_0'(E)$ (this equivalence breaks down in one direction for filters or generalized sequences, see **[29]**). Finally, equi-invertibility of $\{P_n\}$ in a half plane is equivalent to boundedness of S_n in a space $\mathfrak{S}_{0,\omega}'((E))$ for suitable ω. In a somewhat contrived way, some of these equivalences can be extended to the general case. Consider the Banach space $\mathfrak{E}$ of all sequences $\mathfrak{u} = \{u_n\}$ such that $u_n \in E_n$ and

$$\|\mathfrak{u}\|_{\mathfrak{E}} = \sup_n \|u_n\|_n < \infty$$

equipped with the norm $\|\cdot\|_{\mathfrak{E}}$. The sequence $\{G_n\}$, $G_n \in \mathfrak{D}_0'((E_n))$ is $\eta(T)$-uniformly bounded in $0 \le t \le T$ for all T if and only if the map $\varphi \to \mathfrak{G}(\varphi)$ defined by

$$\mathfrak{G}(\varphi)\{u_n\} = \{G_n(\varphi)u_n\} \tag{3.9}$$

defines a distribution in $\mathfrak{D}_0'((\mathfrak{E}))$; (η, ω)-uniform boundedness of $\{G_n\}$ corresponds to the case where $e^{-\omega'\hat{t}}\mathfrak{G} \in \mathfrak{S}_0'((\mathfrak{E}))$ for some ω': this condition for the $(\mathfrak{E})$-valued distribution $\mathfrak{G}$ defined by (3.9) with $G_n = S_n$ is also equivalent to equi-invertibility of the sequence $\{P_n\}$ in a half plane. It is perhaps worth noting that this equivalence between equi-invertibility and boundedness (with half planes naturally replaced by logarithmic regions) does not extend without further restrictions to spaces $\mathfrak{D}'$.

3.1 Example. Let $E = X = E_n = X_n = \mathbf{C}$, $\mathfrak{A}_n = I$, $P_n = e^{n^2}(\delta' - n\delta)$. Then $S_n(\hat{t}) = e^{-n^2}Y_1(t)e^{nt}$. Although $\{S_n\}$ is bounded (in fact, converges to zero) in $\mathfrak{D}'$, $\mathfrak{P}_n(\lambda) = e^{n^2}(\lambda - n)$ so that $\sigma(P_n) = \{n\}$ and there is no logarithmic region Λ contained in all the $\rho(P_n)$.

Note, however, that in the general case the assumption of equi-invertibility of the $\{P_n\}$ implies η-uniform boundedness for all $T > 0$, *for some* $\eta = \eta(T)$. This follows from the "inversion formula" (2.6). It is remarkable that equi-invertibility of the P_n follows simply from η-uniform boundedness of $\{S_n\}$ in (E_n, X_n) for a single interval $0 \le t \le T$ if boundedness conditions are meted out to the P_n. In fact, we have

3.2 Lemma. *Let $P_n \in \mathfrak{S}_0'((X_n;\ E_n)) \cap \mathfrak{D}_0'((E_n;\ X_n))^{-1}$. Assume the sequence $\{P_n\}$ is (η, ω)-uniformly bounded in $t \ge 0$ in the spaces $(X_n;\ E_n)$ for some η, ω. Assume further there exist η', T such that the sequence of inverses $\{S_n\}$ is η'-uniformly bounded in $(E_n;\ X_n)$ in $0 \le t \le T$. Then $\{P_n\}$ is equi-invertible in (E_n) in a logarithmic region Λ.*

The proof can be easily read off that of Theorem 2.5 in **[8]**. Details are omitted.

In what follows, convergence conditions will be forced upon the Laplace transforms $\mathfrak{P}_n(\lambda)$ or on their inverses $\mathfrak{R}_n(\lambda)$. The following simple result will be useful in this connection.

3.3 Lemma. *Let $\mathcal{K}_n \in (X_n;\ E_n)$, $\mathcal{K} \in (X;\ E)$. Assume that $\mathcal{K}$ has an inverse in $(E;\ X)$ and that each $\mathcal{K}_n$ has an inverse in $(E_n;\ X_n)$ such that*

$$\|\mathcal{K}_n^{-1}\|_n \le C \qquad (n \ge 1) \tag{3.10}$$

where $C < \infty$ does not depend on n. Finally, suppose there is a subspace $D \subseteq X$ dense in X such that for every $u \in D$ there exists a sequence $\{u_n\}$, $u_n \in X_n$ with

$$u_n \Rightarrow u, \quad \mathcal{K}_n u_n \Rightarrow \mathcal{K} u \quad \text{as } n \to \infty. \tag{3.11}$$

Then

$$\mathcal{K}_n^{-1} \mathfrak{A}_n u \Rightarrow \mathcal{K}^{-1} u \tag{3.12}$$

for all $u \in E$.

Proof. We take $u \in D$ and write $v = \mathcal{K}u$, $v_n = \mathcal{K}_n u_n$, where $\{u_n\}$ is the sequence provided by the assumptions, so that $v_n \Rightarrow v$. We obtain

$$\begin{aligned} &\|\mathcal{K}_n^{-1}\mathfrak{A}_n v - \mathfrak{A}_n \mathcal{K}^{-1} v\|_n \\ &\quad \leq \|\mathcal{K}_n^{-1} v_n - \mathfrak{A}_n \mathcal{K}^{-1} v\|_n + \|\mathcal{K}_n^{-1}(v_n - \mathfrak{A}_n v)\|_n \\ &\quad \leq \|u_n - \mathfrak{A}_n u\|_n + C\|v_n - \mathfrak{A}_n v\|_n. \end{aligned} \tag{3.13}$$

Since D is dense in X and $\mathcal{K}$ is invertible, the set $\mathcal{K}D$ is dense in E and an obvious approximation argument based on (3.13) shows that (3.12) holds.

4. Convergence results. We denote throughout by P_n a distribution in $\mathcal{S}_0'((X_n;\ E_n)) \cap \mathcal{D}_0'((E_n;\ X_n))^{-1}$ or subspaces thereof and by P a distribution in $\mathcal{S}_0'((X;\ E)) \cap \mathcal{D}_0((E;\ X))^{-1}$; the inverses are $S_n = P_n^{*-1}$, $S = P^{*-1}$.

4.1 THEOREM. *Let the sequence $\{P_n, P\}$ be equi-invertible in a logarithmic region Λ. Assume that*

$$\mathfrak{R}_n(\lambda)\mathfrak{A}_n u \Rightarrow \mathfrak{R}(\lambda) u \qquad (u \in E) \tag{4.1}$$

for $\lambda \in \Lambda$. Then for every $T > 0$ there exists $\eta = \eta(T)$ such that $\{S_n\}$ is η-uniformly bounded and η-strongly uniformly convergent to S in $0 \leq t \leq T$. If (4.1) *holds uniformly with respect to u in $\|u\| \leq 1$ then for every $T > 0$ there exists $\eta = \eta(T)$ such that $\{S_n\}$ is η-uniformly convergent to S in $0 \leq t \leq T$.*

Proof. Let $T > 0$. Consider formula (2.6) with $\gamma > \alpha T + m + 1$ (α the constant in the definition (2.2) of the logarithmic region Λ, m the constant in (3.8)). Writing (2.6) for S and S_n we obtain

$$\begin{aligned} &\|(Y_{-\eta} * S_n \mathfrak{A}_n u)(t) - \mathfrak{A}_n(Y_{-\eta} * Su)(t)\|_n \\ &\quad \leq \frac{1}{2\pi}\int_\Gamma |\lambda|^\eta e^{-(\operatorname{Re}\lambda)t} \|\mathfrak{R}_n(\eta)\mathfrak{A}_n u - \mathfrak{A}_n \mathfrak{R}_n(\lambda) u\|_n \, d|\lambda| \\ &\qquad\qquad (t \leq T, n = 1,2,\ldots). \end{aligned} \tag{4.2}$$

We use now the dominated convergence theorem noting that the integrand is bounded by a constant times $|\lambda|^{\nu}$ with $\nu = \alpha T + m + \eta < -1$. The statement regarding uniform convergence follows in the same way.

4.2 REMARK. The convergence conclusions of Theorem 4.1 can be reinforced if Λ is replaced by larger regions Δ and (4.1) is postulated in Δ. For a half plane Ξ there is (η, ω)-uniform convergence of the $\{S_n\}$ for suitable η, ω; if Δ can be taken to be a reverse logarithmic region $\Omega(\alpha, \beta(\alpha), \omega(\alpha))$ with arbitrarily large α (the constant C in (3.8), *but not* m, may depend on α) there is in addition uniform convergence of $S_n(t)u$ and all derivatives on compacts of $t > 0$. Finally, when Δ is a sector $\Sigma(\varphi + \pi/2, \gamma)$ there exists η, ω such that the sequence $S_{n,\psi}u$ (see the statement of Theorem 2.3) is (η, ω)-strongly uniformly convergent in $t \geq 0$ for some η, ω, uniformly for $|\psi| \leq \varphi' < \varphi$; $S_n(\zeta)u$ is uniformly convergent together with its derivatives of all orders on compacts of $|\zeta| > 0$, $|\arg \zeta| < \varphi$. Corresponding statements hold for uniform convergence.

4.3 REMARK. It is natural to ask whether existence of the limit distribution P in Theorem 4.1 needs to be explicitly postulated or will follow from the assumptions of equi-invertibility of the P_n in a logarithmic region Λ and convergence of the resolvents in Λ. In the present level of generality the answer must be in the negative, since the limit operator $\Re(\lambda)u = \lim \Re_n(\lambda)u$ may not be the resolvent operator of a vector-valued distribution; it may not be one-to-one for some λ or the range of $\Re(\lambda)$, only possible candidate for X, may depend on λ. It is well known that more satisfactory answers exist in particular cases such as $P_n = \delta' \otimes I_n - A_n$ or $P_n = \delta'' \otimes I_n - A_n$ (see §§6 and 7).

4.4 REMARK. Assume that S_n is (η, ω)-uniformly bounded and (η, ω)-strongly uniformly convergent to S in $t \geq 0$. Then the sequence $\{P_n, P\}$ is equi-invertible in a half plane Ξ and (4.1) holds there, with a corresponding statement for uniform convergence; the proof is immediate upon taking Laplace transforms. This converse to Theorem 4.1 does not extend to the general case as Example 3.1 shows. However, a sort of converse can be proved under assumptions of the type of those in Lemma 3.2.

4.5 LEMMA. *Assume the sequence $\{P_n\}$ is (η, ω)-uniformly bounded in $t \geq 0$ in the spaces (X_n, E_n) for some η, ω and that for some η', T the sequence of the inverses $\{S_n\}$ is η'-uniformly bounded in $0 \leq t \leq T$ in $(E_n; X_n)$. Finally assume that for every $T > 0$ there exists $\eta = \eta(T)$ such that $\{S_n\}$ is η-strongly uniformly convergent in $0 \leq t \leq T$ in $(E; X)$ in the* *sense that* $Y_{-\eta} * S_n \mathfrak{A}_n u$ *and* $\mathfrak{A}_n(Y_{-\eta} * Su)$ *coincide with* X_n*-valued continuous functions for* $t \leq T$ *for each* $u \in E$ *and*

$$(4.3) \qquad \left\| \left(Y_{-\eta} * S_n \mathfrak{A}_n u\right)(t) - \mathfrak{A}_n\left(Y_{-\eta} * Su\right)(t)\right\|_{X_n} \to 0 \quad \text{as } n \to \infty$$

uniformly in $0 \leq t \leq T$. *Then* $\{P_n\}$ *is equi-invertible in a logarithmic region* Λ *and* (4.1) *holds in* Λ.

Proof. Lemma 3.2 applies to show that $\{P_n\}$ is equi-invertible in a logarithmic region Λ. We obtain from the formal equation $P_n \mathfrak{A}_n - \mathfrak{A}_n P = -P_n * (S_n \mathfrak{A}_n - \mathfrak{A}_n S) * P$ the equality

$$\begin{aligned} Y_{-3\eta} * P_n \mathfrak{A}_n u &- \mathfrak{A}_n\left(Y_{-3\eta} * P_n u\right) \\ &= -\left(Y_{-\eta} * P_n\right) * \left(Y_{-\eta} * S_n \mathfrak{A}_n - \mathfrak{A}_n\left(Y_{-\eta} * S\right)\right) * \left(Y_{-\eta} * Pu\right) \end{aligned}$$

wherefrom it follows easily that $\{P_n\}$ is η-strongly uniformly convergent in $0 \leq t \leq T$ in $(X_n;\ E_n)$. Since T is arbitrary, the boundedness assumption on the P_n implies that

$$(4.4) \qquad \mathfrak{P}_n(\lambda)\mathfrak{A}_n u \Rightarrow \mathfrak{P}(\lambda)u \quad \text{as } n \to \infty$$

for all $u \in X$. We apply now Lemma 3.3 with $\mathcal{K}_n = \mathfrak{P}_n(\lambda)$, $D = X$, $u_n = \mathfrak{A}_n u$. This concludes the proof.

Although (or because) its assumptions can be easily checked, Theorem 4.1 does not give very significant information on S_n, since the η-strong convergence obtained in each interval $0 \leq t \leq T$ depends on the constants α and m and not on any priori bounds *on* S_n *itself*. In the results that follow, η*-strong uniform convergence on* S_n *in* $0 \leq t \leq T$ *will be deduced from* η*-uniform boundedness in* $0 \leq t \leq T$, with corresponding results for boundedness and convergence in $t \geq 0$. Obviously, this kind of theorem will not hold without certain restrictions on the form of P and of the approximating sequence $\{P_n\}$. To see this, consider the following.

4.6 Example . Let $E = E_n = X = X_n = \mathbf{C}$, $S_n(\hat{t}) = Y_1(\hat{t})(1 + \cos n\hat{t})$. We have $\mathfrak{R}_n(\lambda) = (\mathfrak{L}S_n)(\lambda) = (2\lambda^2 + n^2)/\lambda(\lambda^2 + n^2)$ so that $\mathfrak{P}_n(\lambda) = \mathfrak{R}_n(\lambda)^{-1}$ exists in $\operatorname{Re}\lambda \geq \delta > 0$ and satisfies $|\mathfrak{P}_n(\lambda)| \leq C|\lambda|^3$ there. It follows that there exists $P_n \in \mathfrak{S}_0'$ with $\mathfrak{P}_n(\lambda) = \mathfrak{L}P_n(\lambda) = \mathfrak{R}_n(\lambda)^{-1}$ so that $P_n * S_n = S_n * P_n = \delta$. Obviously, $\mathfrak{R}_n(\lambda) \to 1/\lambda$ and $\mathfrak{P}_n(\lambda) \to \lambda$ uniformly on compacts of $\operatorname{Re}\lambda \geq \delta$ so that $P_n \to \delta' = Y_{-1}$ and $S_n \to Y_1$ in $\mathfrak{S}_0'$. However, $S_n(\hat{t})$, although uniformly bounded in $t \geq 0$, is not even pointwise convergent there.

To prevent this kind of phenomenon, we shall require that P and each P_n "have a leading term" in the following sense: there exists a real number

κ independent of n such that

$$(4.5)\qquad P = Y_{-1-\kappa} \otimes I - Y_{-\kappa} * \Theta = Y_{-\kappa} * (\delta' \otimes I - \Theta),$$
$$P_n = Y_{-\kappa} * (\delta' \otimes I_n - \Theta_n)$$

where $\Theta \in \mathcal{S}_0'((X;\, E))$, $\Theta_n \in \mathcal{S}_0'((X_n;\, E_n))$ for $n \geq 1$. In all later applications Θ (resp. Θ_n) will be an $(X;\, E)$-valued (resp. $(X_n;\, E_n)$-valued) measure, thus $Y_{-\kappa} * \Theta$ will be of "lower order" than $Y_{-1-\kappa} \otimes I$. So will be the Θ_n.

The final ingredient in the statement of Theorem 4.7 below is the requirement that there must exist elements of X to which "Θ can be applied several times" with similar assumptions on X_n, Θ_n. Given an integer $p \geq 1$ and an arbitrary real number ξ we define $D(p, \xi)$ as the set of all $u \in X$ such that

(a) $\Theta u \in \mathfrak{D}_0'(X), \Theta * \Theta u \in \mathfrak{D}_0'(X), \ldots, \Theta^{*(p-1)}u \in \mathfrak{D}_0'(X)$

(b) $Y_{2+\xi} * \Theta u$, $Y_{3+\xi} * \Theta^{*2}u, \ldots, Y_{p+\xi} * \Theta^{*(p-1)}u$ *are E-continuous in* $t \geq 0$.

Note that, although $\Theta^{*p}u$ exists, no conditions are imposed on it.

The spaces $D_n(p, \xi) \subseteq X_n$ are defined in the same way with respect to the Θ_n.

4.7 THEOREM. *Let $P \in \mathcal{S}_0'((X;\, E))$, $P_n \in \mathcal{S}_0'((X_n;\, E_n))$ have a leading term in the sense of* (4.5) *with κ independent of n. Let the sequence $\{P_n, P\}$ be equi-invertible in a logarithmic region Λ and assume the inverses $\{S_n\}$ are η-uniformly bounded in $0 \leq t \leq T < \infty$ with $\eta \leq \kappa$. Finally, assume the space $\tilde{D}(p, \kappa - \eta)$ with $p > \eta + \alpha T + m + 1$ (α the constant in* (2.2) *for Λ, m the constant in* (3.8)) *possesses a subspace $\tilde{D}(p, \kappa - \eta)$ such that* (a) *$\tilde{D}(p, \kappa - \eta)$ is dense in X in the topology of X.* (*b*) *For each $u \in \tilde{D}(p, \kappa - \eta)$ there exists a sequence $\{u_n\}$, $u_n \in D_n(p, \kappa - \eta)$ such that*

$$(4.6)\qquad \left(Y_{j+\kappa-\eta} * \Theta_n^{*(j-1)}u_n\right)(t) \Rightarrow \left(Y_{j+\kappa-\eta} * \Theta^{*(j-1)}u\right)(t) \quad \text{as } n \to \infty$$
$$(1 \leq j \leq p+1)$$

uniformly in $0 \leq t \leq T$; moreover, if $\mathfrak{f}_j$ (resp. *$\mathfrak{f}_{n,j}$*) *denotes the Laplace transform of $\Theta^{*j}u$* (resp. *$\Theta_n^{*j}u_n$*),

$$(4.7)\qquad \left\|\mathfrak{f}_{n,p}(\lambda)\right\|_n \leq C \qquad (\operatorname{Re}\lambda \geq \omega, n = 1, 2, \ldots)$$

where C, ω do not depend on n, and

$$(4.8)\quad \mathfrak{f}_{n,1}(\lambda) \Rightarrow \mathfrak{f}_1(\lambda), \quad \mathfrak{f}_{n,p}(\lambda) \Rightarrow \mathfrak{f}_p(\lambda) \quad \text{as } n \to \infty \qquad (\operatorname{Re}\lambda \geq \omega).$$

Under these hypotheses we conclude that the sequence $\{S_n\}$ is η-strongly uniformly convergent in $0 \leq t \leq T$ to S, that is,

$$(4.9)\qquad \left(Y_{-\eta} * S_n\mathfrak{A}_n u\right)(t) \Rightarrow \left(Y_{-\eta} * Su\right)(t) \quad \text{as } n \to \infty$$

uniformly in $0 \leq t \leq T$ for all $u \in E$.

Proof. Since $S_n * (Y_{-1-\kappa} \otimes I_n - Y_{-\kappa} * \Theta_n) = \delta \otimes I_n$, convolving both sides with $Y_{1+\kappa-\eta}$ we obtain

$$(4.10) \qquad (Y_{-\eta} * S_n)u_n = Y_{1+\kappa-\eta} \otimes u_n + Y_1 * (Y_{-\eta} * S_n) * \Theta_n u_n \qquad (u_n \in X_n).$$

On the other hand, if U_n is an arbitrary distribution in $\mathfrak{D}_0'(X_n)$ a similar computation yields

$$(4.11) \quad (Y_{-\eta} * S_n) * U_n = Y_{1+\kappa-\eta} * U_n + Y_1 * (Y_{-\eta} * S_n) * \Theta_n * U_n.$$

Assume now that $u \in \tilde{D}(p, \kappa - \eta)$ and let $\{u_n\}$ be the sequence postulated in the statement of Theorem 4.7. Making use of (4.10) in (4.11) (with $U_n = \Theta_n u_n$) and iterating the procedure, we obtain

$$(4.12) \quad \begin{aligned} (Y_{-\eta} * S_n)u_n &= Y_{1+\kappa-\eta} \otimes u_n + Y_{2+\kappa-\eta} * \Theta_n u_n \\ &\quad + Y_2 * (Y_{-\eta} * S_n) * \Theta^{*2} u_n \\ &= \sum_{j=1}^{p} Y_{j+\kappa-\eta} * \Theta_n^{*(j-1)} u_n + (Y_{p-\eta} * S_n) * \Theta_n^{*p} u_n. \end{aligned}$$

We make now use of the representation (2.6), obtaining

$$(4.13) \quad \begin{aligned} (Y_{-\eta} * S_n u_n)(t) &= \sum_{j=1}^{p} (Y_{j+\kappa-\eta} * \Theta^{*(j-1)} u_n)(t) \\ &\quad + \frac{1}{2\pi i} \int_\Gamma \lambda^{\eta-p} e^{\lambda t} \mathfrak{R}_n(\lambda) \mathfrak{f}_{p,n}(\lambda)\, d\lambda \\ &\qquad (0 \le t \le T,\ n = 1, 2, \dots). \end{aligned}$$

We apply Lemma 3.3 to $\mathcal{K} = \mathfrak{P}(\lambda)$, $\mathcal{K}_n = \mathfrak{P}_n(\lambda)$; since, in view of the first condition (4.8) we have $\mathfrak{P}_n(\lambda)u \Rightarrow \mathfrak{P}(\lambda)u$ for $u \in \tilde{D}$ it follows that (4.1) holds. We note now that

$$\begin{aligned} &\mathfrak{R}_n(\lambda)\mathfrak{f}_{n,p}(\lambda) - \mathfrak{A}_n \mathfrak{R}(\lambda)\mathfrak{f}_p(\lambda) \\ &\quad = \mathfrak{R}_n(\lambda)(\mathfrak{f}_{n,p}(\lambda) - \mathfrak{A}_n \mathfrak{f}_p(\lambda)) + (\mathfrak{R}_n(\lambda)\mathfrak{A}_n - \mathfrak{A}_n \mathfrak{R}(\lambda))\mathfrak{f}_p(\lambda) \end{aligned}$$

and work with the integral (4.13) much in the same way as in Theorem 4.1 (see (4.2)); again the integrand is bounded by a constant times $|\lambda|^\nu$ with $\nu = \eta + \alpha T + m - p < -1$. Using the dominated convergence theorem we deduce that $(Y_{-\eta} * S_n u_n)(t) \Rightarrow (Y_{-\eta} * Su)(t)$ uniformly in $0 \le t \le T$. Observe next that

$$\begin{aligned} &Y_{-\eta} * S_n \mathfrak{A}_n u - \mathfrak{A}_n(Y_{-\eta} * Su) \\ &\quad = Y_{-\eta} * S_n(\mathfrak{A}_n u - u_n) + (Y_{-\eta} * S_n u_n - \mathfrak{A}_n(Y_{-\eta} * Su)), \end{aligned}$$

therefore (4.9) stands proved for $u \in \tilde{D}$ in view of the π-uniform boundedness assumption. A similar argument extends (4.9) from $u \in \tilde{D}$ to $u \in E$ since $\tilde{D}$ is dense in X, thus in E.

4.8 REMARK. If the $\{S_n\}$ are (η, ω)-uniformly bounded in $t \geq 0$ then the $\{P_n\}$ are equi-invertible in a half plane Ξ (see §3). No significant improvement in Theorem 4.3 occurs, except that the condition on p is $p > \eta + m + 1$, good for all $T > 0$.

4.9 REMARK. The convergence conclusions of Theorem 4.7 can be reinforced in case the P_n belong to the subclasses of $\mathfrak{D}_0'((E;\ X))^{-1}$ introduced in §2. We look first at the class $\mathcal{C}_+^\infty((E;\ X))^{-1}$ of abstract parabolic distributions, where the hypotheses are reinforced as follows: (I) For every $\alpha > 0$ the sequence $\{P_n\}$ is equi-invertible in a reverse logarithmic region $\Omega(\alpha, \beta, \omega)$ where β, ω may depend on α; the constant C in (3.7) (*but not* m) may also depend on α. (II) The Laplace transforms $\mathfrak{f}_j$, $\mathfrak{f}_{n,j}$ can be analytically extended to the reverse logarithmic regions $\Omega(\alpha, \beta, \omega)$ above with preservation of both relations (4.8). Then the convergence conclusions of Theorem 4.7 can be supplemented with uniform convergence of $S_n(\cdot)u$ and all derivatives on compact subsets of $t > 0$; the argument is the same in Remark 4.2.

Consider now the class $\mathfrak{A}(\varphi;\ (E;\ X))^{-1}$. Assume the sequence $\{P_n\}$ is equi-invertible in a sector $\Sigma = \Sigma(\varphi + \pi/2, \gamma)$ and that $\mathfrak{f}_j$, $\mathfrak{f}_{n,j}$ can be analytically extended to Σ with preservation of both relations (4.8). Then we obtain convergence of $S_n(\zeta)$ uniformly on compacts of $\Sigma(\varphi') \cap \{\zeta;\ |\zeta| \geq \varepsilon\}$ for every $\varepsilon > 0$. Convergence statements in the whole sector $\Sigma(\varphi)$ may be obtained under the equi-invertibility assumption above. Assume that (III) For a $T > 0$ there exists η such that $Y_{-\eta} * S_{n,\psi}$ coincides with a jointly continuous function of t, ψ in $t \leq T$, $|\psi| \leq \varphi$ and

$$\|(Y_{-\eta} * S_{n,\psi}u)\|_n \leq C\|u\|_n \tag{4.14}$$

there, for C independent of n (see the definition of $S_{n,\psi}$ in §2). (IV) The convergence relations (4.6) can be analytically extended to the sector $\Sigma_+(\varphi)$ (that is, $Y_{j+k-\eta} * \Theta^{*(j-1)}u$, $Y_{j+k-\eta} * \Theta_n^{*(j-1)}u_n$ admit analytic extensions to the sector $\Sigma_+(\varphi)$ and convergence is uniform in $|\arg\zeta| \leq \varphi$, $|\zeta| \leq T$). (V) The functions $\mathfrak{f}_1$, $\mathfrak{f}_{n,1}$, $\mathfrak{f}_p$, $\mathfrak{f}_{n,p}$ can be analytically extended to the sector $\Sigma(\varphi + \pi/2, \gamma)$ with preservation of (4.7) and of both relations (4.8). Then the convergence relation (4.9) can be extended to $|\zeta| \leq T$, $|\arg\zeta| \leq \varphi$ and is uniform there. This convergence property will be called *η-strong uniform convergence* of $\{S_n\}$ in $|\arg\zeta| \leq \varphi$; likewise, (4.14) is *η-uniform boundedness* of S in $|\arg\zeta| \leq \varphi$, $|\zeta| \leq T$.

We point out finally that Theorem 4.7 contains results of Chen and Grimmer [**2**].

5. Convergence in other intervals. The requirement that $\eta \leq \kappa$ in Theorem 4.3 is essential; if $\eta > \kappa$, $Y_{1+\kappa-\eta}$ is unbounded near the origin and $D(p, \kappa - \eta)$ is empty. Moreover, the hypothesis of η-uniform boundedness of $\{S_n\}$ in any interval $0 \leq t \leq T$ is in general contradictory if $\eta > \kappa$ as we see taking $E = E_n = X = X_n = \mathbf{C}$, $P_n = P = \delta' = Y_{-1}$. Nevertheless, the restriction that $\eta \leq \kappa$ can be totally lifted if η-uniform boundedness and convergence are formulated in intervals $\varepsilon \leq t \leq T$, $\varepsilon > 0$ rather than $0 \leq t \leq T$. Theorem 4.3 is modified as follows: the convergence relations (4.6) are postulated in $\varepsilon \leq t \leq T$ and the convergence conclusion is likewise obtained in $\varepsilon \leq t \leq T$, the other hypotheses remaining unchanged. We omit the details.

6. Application. $P = \delta' \otimes I - \delta \otimes A$. Here A is a closed, densely defined operator in E and $X = D(A)$ equipped with the graph norm. We also assume $P_n = \delta' \otimes I_n - \delta \otimes A_n$ with $X_n = D(A_n)$ and A_n having similar properties in E_n (so that $\Theta = \delta \otimes A$, $\Theta_n = \delta \otimes A_n$). The assumption that $P \in \mathfrak{D}_0'((X; E))^{-1}$ simply means that A generates a regular distribution semigroup in the sense of Lions **[19]**, a similar statement holding for A_n; we have $\mathfrak{P}(\lambda) = \lambda I - A$, $\mathfrak{P}_n(\lambda) = \lambda I_n - A_n$, $\mathfrak{R}(\lambda) = (\lambda I - A)^{-1} = R(\lambda; A)$, $\mathfrak{R}_n(\lambda) = R(\lambda; A_n)$ hence equi-invertibility of $\{P_n\}$ in a region Δ signifies existence of $R(\lambda; A_n)$ in Δ for all n and that an inequality of the form

$$\|R(\lambda; A_n)\|_n \leq C(1 + |\lambda|)^m \qquad (\lambda \in \Delta, n \geq 1) \tag{6.1}$$

holds, with C and m independent of n. We consider first some applications of Theorem 4.1.

(I) *Exponential formulas for distribution semigroups*: **[11]**. Somewhat imprecisely, an *exponential formula* or *representation theorem* is a formula that justifies writing $S(\hat{t}) = P^{*-1} = e^{\hat{t}A}$. The "inverse Laplace transform" (2.6) obviously deserves the title. Other formulas look like

$$S(\hat{t}) = \lim Y_1(\hat{t})e^{\hat{t}A_n} \tag{6.2}$$

where the A_n are, say, bounded operators and the limit is understood in the sense of distributions; usually one takes $E = E_n = X = X_n$. Of special interest is the *Yosida approximation* where $A_\nu = \nu A R(\nu; A) = \nu^2 R(\nu; A) - \nu I$, ν sufficiently large (the subindexing by a continuous variable causes no problems). A simple computation shows that if λ is a complex number $\neq -\nu$ and $\lambda\nu(\lambda + \nu)^{-1} \in \rho(A)$ then $\lambda \in \rho(A_\nu)$ and

$$R(\lambda; A_\nu) = \frac{1}{\lambda + \nu}\left[I + \frac{\nu^2}{\lambda + \nu} R\left(\frac{\lambda\nu}{\lambda + \nu}; A\right)\right]. \tag{6.3}$$

Given a region Δ and a positive real number σ denote by $\Delta_\sigma = \{\lambda;\ \lambda - \sigma \in \Delta\}$, the *right translate* of Δ. It is obvious that translates of logarithmic regions, half planes, reverse logarithmic regions and sectors are regions of the same type; the parameter α and the angle φ in the last two cases remain unchanged. Let Λ be a logarithmic region. For every $\nu \geq 0$ there exists $\sigma(\nu) > 0$, $\sigma(\nu)\downarrow 0$ as $\nu \to \infty$ such that the function $\zeta_\nu(\lambda) = \lambda\nu(\lambda + \nu)^{-1}$ maps $\Lambda_{\sigma(\nu)}$ into Λ (see [**3**, p. 541]); moreover, $|\zeta_\nu(\lambda)| \leq \nu|\lambda|(\nu + \omega)$ thus it follows from (6.3) that the sequence $\{P_\nu\} = \{\delta' \otimes I - \delta \otimes A_\nu\}$ is equi-invertible in a suitable logarithmic region Λ_σ. Finally, since $\zeta_\nu(\lambda) \to \lambda$ as $\nu \to \infty$ it follows again from (6.3) that $R(\lambda;\ A_n) \to R(\lambda;\ A)$ in (E) for every $\lambda \in \Lambda$. Consequently, it follows from Theorem 4.1 that (6.2) holds for the $\{A_\nu\}$ in the topology of $\mathcal{D}'(E)$; this was proved by similar means in [**8**]. Since the maps $\zeta_n(\lambda)$ enjoy the same properties in relation to half planes, reverse logarithmic regions and sectors, the improvements to the convergence pointed out in Remark 4.2 hold if A belongs to $\mathcal{S}_0'((E;\ X))^{-1}$, $\mathcal{C}^\infty((E;\ X))^{-1}$ or $\mathcal{Q}(\varphi;(E;\ X))^{-1}$.

In order to apply Theorem 4.7 we check first the hypotheses independent of η-uniform boundedness. It is easily verified that the set $D(p, \xi)$ coincides with $D(A^p)$, the domain of A^p, for any $\xi \geq 0$; note that

$$D(A^p) = R(\lambda; A)^p E \qquad (\lambda \in \rho(A))$$

is dense in $X = D(A)$ in the topology of X. The same considerations apply verbatim to every P_n. The convergence assumptions in Theorem 4.7 can all be derived from either convergence of A_n or of the resolvents $R(\lambda;\ A_n)$ for a single λ. This is made explicit in the following result.

6.1 Lemma. *Let $\{P, P_n\}$ be equi-invertible in Δ. Assume that*

$$R(\lambda; A_n)\mathfrak{A}_n u \Rightarrow R(\lambda; A)u \qquad (n \to \infty) \tag{6.4}$$

for a single $\lambda = \lambda_0 \in \Delta$ and all $u \in E$. Then (a) (6.4) *holds for every $\lambda \in \Delta$,* (b) *Given an arbitrary integer p and a $u \in D(A^p)$ there exists a sequence $\{u_n\}$, $u_n \in D(A_n^p)$ such that*

$$u_n \Rightarrow u, \quad A_n^k u_n \Rightarrow A_u^k \tag{6.5}$$

for $1 \leq k \leq p$. Conversely, (6.5) *for $k = 1$ in a subspace $D \subseteq X$ dense in X implies* (6.4) *for $\lambda \in \Delta$, $u \in E$.*

Proof. We have

$$\begin{aligned} R(\lambda_0; A_n)^2\mathfrak{A}_n u &- \mathfrak{A}_n R(\lambda_0; A)^2 u \\ &= R(\lambda_0; A_n)(R(\lambda_0; A_n)\mathfrak{A}_n u - \mathfrak{A}_n R(\lambda_0; A)u) \\ &\quad + (R(\lambda_0; A_n)\mathfrak{A}_n - \mathfrak{A}_n R(\lambda_0; A))R(\lambda_0; A)u \end{aligned}$$

so that in view of (6.1) and (6.4) we have $R(\lambda_0; A_n)^2\mathfrak{A}_n u \Rightarrow R(\lambda_0; A)^2 u$. Arguing in the same way we show that

$$(6.6) \qquad R(\lambda_0; A_n)^k \mathfrak{A}_n u \Rightarrow R(\lambda_0; A)^k u$$

for $u \in E$, $k = 1, 2, \ldots$. We use now the Taylor series of the resolvents in $|\lambda - \lambda_0| < r = \{\max_{n\geq 1}(\|R(\lambda_0; A)\|, \|R(\lambda; A_n)\|)\}^{-1}$; since

$$R(\lambda; A_n)\mathfrak{A}_n - \mathfrak{A}_n R(\lambda; A) = \Sigma(\lambda_0 - \lambda)^k \big(R(\lambda_0; A_n)^{k+1}\mathfrak{A}_n - \mathfrak{A}_n R(\lambda_0; A)^{k+1}\big)$$

it follows from (6.4) that the convergence relation (6.2) can be extended to $|\lambda - \lambda_0| < r$. An obvious argument using the connectedness of Δ and the bound (6.1) then shows that (6.4) holds for every $\lambda \in \Delta$, thus proving (a). To show (b), we fix $\lambda \in \Delta$ and set $u_n = R(\lambda; A_n)^p \mathfrak{A}_n u$; we note that, if $1 \leq k \leq p$,

$$A^k u_n = A^k R(\lambda; A_n)^p u = -A^{k-1}R(\lambda; A_n)^{p-1}u + \lambda A^{k-1}R(\lambda; A_n)^p u$$

and use an obvious inductive reasoning.

The fact that (6.5) for $k = 1$ implies (6.4) follows from Lemma 3.3: we take $\mathcal{K}_n = \lambda I - A_n$.

A last observation concerns Lemma 3.2. Note that $Y_2 * P_n = Y_1 \otimes I_n - Y_2 \otimes A_n$, hence $\{P_n\}$ is $(-2, \omega)$-uniformly bounded in $t \geq 0$ in the spaces $(X_n; E_n)$ for any $\omega > 0$. Assume that $\{S_n\}$ is η-uniformly bounded in $0 \leq t \leq T$. Since $A_n S_n = \delta' * S_n - \delta \otimes I_n$ it follows that $A_n(Y_{-\eta+1} * S_n) = Y_{-\eta} * S_n - Y_{-\eta+1} \otimes I_n$. Accordingly, $\{S_n\}$ is $(\eta + 1)$-uniformly bounded in $0 \leq t \leq T$ in the spaces $(E_n; X_n)$. An application of Lemma 3.2 then completes the proof of the following result.

6.2 LEMMA. *Assume each* $P_n = \delta' \otimes I - \delta \otimes A_n$ *belongs to* $\mathfrak{D}_0'((E_n; X_n))^{-1}$, *and that the sequence* $\{S_n\}$ *is* η*-uniformly bounded in* $0 \leq t \leq T$ *for some* η, T $(T > 0)$. *Then* $\{P_n\}$ *is equi-invertible in a logarithmic region* Λ.

We indicate below how several results in the literature can be obtained in substantially generalized versions as particular cases of Theorem 4.7 and of the preceding observations.

(II) *The Trotter-Kato theorem for distribution semigroups.* In [**32**], Trotter assumes that A and the A_n generate strongly continuous semigroups (i.e. that $S = (\delta' \otimes I - \delta \otimes A)^{*-1}$, $S_n = (\delta' \otimes I - \delta \otimes A)^{*-1}$ are strongly continuous in the spaces where they live). The boundedness assumption on the $\{S_n\}$ corresponds to our $(0, \omega)$-uniformly boundedness in $t \geq 0$, which has been seen to imply (see §3) that the $\{P_n\}$ are equi-invertible in a half plane Ξ. The convergence conditions are (6.4) at a single point of Ξ or the first condition (6.5) in a dense subset of X. It follows from Theorem 4.7, Lemma 6.1 and Lemma 6.2 that the following generalization of Theorem 5.1 in [**32**] holds:

6.3 THEOREM. *Let* $P = \delta' \otimes I - \delta \otimes A \in \mathfrak{D}_0'((E;\ X))^{-1}$, $P_n = \delta' \otimes I_n - \delta \otimes A_n \in \mathfrak{D}_0'((E_n;\ X_n))^{-1}$. *Assume the* $\{S_n\}$ *are* η*-uniformly bounded in* $0 \le t \le T$ *for some* $T > 0$ *and* $\eta \ge 0$ *and*

$$R(\lambda_0; A_n)\mathfrak{A}_n u \Rightarrow R(\lambda_0; A)u \tag{6.7}$$

for all $u \in E$ *and a fixed* λ_0 *in* Λ (Λ *the region in Lemma* 6.2) *or that there exists a subspace* $D \subseteq X$ *dense in* X *and such that for each* $u \in D$ *there is a sequence* $\{u_n\}$, $u_n \in X_n$ *with*

$$u_n \Rightarrow u, \quad A_n u_n \Rightarrow Au. \tag{6.8}$$

Then $\{S_n\}$ *is* η*-strongly uniformly convergent to* S *in* $0 \le t \le T$, *that is, for each* $u \in E$,

$$(Y_{-\eta} * S_n \mathfrak{A}_n u)(t) \Rightarrow (Y_{-\eta} * Su)(t) \tag{6.9}$$

uniformly in $0 \le t \le T$.

6.4 REMARK. (η, ω)-uniform boundedness in $t \ge 0$ does not guarantee (η, ω)-uniform convergence even if $\eta = 0$. To see this let $X = E = E_n$ be a separable Hilbert space, $\{e_k;\ k \ge 1\}$ a complete orthonormal system in E, $A = 0$, $A_n(\Sigma\, c_k e_k) = \Sigma(ik/n)c_k e_k$, $X_n = D(A_n)$ (note that all the X_n coincide). A_n generates a strongly continuous semigroup S_n given by $S_n(\Sigma\, c_k e_k) = \Sigma \exp(ikt/n)c_k e_k$ and the $\{S_n\}$ satisfy $\|S_n(t)\| = 1$ in $t \ge 0$. Either (6.7) or (6.8) are easily checked. However, $S_n(t)$ does not converge uniformly to $S(t) = I$ in $t \ge 0$.

It should be pointed out that in Trotter's Theorem 5.1 in [**32**] (as completed by Kato [**12**]) the existence of A and the fact that it is a semigroup generator are not postulated as in Theorem 6.3 but *proved*, i.e. A is obtained as the operator whose resolvent is $R(\lambda)u = \lim R(\lambda;\ A_n)\mathfrak{A}_n$. However, some reinforcement of the assumptions is necessary. In the case considered by Trotter and Kato ((0, ω)-uniform boundedness in $t \ge 0$) we have

$$\|R(\lambda; A)\| \le C(\operatorname{Re}\lambda - \omega)^{-1} \qquad (\operatorname{Re}\lambda \ge \omega) \tag{6.10}$$

and $R(\lambda)$ is the resolvent of a semigroup generator A if we add the assumption that

$$\nu R(\nu; A)A_n u \Rightarrow u \quad \text{as } \nu \to \infty \tag{6.11}$$

uniformly with respect to n (see [**13**, p. 512]); this condition is used to show that the common nullspace N of all the operators $R(\lambda)$ reduces to zero, or, equivalently, that R, the common range of all the $R(\lambda)$ is dense in E (since $\mathrm{Cl}(R) \cap N = \{0\}$). However, inequality (6.10), at least for λ real, is necessary for the proof and it is not clear whether any reasonable analogue of (6.11) would do the trick under the less stringent bound (6.1). Nevertheless, it is possible to impose somewhat contrived conditions on the $\{P_n\}$ that entirely dispense with assumptions on existence or properties of A, as seen below.

(III) *The Trotter-Kato theorem for distribution semigroups "in the absence of A."* Consider Theorem 6.3 with the following modifications; the operator A is omitted from the statement and condition (6.7) is weakened to: for every $u \in E$ there exists $v \in E$ such that

$$R(\lambda_0; A_n)\mathfrak{A}_n u \Rightarrow v \tag{6.12}$$

where λ_0 is a fixed element of the logarithmic region Λ in which the $\{P_n\}$ are equi-invertible (see Lemma 6.2). Working in the style of Lemma 6.1 we can extend the limit relation (6.12) to arbitrary $\lambda \in \Lambda$ and use it to define an (E)-valued holomorphic function $R(\lambda)$ $(R(\lambda)u = v)$ that satisfies $R(\mu) - R(\lambda) = (\lambda - \mu)R(\lambda)R(\mu)$, i.e. a *pseudo-resolvent* ([**34**, p. 215]). We impose the following additional condition on R: *for some* $\lambda \in R(\lambda)$ *is one-to-one and has dense domain*. This is known to imply that there exists a closed, densely defined operator A in E such that $R(\lambda) = R(\lambda; A)$. From this point on we can apply Theorem 6.3 in its original form and obtain (6.9). Theorems of this type were obtained by Takahashi and Oharu [**31**].

A natural generalization of the preceding argument consists of giving up the assumption of denseness of $R(\lambda)$ (although we still require the $R(\lambda)$ to be one-to-one). In this case the operator A is not densely defined $(D(A) = R)$ but the convergence relation (6.4) can still be proved in $E_0 = AR$. We leave the easy checking of the details to the reader. The result obtained generalizes theorems of Takahashi-Oharu [**31**] and Cannon [**1**].

Similar generalizations of Theorem 6.3 can be obtained under weakened versions of condition (6.8). This has been done for strongly continuous semigroups by Kurtz [**16**], [**17**]. Given a sequence of (in general unbounded) operators A_n in E_n, Kurtz defines the *extended limit* A of the A_n (in symbols, $A = \text{ex-lim } A_n$) as the operator in E whose domain consists of all $u \in E$ such that there exists a sequence $\{u_n\}$, $u_n \in E_n$ with

$$u_n \Rightarrow u, \quad A_n u_n \Rightarrow v \tag{6.13}$$

and $Au = v$. In the general case, the operator A *may be multivalued*, i.e. v may depend on the sequence $\{u_n\}$. Reasonable conditions on the A_n that guarantee that A is one-valued have been given by Kurtz ([**16**, p. 355]) but they are essentially based on inequalities of the type of (6.10) holding uniformly for all the A_n and it is not clear whether suitable analogues will hold under the more general bound (6.1).

(IV) *Convergence results for semigroups strongly continuous in* $t > 0$. Diverse generalizations of the Trotter-Kato theorem have been proved for classes of semigroups strongly continuous in $t > 0$ but not in $t \geq 0$, for instance for Hille's class A (see Oharu-Sunouchi [**22**]) or for Zabreiko-Zafievskii's class $A^{0,\alpha}$ (see Ponomarev [**25**]); for the definition of the classes see [**35**] and [**36**]. These results can be embraced by the present theory: we limit ourselves here to the class $(1, A)$ consisting of all semigroups $\{S(t);\ t > 0\}$ strongly continuous for $t > 0$ such that $\|S(\cdot)u\|$ is summable near zero for every $u \in E$ and such that $S(t)u \to u$ in the sense of Abel for every $u \in E$, that is,

$$\lim_{\lambda \to \infty} \lambda \int_0^\infty e^{-\lambda t} S(t)u\, dt = u \qquad (u \in E)$$

(note that strong continuity in $t > 0$ implies exponential growth of $\|S(t)\|$). Finally we assume that the union of all subspaces $S(t)E$, $t > 0$ is dense in E. Under these assumptions the Laplace transform $R(\lambda) = \mathfrak{L}S(\lambda)$ is a pseudo-resolvent in $\operatorname{Re}\lambda > \omega$ such that every $R(\lambda)$ is one-to-one and has dense domain, so that there exists a closed, densely defined operator A with $R(\lambda; A) = R(\lambda)$. We easily check that this means that $P = \delta' \otimes I - \delta \otimes A \in \mathfrak{D}_0'((E;\ D(A))^{-1}$ and $S = P^{*-1}$, thus all the results in §4 may be applied. However, since S may possess a singularity at $t = 0$, those in §5 are perhaps of more interest.

(V) *Convergence results for analytic semigroups and distribution semigroups*. Let the operator A be such that $\delta' \otimes I - \delta \otimes A \in \mathfrak{A}(\varphi;\ (E;\ D(A)))^{-1}$ for $0 < \varphi \leq \pi$ (see §2) and let A_n be operators enjoying similar properties in their home spaces E_n. The following result obtains convergence in whole sectors of the complex plane: in it, as in §2, we denote by S_ψ the distribution that coincides with $S(te^{i\psi})$ for $t > 0$ and with $t < 0$, and by $S_{n,\psi}$ the distributions similarly defined from the S_n.

6.5. THEOREM. *Let* $T > 0$, $\eta > 0$ *be such that* $\{S_n\}$ *is* η*-uniformly bounded in* $|\arg \zeta| \leq \varphi$, $|\zeta| \leq T$; *moreover, let the estimate*

$$\|S_n(\zeta)\|_n \leq Ce^{\omega|\zeta|}$$

hold in $|\arg \zeta| \leq \varphi$, $|\zeta| \geq \varepsilon$ *with* C *independent of* n. *Finally, assume there exists a subspace* $D \subseteq X$ *dense in* X *such that for each* $u \in D$ *there is a sequence* $\{u_n\}$, $u_n \in X_n$ *such that* (6.8) *holds. Then, for each* $\varphi' < \varphi$, $\{S_n\}$ *is*

η-strongly uniformly convergent to S in $|\arg\zeta| \leq \varphi' < \varphi$, $|\zeta| \leq T$ *and*

$$S_n(\zeta)\mathfrak{A}_n U \Rightarrow S(\zeta)u \qquad (u \in E)$$

uniformly on compacts of $|\arg\zeta| \leq \varphi'$, $|\zeta| \geq \varepsilon$ *for every* $\varepsilon > 0$.

We sketch the proof. Both boundedness conditions guarantee equi-invertibility in a sector $\Sigma = \Sigma(\varphi + \pi/2, \gamma)$ (by an obvious deformation-of-contour argument). On the other hand, the convergence assumption on the A_n and Lemma 6.3 produce the convergence relation (6.7) in Σ thus the conclusion on convergence in $\varepsilon \leq |\zeta| \leq \varepsilon^{-1}$ follows from Remark 4.8. The remaining convergence statement follows then from Lemma 6.3 and Remark 4.8.

Clearly, the same conclusion can be obtained if (6.7) is *assumed* for a single $\lambda_0 \in \Sigma$.

Theorem 6.5 contains results of Piskarev **[23]** where the S_n and S are analytic semigroups (i.e. strong continuity in $|\arg\zeta| \leq \varphi$ is assumed).

7. Application. $P = \delta'' \otimes I - \delta' \otimes B - \delta \otimes A$. We consider first the case $B = 0$. As in the previous section A is closed and densely defined in E; we also assume $P_n = \delta'' \otimes I_n - \delta \otimes A_n$ with A_n closed and densely defined in E_n. The spaces X, X_n are defined as in Section 6. We have $P = Y_{-1} * (\delta' \otimes I - Y_1 \otimes A)$ with a similar equality for each P_n so that $\Theta = Y_1 \otimes A$, $\Theta_n = Y_1 \otimes A_n$; since $\kappa = 1$ in (4.5), the allowable range of η in Theorem 4.7 is $\eta \leq 1$. We have $\mathfrak{P}_n(\lambda) = \lambda^2 I - A_n$, $\mathfrak{R}_n(\lambda) = \mathfrak{R}(\lambda^2; A_n)$ so that equi-invertibility of the $\{P_n\}$ in a region Δ means existence of $R(\lambda^2; A_n)$ for $\lambda \in \Delta$ for all n and

$$\|R(\lambda^2; A_n)\|_n \leq C(1 + |\lambda|)^m \qquad (\lambda \in \Delta, n \geq 1). \tag{7.1}$$

All the results in §6 have immediate counterparts here. We note that if A is a bounded, everywhere defined operator in E then $P \in \delta'' \otimes I - \delta \otimes A \in \mathfrak{D}_0'((E))$ with

$$\begin{aligned} S(t) &= Y_1(t)A^{-1/2}\sin tA^{1/2} \\ &= Y_1(t)\sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!}A^k. \end{aligned} \tag{7.2}$$

(the use of square roots is purely symbolic here). Accordingly, we may expect the analogue of the exponential formula (6.2) to be the "sine formula"

$$S(\hat{t}) = \lim Y_1(\hat{t})A_n^{-1/2}\sin(tA_n^{1/2}) \tag{7.3}$$

with the corresponding "cosine formula" for the derivative $C = S'$,

$$C(\hat{t}) = \lim Y_1(t)\cos(tA_n^{1/2}). \tag{7.4}$$

Both can be established for arbitrary $\delta'' \otimes I - \delta \otimes A \in \mathfrak{D}_0'((E;\ D(A)))^{-1}$ and $A_\nu = \nu AR(\nu;\ A)$ by the methods in §6 (essentially the same method was used in **[8]**, where these formulas appear for the first time).

We check next the hypotheses necessary for the application of Theorem 4.7. Once again $D(p, \xi) = D(A^p)$ for all $\xi \geq 0$. Lemma 6.1 applies verbatim and Lemma 6.2 admits an obvious analogue. The case in §6 where A generates a strongly continuous semigroup also has a counterpart here. Assume $C = S'$ is strongly continuous in $t > 0$. Then we can show that C, extended evenly to all t ($C(t) = C(|t|)$) is a *cosine function* or *cosine operator function* in the sense of **[30]** (i.e. it satisfies the "cosine functional equations" $C(0) = I$, $C(s+t) + C(s-t) = 2C(s)C(t)$) and A is its infinitesimal generator, in the sense that $Au = C''(0)u$, the domain of A consisting of all u such that $u \to C(t)u$ is twice continuously differentiable in $(-\infty, \infty)$; conversely, if A generates a strongly continuous cosine function $C(\cdot)$ then $P = \delta'' \otimes I - \delta \otimes A \in \mathfrak{D}_0'(E;\ D(A)))^{-1}$ and $S = P^{*-1}$ is given by $S(t)u = \int_0^t C(s)u\,ds$. In the general case ($P = \delta'' \otimes I - \delta \otimes A \in \mathfrak{D}_0'((E\colon\ D(A)))^{-1}$) we may say that P "generates a distribution cosine function."

(I) *Convergence theorems for distribution cosine functions.* The following result is an obvious analogue of Theorem 6.3.

7.1 THEOREM. *Let* $P = \delta'' \otimes I - \delta \otimes A \in \mathfrak{D}_0'((E;\ X))^{-1}$, $P_n = \delta'' \otimes I_n - \delta \otimes A_n \in \mathfrak{D}_0'((E_n;\ X_n))^{-1}$. *Assume the* $\{S_n\}$ *are* η*-uniformly bounded in* $0 \leq t \leq T$ *with* $T > 0$ *and* $\eta \leq 1$ *and*

$$R(\lambda_0^2; A_n)\mathfrak{A}_n u \Rightarrow R(\lambda_0^2; A)u \qquad (u \in E) \tag{7.5}$$

for a fixed λ_0 *in the logarithmic region* Λ *where* $\{P_n, P\}$ *is equi-invertible or that there exists a subspace* $D \subseteq X$ *dense in* X *and such that for each* $u \in D$ *there is a sequence* $\{u_n\}$, $u_n \in X_n$ *such that*

$$u_n \Rightarrow u, \quad A_n u_n \Rightarrow Au. \tag{7.6}$$

Then $\{S_n\}$ *is* η*-strongly uniformly convergent to* S *in* $0 \leq t \leq T$, *that is, for each* $u \in E$

$$\left(Y_{-\eta} * S_n \mathfrak{A}_n u\right)(t) \Rightarrow \left(Y_{-\eta} * Su\right)(t) \tag{7.7}$$

uniformly in $0 \leq t \leq T$.

Theorem 7.1 considerably generalizes results of Konishi **[14]**, Goldstein **[10]** and Piskarev **[23]** where S is assumed $(1, \omega)$-uniformly bounded in $t \geq 0$ (that is, C is $(0, \omega)$-uniformly bounded in $t \geq 0$).

The case $P = \delta'' \otimes I - \delta' \otimes B - \delta \otimes A$ is substantially more complicated. Here A and B are assumed closed with $D(A) \cap D(B)$ dense in E

and we define $X = D(A) \cap D(B)$ endowed with the "joint graph norm" $\|u\|_X = \|u\| + \|Au\| + \|Bu\|$; A_n and B_n are subjected to the same requirements and S_n is correspondingly defined. We can write $P = Y_{-1} * (\delta' \otimes I - \delta \otimes B - Y_1 \otimes A)$ with a similar equality for each P_n so that $\Theta = \delta \otimes B + Y_1 \otimes A$, $\Theta_n = \delta \otimes B_n + Y_1 \otimes A_n$; again $\eta \leq 1$ is the allowable range. The equi-invertibility assumption translates into the requirement that $(\lambda^2 I_n - \lambda B_n - A_n)^{-1}$ should exist (as an operator in $(E_n; X_n)$) and

$$\left\|\left(\lambda^2 I_n - \lambda B_n - A_n\right)^{-1}\right\|_n \leq C(1 + |\lambda|)^m \qquad (\lambda \in \Delta) \tag{7.8}$$

with C, m independent of n. To discuss the applications of Theorem 4.7 we must identify the spaces $D(p, \xi)$. It is plain that if $\xi \geq 0$, $u \in D(1, \xi)$ if and only if $u \in D(A) \cap D(B)$, $u \in D(2, \xi)$ if and only if $u \in D(A) \cap D(B)$ and $Au, Bu \in D(A) \cap D(B)$, etc.: in general, $u \in D(p, \xi)$ if and only if $u \in D(L_1 L_2 \cdots L_p)$ where $L_j = A$ or $L_j = B$. Unlike in the cases hitherto considered, the subspaces $D(p, \xi)$ may reduce to $\{0\}$ if $p > 1$.

7.2 Example. Let B, say, be a selfadjoint operator in a Hilbert space H such that $B \leq 0$ (so that $\|R(\lambda; B)\| \leq (\cos\varphi)^{-1}|\lambda|^{-1}$ in any sector $\Sigma_+(\varphi + \pi/2)$) and let A be a bounded operator in E. Noting that $\lambda^2 I - \lambda B - A = (I - \lambda^{-1}AR(\lambda; B))\lambda(\lambda I - B)$ we see that $\mathfrak{R}(\lambda) = \lambda^{-1}R(\lambda; B)(I - \lambda^{-1}AR(\lambda; B))^{-1}$ exists as a bounded operator in $(E; X)$ in the intersection of the sector $\Sigma_+(\varphi + \pi/2)$ with $|\lambda| > 2\|A\|^{1/2}(\cos\varphi)^{-1/2}$ and satisfies $\|R(\lambda)\| \leq C|\lambda|^{-2}$ there, hence $P \in \mathfrak{D}_0'((E; X))^{-1}$. Consider now the following particular instance of this situation: $H = L^2(0, 1)$, $Bu = u''$ with domain $D(A)$ consisting of all differentiable u with u' absolutely continuous and $u'' \in H$ satisfying the boundary conditions $u(0) = u(1) = 0$, and A is the multiplication operator $(Au)(x) = \chi(x)u(x)$ with χ the characteristic function of the Cantor set in $0 \leq t \leq 1$. Then an element of the form Au can belong to the domain of B only if $u = 0$, i.e. $D(BA) = \{0\}$ showing in particular that $D(2, \xi) = D(A^2) \cap D(AB) \cap D(BA) \cap D(B^2) = \{0\}$.

Since the version of Theorem 4.7 applicable here parallels closely that for a different distribution in the next section, we omit it.

8. Application. $P = \delta'(\hat{t}) \otimes I - \delta(\hat{t}) \otimes A - \delta(\hat{t} - h) \otimes B$. This case has many features in common with the one considered in the previous section: in particular, the assumptions on A, B and the definition of X are the same. We have here $\kappa = 0$, $\Theta = \delta(\hat{t}) \otimes A + \delta(\hat{t} - h) \otimes B$. Finally, $\mathfrak{P}(\lambda) = \lambda I - A - e^{-\lambda h}B$ so that $\mathfrak{R}(\lambda) = (\lambda I - A - e^{-\lambda h})^{-1}$.

The same observations apply to X_n, Θ_n, $\mathfrak{P}_n$, $\mathfrak{R}_n$; equi-invertibility translates into existence of $(\lambda I_n - A_n - e^{-\lambda h}B_n)^{-1}$ (as an operator in $(E_n;\, X_n)$) and the estimate

$$\left\| (\lambda I_n - A_n - e^{-\lambda h}B_n)^{-1} \right\|_n \leq C(1 + |\lambda|)^m \qquad (\lambda \in \Delta) \tag{8.1}$$

with C, m independent of n. The identification of the subspaces $D(p, \xi)$ is the same as that for $P = \delta'' \otimes I - \delta' \otimes B - \delta \otimes A$ thus a slight modification of Example 7.2 produces a P with $D(p, \xi) = \{0\}$ for $p > 1$; the roles of A and B are now reversed and we write

$$\lambda I - A - e^{-\lambda h}B = \left(I - e^{-\lambda h}BR(\lambda;\, A)\right)(\lambda I - A).$$

Hence $\mathfrak{R}(\lambda) = R(\lambda;\, A)(I - e^{-\lambda h}BR(\lambda;\, A))^{-1}$ in some half plane $\operatorname{Re}\lambda \geq \omega$ and $\|R(\lambda)\| \leq C(\operatorname{Re}\lambda - \omega)^{-1}$ there, so that $P \in \mathfrak{D}_0'((E;\, X))^{-1}$.

We limit ourselves to a sample application of Theorem 4.7.

8.1 THEOREM. *Let $\{P_n, P\}$ be equi-invertible in a half plane Ξ with $m = 0$ in (8.1). Assume $\{S_n\}$ is η-uniformly bounded in $0 \leq t \leq T$ with $\eta < 0$ and that there exists a subspace $D \subseteq X$ dense in X such that for each $u \in D$ there is a sequence $\{u_n\}$, $u_n \in X_n$ with*

$$u_n \Rightarrow u, \quad A_n u_n \Rightarrow Au, \quad B_n u_n \Rightarrow Bu. \tag{8.2}$$

Then S_n is η-strongly uniformly convergent to S in $0 \leq t \leq T$.

Unfortunately, Theorem 8.1 leaves out the most interesting case, namely $\eta = 0$. To get at it one has to assume that $D \subseteq D(A^2) \cap D(AB) \cap D(BA) \cap D(B^2)$, that $u_n \in D(A_n^2) \cap D(A_n B_n) \cap D(B_n A_n) \cap D(B^2)$ and supplement 8.2 with the conditions $A_n^2 u_n \Rightarrow A^2 u$, $A_n B_n u_n \Rightarrow ABu$, $B_n A_n u_n \Rightarrow BAu$, $B_n^2 u_n \Rightarrow B^2 u$. We omit the details. The case of several time delays $(P = \delta'(\hat{t}) \otimes I - \delta(\hat{t}) \otimes A - \sum \delta(\hat{t} - h_k) \otimes B_k)$ is handled in the same way.

9. Finite difference approximations. We go one step further by approximating δ' itself by a finite difference expression. The results here are, not unexpectedly, adaptations of those in §4. To evade the question of finite difference approximation of derivatives of fractional order we shall limit ourselves to distributions of the form (4.5) with $\kappa = 0$ in this section and with $\kappa = 1$ in the next. We assume then that

$$P = \delta' \otimes I - \Theta \tag{9.1}$$

where $\Theta \in \mathcal{S}_0'((X;\, E))$. We shall only consider the usual difference approximation $D_n = \tau_n^{-1}(\delta(\hat{t}) - \delta(\hat{t} - \tau_n))$ for δ', where $\{\tau_n\}$ is a decreasing sequence of positive numbers tending to zero. The approximating distributions will be assumed of the form

$$\mathfrak{C}_n = D_n \otimes I_n - \Theta_n \qquad (n \geq 1) \tag{9.2}$$

where Θ, Θ_n will eventually be forced to satisfy the assumptions in Theorem 4.3. We write henceforth $\mathfrak{S}_n = \mathfrak{C}_n^{*-1}$ (when $\mathfrak{C}_n \in \mathfrak{D}_0'((E;X))^{-1}$).

We note in passing that Theorem 4.1 applies perfectly well to the present situation: if the sequence $\{\mathfrak{C}_n, P\}$ is equi-invertible in a logarithmic region Λ and if

$$\mathfrak{L}\mathfrak{C}_n(\lambda)^{-1} \Rightarrow P(\lambda)$$

for $\lambda \in \Lambda$ then, given $T > 0$ there exists $\eta = \eta(T)$ such that $\{\mathfrak{S}_n\}$ is η-strongly uniformly convergent to S in $0 \leq t \leq T$. So does Remark 4.2, but, for reasons made clear at the end of this section, the hypotheses turn out to be in general contradictory. On the other hand, direct application of the other results in §4 is not practicable: if the $\mathfrak{C}_n$ are written in the form (4.5), the Θ_n will fail to satisfy assumptions like those in Theorem 4.7, hence some modifications will be necessary. We begin by writing the analogue of (4.13) where $\kappa = 0$ and δ' is replaced by D_n. We proceed formally, starting from the obvious equality

$$\begin{aligned} Y_{-\eta} * \mathfrak{S}_n u_n = {} & D_n^{*-1} * Y_{-\eta} \otimes u_n \\ & + D_n^{*-1} * \left(Y_{-\eta} * \mathfrak{S}_n\right) * \Theta_n u_n \qquad (u_n \in X_n) \end{aligned}$$

and its extension

$$Y_{-\eta} * \mathfrak{S}_n * F_n = Y_{-\eta} * D_n^{*-1} * F_n + D_n^{*-1} * \left(Y_{-\eta} * \mathfrak{S}_n\right) * \Theta_n * F_n$$

for $F_n \in \mathfrak{D}_0'((X_n))$. Feeding the second into the first and iterating we obtain

(9.3)

$$\begin{aligned} \left(Y_{-\eta} * \mathfrak{S}_n\right)u_n &= \left(D_n^{*-1} * Y_{-\eta}\right) \otimes u_n \\ &\quad + \left(D_n^{*-2} * Y_{-\eta}\right) * \Theta u_n + D_n^{*-2} * \left(Y_{-\eta} * \mathfrak{S}_n\right) * \Theta_n^{*2} u_n \\ &= \cdots = \sum_{j=1}^{p} D_n^{*-j} * Y_{-\eta} * \Theta_n^{*(j-1)} u_n \\ &\quad + D_n^{*-p} * \left(Y_{-\eta} * \mathfrak{S}_n\right) * \Theta_n^{*p} u_n \end{aligned}$$

(we note in passing that

$$D_n^{*-1} = \tau_n \sum_{j=0}^{\infty} \delta\left(\hat{t} - j\tau_n\right)\big). \tag{9.4}$$

The last term in (9.3) cannot in general be expressed as an inverse Laplace transform as in (4.14) due to the fact that $\mathfrak{L}D_n(\lambda)^{-1} = \tau_n(1 - e^{-\lambda\tau_n})^{-1}$ does not die down as $|\lambda| \to \infty$, thus compromising convergence of the integral. To straighten out this problem we introduce the "averaging distribution"

$$(9.5)\quad \chi_n = D_n * \delta'^{-1} = D_n * Y_1 = \tau_n^{-1}\big(Y_1(\hat{t}) - Y_1(\hat{t} - \tau_n)\big) = \tau_n^{-1}\xi_n(t)$$

where ξ_n is the characteristic function of the interval $0 \leq t \leq \tau_n$. Convolution of both sides of (9.3) with χ_n^{*p} produces the formula

$$(9.6)\qquad \chi_n^{*p} * \big(Y_{-\eta} * \mathfrak{S}_n\big)u_n = \sum_{j=1}^{p} D_n^{*(p-j)} * Y_{p-\eta} * \Theta_n^{*(j-1)}u_n + Y_p * \big(Y_{-\eta} * \mathfrak{S}_n\big) * \Theta_n^{*p}u_n.$$

The main result for finite difference approximations follows:

9.1 THEOREM. *Let $P \in \mathfrak{S}_0'((X;\, E))$ be given by* (9.1) *and let $\mathfrak{C}_n \in \mathfrak{S}_0'((X_n;\, E_n))$ be given by* (9.2). *Let the sequence $(\mathfrak{C}_n, P)$ be equi-invertible in a logarithmic region Λ. Given $p > \eta + \alpha T + m + 1$ (α, m as in Theorem* 4.7) *assume that the sequence $\{\chi_n^{*p} * \mathfrak{S}_n\}$ is η-uniformly bounded in $0 \leq t \leq T < \infty$ with $\eta < 0$ and that the space $D(p, -\eta)$ possesses a subspace $\tilde{D}(p, -\eta)$ satisfying* (a) *and* (b) *in Theorem* 4.7. *Then the sequence $\chi_n^{*p} * \mathfrak{S}_n$ is η-strongly uniform convergent to S_n in $0 \leq t \leq T$.*

For the proof we only have to observe that (9.6) can be written

$$(9.7)\quad Y_{-\eta} * (\chi_n^{*p} * \mathfrak{S}_n u)(t) = \sum_{j=1}^{p} \chi_n^{*(p-j)} * \big(Y_{j-\eta} * \Theta^{*(j-1)}u_n\big)(t) + \frac{1}{2\pi i}\int_\Gamma \lambda^{\eta-p}e^{\lambda t}\mathfrak{L}\mathfrak{C}_n(\lambda)^{-1}\mathfrak{f}_{p,n}(\lambda)\,d\lambda$$

and the proof ends just like that of Theorem 4.3.

9.2. REMARK. Theorem 9.1 does not have much practical value unless $\mathfrak{S}_n = \mathfrak{C}_n^{*-1}$ is amenable to reasonably explicit computation; this can be achieved for instance by "discretizing" Θ as well. To simplify, assume that $E_n = X_n$ and that Θ_n is of the form

$$(9.8)\qquad \Theta_n = \sum_{j=0}^{r} \delta\big(\hat{t} - j\tau_n\big) \otimes A_{nj}$$

where the A_{nj} are bounded operators in E_n and r depends in general on n. The approximation is called *explicit* if $A_{n,0}=0$ for all n, *implicit* otherwise. If $\{\Theta_n\}$ is explicit then

$$\begin{aligned}\mathfrak{C}_n = D_n \otimes I_n - \Theta_n = \tau_n^{-1}\big(\delta(\hat{t}) \otimes I_n - \delta(\hat{t}-\tau_n) \otimes (I_n + \tau_n A_{n1}) \\ -\tau_n\delta(\hat{t}-2\tau_n) \otimes A_{n2} - \cdots - \tau_n\delta(\hat{t}-r\tau_n) \otimes A_{nr}\big) \\ = \tau_n^{-1}(\delta \otimes I - \nu_n),\end{aligned}$$

where ν_n is a measure with support in $\tau_n, 2\tau_n, \ldots, r\tau_n$; accordingly $\mathfrak{S}_n = \mathfrak{C}_n^{*-1} = \tau_n(\delta \otimes I_n + \nu_n + \nu_n * \nu_n + \cdots)$, a series which is always convergent since ν_n^{*k} has support in $t \geq k\tau_n$. Moreover, it is obvious that we can write

$$\mathfrak{S}_n = \mathfrak{C}_n^{*-1} = \tau_n \sum_{j=0}^{\infty} \delta(\hat{t}-j\tau_n) \otimes S_{nj} \tag{9.9}$$

where the S_{nj} are bounded operators in E_n.

In the implicit case we have

$$\begin{aligned}\mathfrak{C}_n = \tau_n^{-1}\big(\delta(\hat{t}) \otimes (I_n - \tau_n A_{n,0}) - \delta(\hat{t}-\tau_n) \otimes (I_n + \tau_n A_{n,1}) \\ -\tau_n\delta(\hat{t}-2\tau_n) \otimes A_{n,2} - \cdots - \tau_n\delta(\hat{t}-r\tau_n) \otimes A_{nr}\big)\end{aligned}$$

thus $\mathfrak{S}_n = \mathfrak{C}_n^{*-1}$ will exist if $\tau_n\|A_{n,0}\| < 1$. It is again obvious that $\mathfrak{S}_n$ will be given by an expression of the type of (9.9).

We examine briefly the conclusions of Theorem 9.1 in the case where $\mathfrak{S}_n$ admits a representation of the form (9.9). The approximation usually handled in practice is not $\mathfrak{S}_n$ itself but the (piecewise constant) first order average

$$(\chi_n * \mathfrak{S}_n)(t) = \sum_{j=0}^{\infty} \xi_n(\hat{t}-j\tau_n) \otimes S_{nj}. \tag{9.10}$$

9.3 Remark. It is of obvious interest to inquire whether η-uniform convergence of $\chi_n * \mathfrak{S}_n$ (rather than of $\chi_n^{*p} * \mathfrak{S}_n$) can be squeezed out of Theorem 9.1. The answer is in doubt, but a particular case is easy to consider and will be of use later. In fact, assume $\mathfrak{S}_n$ is of the form (9.9) and that $\{\chi_2^{*2} * \mathfrak{S}_n\}$ is strongly uniformly convergent in $0 \leq t \leq T$. Then $\{\chi_n * \mathfrak{S}_n\}$ is as well strongly uniformly convergent in $0 \leq t \leq T$; it suffices to observe that

$$\big(\chi_n^{*2} * \mathfrak{S}_n\big)\big((j+1/2)\tau_n\big) = (\chi_n * \mathfrak{S}_n)\big((j+1/2)\tau_n\big), \qquad j = 0, 1, \ldots.$$

The result does not extend to $p > 2$ as the example $S_{nj} = (-1)^j S$ shows.

9.4 REMARK. Results of the type of those in Remark 4.8 cannot be expected here, since $\mathfrak{S}_n$ cannot belong to $\mathcal{C}_+^\infty$, much less to any class $\mathcal{Q}(\varphi)$.

10. Finite difference approximations (continuation). We examine here briefly "second order equations"

$$P = \delta' * (\delta' \otimes I - \Theta). \tag{10.1}$$

The sequence of approximating distributions is now

$$\mathfrak{C}_n = D_n * (D_n \otimes I_n - \Theta_n). \tag{10.2}$$

As in §9 we write $\mathfrak{S}_n = \mathfrak{C}_n^{*-1}$ when the inverse exists. The basic equality (9.3) suffers no change save convolution of both sides by D_n^{*-1}:

$$\begin{aligned}(Y_{-\eta} * \mathfrak{S}_n)u_n = {} & \sum_{j=1}^{p} D_n^{*-(j+1)} * Y_{-\eta} * \Theta_n^{*(j-1)}u_n \\ & + D_n^{*-(p+1)} * (Y_{-\eta} * \mathfrak{S}_n) * \Theta_n^{*p}u_n\end{aligned}$$

thus averaging of order $p + 1$ (that is, convolution by $\chi_n^{*(p+1)}$) is necessary to remove the last traces of D_n from the last term. After this is done, we obtain the formula

$$\begin{aligned}Y_{-\eta} * (\chi_n^{*(p+1)} * \mathfrak{S}_n u)(t) = {} & \sum_{j=1}^{p} \chi_n^{*(p-j)} * (Y_{j+1-\eta} * \Theta_n^{*(j-1)}u_n)(t) \\ & + \frac{1}{2\pi i}\int_\Gamma \lambda^{\eta-p-1}e^{\lambda t}\mathfrak{L}\mathfrak{C}(\lambda)^{-1}\mathfrak{f}_{p,n}(\lambda)\,d\lambda.\end{aligned}$$

An obvious analog of Theorem 9.1 results. To avoid tiresome repetition we only point out the necessary modifications. This time we only need $p > \eta + \alpha T + m$ and the allowable range of η is $\eta \leq 1$. The conclusion is η-strong uniform convergence of the $(p + 1)$th order average $\chi_n^{*(p+1)} * \mathfrak{S}_n$.

11. Applications. $P = \delta' \otimes I - \delta \otimes A$. The assumptions on A are those in §6 as in the definition of X. The operator A_n is densely defined and closed and $X_n = D(A_n)$ endowed with the graph norm. The sequence of approximating distributions is (9.2) and we only consider the cases

$$\Theta_n = \delta(\hat{t} - \tau_n) \otimes A_n, \qquad \Theta_n = \delta(\hat{t}) \otimes A_n, \tag{11.1}$$

the first implicit, the second explicit (to lighten the notation we have eliminated the double indices in (9.8)). Assuming (as is often the case in practice) that each A_n is a bounded operator we obtain the formula

$$\mathfrak{S}_n = \tau_n \sum_{j=0}^{\infty} \delta(\hat{t} - j\tau_n) \otimes (I_n + \tau_n A_n)^j \tag{11.2}$$

for $\mathfrak{S}_n = \mathfrak{C}_n^{*-1}$ with $\mathfrak{C}_n = D_n \otimes I_n - \delta(\hat{t} - \tau_n) \otimes A_n$. Since $\mathfrak{L}\mathfrak{C}(\lambda) = \tau_n^{-1}(1 - e^{-\lambda\tau_n})I_n - e^{-\lambda\tau_n}A_n$ we have

$$\mathfrak{L}\mathfrak{C}_n(\lambda)^{-1} = e^{\lambda\tau_n}R\big(\tau_n^{-1}(e^{\lambda\tau_n} - 1); A_n\big).$$

When $\mathfrak{C}_n = D_n \otimes I_n - \delta(\hat{t}) \otimes A_n$ we do not require that A_n be bounded but merely that $I_n - \tau_n A_n$ be invertible for τ_n sufficiently small. We have

$$\mathfrak{S}_n = \tau_n \sum_{j=0}^{\infty} \delta(\hat{t} - j\tau_n) \otimes (I_n - \tau_n A_n)^{-(j+1)} \tag{11.3}$$

and, since $\mathfrak{L}\mathfrak{C}_n(\lambda) = \tau_n^{-1}(1 - e^{-\lambda\tau_n})I_n - A_n$ we have $\mathfrak{L}\mathfrak{C}_n(\lambda)^{-1} = R(\tau_n^{-1}(1 - e^{-\lambda\tau_n}); A_n)$. As in previous sections, we write $P_n = \delta' \otimes I_n - \delta \otimes A_n$. The next result shows that equi-invertibility of the $\mathfrak{C}_n$ can be deduced from the corresponding property for the P_n, at least when Λ is a logarithmic region or a half plane and Θ_n is given by the second equality (11.1).

11.1 Theorem. *Let the sequence* $\{P_n\}$ *be equi-invertible in a logarithmic region* (resp. *a half plane*). *Then the sequence* $\{\mathfrak{C}_n\}$, *where*

$$\mathfrak{C}_n = D_n \otimes I_n - \delta \otimes A_n \tag{11.4}$$

is a well equi-invertible in a logarithmic region Λ (resp. *a half plane* Ξ). *Moreover, if*

$$R(\lambda; A_n)\mathfrak{A}_n u \Rightarrow R(\lambda; A)u \qquad (\lambda \in \Lambda) \tag{11.5}$$

for $u \in E$ [*uniformly in* $\|u\| \leq 1$] *then*

$$\mathfrak{L}\mathfrak{C}_n(\lambda)^{-1}\mathfrak{A}_n u \Rightarrow R(\lambda; A)u \qquad (\lambda \in \Lambda) \tag{11.6}$$

for $u \in E$ [*uniformly in* $\|u\| \leq 1$] *with a corresponding statement for* Ξ.

Since $\mathfrak{L}\mathfrak{C}_n(\lambda) = R(\tau_n^{-1}(1 - e^{-\lambda\tau_n}); A)$ it is clear that the proof of Theorem 11.1 will follow from mapping properties of the functions

$$\zeta_n(\lambda) = \frac{1}{\tau_n}(1 - e^{-\lambda\tau_n}) = \int_0^\lambda e^{-z\tau_n}\,dz \tag{11.7}$$

made explicit in the following result. We use the notation Λ_σ for the translate $\{\lambda; \lambda - \sigma \in \Lambda\}$ (see §6, (I)).

11.2 Lemma. (a) *Let* Λ *be a logarithmic region,* $\sigma > 0$. *Then there exists* $m = m(\sigma)$ *such that* ζ_n, $n \geq m$ *maps* Λ_σ *into* Λ. *The same result holds for half planes* $\operatorname{Re}\lambda \geq \omega > 0$. (b) *We have*

$$|\zeta_n(\lambda)| \leq |\lambda|, \quad \zeta_n(\lambda) \to \lambda \quad \textit{as } n \to \infty \tag{11.8}$$

for all λ, *the second relation uniform on compacts of the complex plane.*

Proof. Both relations (11.8) are obvious, the first following from the second representation (11.7) for ζ_n. The particular case of (a) corresponding to half planes is immediate since ζ_n maps $\operatorname{Re}\lambda \geq \omega$ into the circle $|\zeta - \tau_n^{-1}| \leq \tau_n^{-1}e^{-\omega\tau_n}$. To complete the proof we use an idea in [**3**, p. 541]. Note that since $\omega > 0$ the function $\zeta_n(\lambda)$ does not vanish in Λ and $\zeta_n(\lambda)^{-1} \to 1/\lambda$ uniformly in Λ as $n \to \infty$. Call Λ^{-1} the set $\{\lambda;\ 1/\lambda \in \Lambda\}$ and define $(\Lambda_\sigma)^{-1}$ similarly. Since Λ^{-1} is a compact neighborhood of $(\Lambda_\sigma)^{-1}$ there exists an integer m such that the function $\varphi_n(\lambda) = \zeta_n(\lambda^{-1})^{-1}$ maps $(\Lambda_\sigma)^{-1}$ into Λ^{-1} for $n \geq m$. This relation is equivalent to $\zeta_n\Lambda_\sigma \subseteq \Lambda$ for $n \geq m$, which ends the proof of Lemma 10.2.

11.3 Remark. Equi-invertibility of the $\mathfrak{C}_n$ even in a half-plane does not guarantee equi-invertibility of the P_n in any reasonable region. An obvious example is that where $E = \mathbf{C}$, A_n the operator of multiplication by a_n with $a_n > \tau_n^{-1}(1 + e^{-\omega\tau_n})$. The mapping properties of ζ_n in half planes pointed out in the proof of Lemma 11.2 imply that $\|\mathfrak{L}\mathfrak{C}_n(\lambda)^{-1}\| \leq (a_n - \tau_n^{-1}(1 + e^{-\omega\tau_n}))^{-1}$ if $\operatorname{Re}\lambda \geq \omega$ (which can be kept bounded) but the P_n are not equi-invertible in any logarithmic region.

(I) *The Hille approximation for distribution semigroups.* Here we take $E_n = E$, $X_n = X = D(A)$ $\mathfrak{C}_n = D_n \otimes I - \delta \otimes A$ (so that $\mathfrak{S}_n$ is given by (11.3)) with $A_n = A$; since $R(\lambda; A)$ exists for λ real and sufficiently large $I - \tau_n A$ will be invertible for n large enough. As observed above, $\mathfrak{L}\mathfrak{C}_n(\lambda)^{-1} = R(\tau_n^{-1}(1 - e^{-\lambda\tau_n}); A)$; hence by Theorem 11.1 the sequence $\{\mathfrak{C}_n\}$ is equi-invertible and (11.6) holds uniformly for $\|u\| \leq 1$. Thus, by Theorem 4.1 $\mathfrak{S}_n$ is $\eta(T)$-uniformly convergent to $S = P^{*-1}$ in $(E;\ X)$ in $0 \leq t \leq T$ for all $T > 0$ or, equivalently, in $\mathfrak{D}'((E;\ X))$. We have then proved the following result.

11.4 Theorem. *Let A be the infinitesimal generator of a regular distribution semigroup S in E, $\{\tau_n\}$ a decreasing sequence of positive numbers with $\tau_n \to 0$. Then*

$$(11.9)\quad \chi_n * \mathfrak{S}_n = \sum_{j=0}^{\infty} \xi_n(t - j\tau_n) \otimes (I - \tau_n A)^{-(j+1)} \to S \quad \text{as } n \to \infty$$

in $\mathfrak{D}'((E;\ D(A))$.

We have used here the fact that $\chi_n * \mathfrak{S}_n$ is convergent in $(E;\ X)$ if $\mathfrak{S}_n$ is (immediate verification). This theorem is a natural generalization of a result of Hille (see [**11**, p. 352]) where S is a strongly continuous semigroup and $\chi_n * \mathfrak{S}_n$ converges strongly to S uniformly on compacts of

$t \geq 0$; this is easily seen to be equivalent to the more familiar formula

$$S(t)u = \lim_{n\to\infty} \left(I + \frac{t}{n}A\right)^{-(n+1)} u, \tag{11.10}$$

convergence as described above. More on this will be found below.

We return to the general set-up. A first tool in the applications of Theorem 9.1 is a rather obvious analogue of Lemma 6.1.

11.5 LEMMA. *Let* $\{\mathfrak{C}_n, P\}$ *be equi-invertible in a region* Δ. *Assume that*

$$\mathfrak{L}\mathfrak{C}_n(\lambda)^{-1}\mathfrak{A}_n u \Rightarrow R(\lambda; A)u \quad \text{as } n \to \infty. \tag{11.11}$$

for a single $\lambda = \lambda_0 \in \Delta$ *and all* $u \in E$. *Then* (a) (11.11) *holds for every* $\lambda \in \Delta$. (b) *Given an arbitrary integer* p *and a* $u \in D(A^p)$ *there exists a sequence* $\{u_n\}$, $u_n \in D(A_n^p)$ *such that*

$$u_n \Rightarrow u, \quad A_n^k u_n \Rightarrow A^k u \tag{11.12}$$

for $1 \leq k \leq p$. *Conversely,* (11.12) *for* $k = 1$ *in a subspace* $D \subseteq X$ *dense in* X *implies* (11.11) *for* $\lambda \in \Delta$, $u \in E$.

The proof follows closely that of Lemma 6.1. We consider first the case $\mathfrak{C}_n = D_n \otimes I_n - \delta \otimes A_n$ so that $\mathfrak{L}\mathfrak{C}_n(\lambda)^{-1} = R(\zeta_n(\lambda); A_n)$ with $\zeta_n(\lambda) = \tau_n^{-1}(1 - e^{-\lambda\tau_n})$. We begin by extending the convergence relation (11.11) to powers ($R(\zeta_n(\lambda_0); A_n)^k\mathfrak{A}_n u \Rightarrow R(\lambda_0; A)^k u$ as $n \to \infty$) and use then the power series of the resolvent for extension of (11.11) to arbitrary $\lambda \in \Delta$; the sequence $\{u_n\}$ in (11.12) is obviously $u_n = R(\zeta_n(\lambda); A_n)\mathfrak{A}_n u$ for any $\lambda \in \Delta$. The converse follows from Lemma 3.3 with $\mathfrak{K}_n = \zeta_n(\lambda)I - A_n$. In the explicit case $\Theta_n = \delta(t - \tau_n) \otimes A_n$ the proof is essentially similar.

There is of course an analogue of Lemma 6.2:

11.6 LEMMA. *Assume each* $\mathfrak{C}_n = D_n \otimes I_n - \delta \otimes A_n$ *belongs to* $\mathfrak{D}_0'((E_n;\ X_n))^{-1}$ *and that the sequence* $\{\mathfrak{S}_n\} = \{\mathfrak{C}_n^{*-1}\}$ *is* η*-uniformly bounded in* $0 \leq t \leq T$ *for some* η, T ($T > 0$). *Then* $\{\mathfrak{C}_n\}$ *is equi-invertible in a logarithmic region* Λ. *The result is as well true for* $\mathfrak{C}_n = D_n \otimes I_n - \delta(t - \tau_n) \otimes A_n$.

Proof. We only have to show that the hypotheses imply η'-uniform boundedness of $\{\mathfrak{S}_n\}$ in $0 \leq t \leq T$ in the spaces $(E_n;\ X_n)$ (in view of Lemma 3.2). To do this, observe that $A_n\mathfrak{S}_n = D_n * \mathfrak{S}_n - \delta \otimes I_n$ so that $A_n(Y_{-\eta+1} * \mathfrak{S}_n) = \chi_n * Y_{-\eta} * \mathfrak{S}_n - Y_{-\eta+1} \otimes I_n$. In case $\mathfrak{C}_n = D_n \otimes I_n - \delta(t - \tau_n) \otimes A_n$ the pertinent equality is $A_n(Y_{-\eta+1} * \mathfrak{S}_n) = \delta(\hat{t} + \tau_n) * \chi_n * Y_{-\eta} * \mathfrak{S}_n - \delta(\hat{t} + \tau_n) * Y_{-\eta+1} \otimes I_n$.

(II) *The Lax equivalence theorem for distribution semigroups.* We consider approximations of the form

$$\mathfrak{F}_n = \sum_{j=0}^{\infty} \xi_n(\hat{t} - j\tau_n) \otimes S_n^j \tag{11.13}$$

to $S = (\delta' \otimes I - \delta \otimes A)^{*-1}$, where the S_n are bounded operators in E_n. Obviously $\mathfrak{F}_n = \chi_n * \mathfrak{S}_n$ with

$$\mathfrak{S}_n = \mathfrak{C}_n^{*-1}, \quad \mathfrak{C}_n = D_n \otimes I_n - \delta(\hat{t} - \tau_n) \otimes A_n, \quad A_n = \tau_n^{-1}(S_n - I).$$

As seen above, $\mathfrak{L}\mathfrak{C}_n(\lambda)^{-1} = e^{\lambda\tau_n} R(\tau_n^{-1}(e^{\lambda\tau_n} - 1);\ A_n)$. In case the approximation is

$$\mathfrak{G}_n = \sum_{j=0}^{\infty} \xi_n(\hat{t} - j\tau_n) \otimes S_n^{j+1} \tag{11.14}$$

we have $\mathfrak{G}_n = \chi_n * \mathfrak{S}_n = \chi_n * \mathfrak{C}_n^{*-1}$ with $\mathfrak{C}_n = D_n \otimes I_n - \delta \otimes A_n$, $A_n = \tau^{-1}(I - S_n^{-1})$ (note that A_n may well be unbounded here).

11.7 THEOREM. *Let $P \in \delta' \otimes I - \delta \otimes A \in \mathfrak{D}_0'((E;\ X))^{-1}$. Let $\eta \geq 0$ and assume that for some integer $r \geq 0$ the sequence $\{\chi_n^{*r} * \mathfrak{F}_n\}$ is η-uniformly bounded in $0 < t < T$ and there exists a subspace $D \subseteq X$ dense in X such that, for every $u \in D$ there is a sequence $\{u_n\}$, $u_n \in E_n$ such that*

$$u_n \Rightarrow u, \qquad \tau_n^{-1}(S_n - I)u_n \Rightarrow u. \tag{11.15}$$

Then, if $p > \max(r - 1, \eta + \alpha T + m)$, $\chi_n^{*p} * \mathfrak{F}_n$ is η-strongly uniformly convergent to $S = (\delta' \otimes I - \delta \otimes A)^{*-1}$ in $0 \leq t \leq T$. The result is also valid for approximations of the form (11.14), *the second assumption* (11.15) *modified to $\tau_n^{-1}(I - S_n^{-1})u_n \Rightarrow Au$, $u_n \in D(S_n^{-1})$; here we assume that each S_n is one-to-one and that S_nE_n is dense in E_n.*

The proof is a direct consequence of Theorem 9.5; the rest of the assumptions are those in Theorem 4.7 and have already been checked in §6. The range of p $(p > \eta + \alpha T + m)$ results from the fact that $\mathfrak{F}_n = \chi_n * \mathfrak{C}_n^{*-1}$ with a similar equality for $\mathfrak{G}_n$.

Theorem 11.7 is a generalization of Theorem 3.3 in **[18]** (see also **[26]**); approximations are only considered there in the space E (i.e. $E_n = E$, $X_n = X$) and $\eta = 0$, that is boundedness is uniform and convergence is strong. To be sure, Theorem 11.7 only guarantees in this case that $\chi_n^{*p} * \mathfrak{F}_n$ will be strongly convergent for some p; but when $E = E_n$, uniform boundedness of $\mathfrak{F}_n$ or $\mathfrak{G}_n$ in $0 \leq t \leq T$ is easily seen to imply that $\|\mathfrak{F}_n(t)\|$, $\|\mathfrak{G}_n(t)\| \leq Ce^{\omega t}$ $(t \geq 0)$ which in turn implies equi-invertibility in a half plane $(\alpha = 0)$ with $m = 0$. We can take then $p = 1$ and reduce it to $p = 0$ via Remark 9.2. The same results hold in the general case when equi-invertibility in a half-plane Ξ with $m = 0$ is assumed.

As an application of Theorem 11.7 and the previous remarks we can obtain easily Hille's exponential formula (11.10) for a strongly continuous semigroup $S(t)$. Let A be its infinitesimal generator. Take $E_n = E$, $A_n = A$ and consider $\mathfrak{G}_n$ with $S_n = (I - n^{-1}A)^{-1}$. It follows easily from the inequalities in the Hille-Yosida theorem that $\|\mathfrak{G}_n(t)\| \leq Ce^{\omega t}$ in $t \geq 0$ while $n(I - S_n^{-1}) \to A$ thus we obtain strong convergence uniformly in every interval $0 \leq t \leq T$.

12. Application. $P = \delta'' \otimes I - \delta \otimes A$. Three equally natural sequences of approximating distributions suggest themselves, namely

$$\begin{aligned} \mathfrak{C}_n &= D_n^{*2} \otimes I_n - \delta(\hat{t} - 2\tau_n) \otimes A_n, \\ \mathfrak{C}_n &= D_n^{*2} \otimes I_n - \delta(\hat{t} - \tau_n) \otimes A_n, \\ \mathfrak{C}_n &= D_n^{*2} \otimes I_n - \delta \otimes A_n. \end{aligned} \tag{12.1}$$

The first two are explicit in the sense that computation of operator inverses is not necessary to calculate $\mathfrak{S}_n = \mathfrak{C}_n^{*-1}$: in both cases we assume A_n bounded and we have

$$\mathfrak{S}_n = \tau_n^2 \sum_{j=0}^{\infty} \delta(t - j\tau_n) \otimes S_{nj} \tag{12.2}$$

with $S_{nj} \in (E_n)$ and $S_{n,0} = I$. In the third case we do not assume A_n bounded but merely that $(I - \tau_n^2 A_n)^{-1}$ exists for τ_n sufficiently small; $\mathfrak{S}_n$ is again given by an expression of the form (12.2) but now $(S_{n,0}) = (I - \tau_n^2 A_n)^{-1}$. The distributions Θ_n corresponding to the three choices of $\mathfrak{C}_n$ are

$$\begin{aligned} \Theta_n &= D_n^{*-1}(\hat{t} - 2\tau_n) \otimes A_n, \\ \Theta_n &= D_n^{*-1}(\hat{t} - \tau_n) \otimes A_n, \\ \Theta_n &= D_n^{*-1} \otimes A_n \end{aligned} \tag{12.3}$$

respectively (see (9.4) for D_n^{*-1}). All the results in the previous section have obvious counterparts in the present situation. We limit ourselves to pointing out the necessary changes.

THEOREM 11.1. *The distribution* $\mathfrak{C}_n$ *is now* $D_n^{*2} \otimes I_n - \delta \otimes A_n$ *so that* $\mathfrak{L}\mathfrak{C}_n(\lambda) = R(\zeta_n(\lambda)^2; A)$. *Lemma* 11.1 *holds for the functions* ζ_n^2 (*the proof is essentially the same*).

THEOREM 11.4. *Holds without changes for* $S = (\delta'' \otimes I - \delta \otimes A)^{*-1}$ *and* $\mathfrak{S}_n = (D_n^{*2} \otimes I_n - \delta \otimes A)^{*-1}$.

LEMMA 11.5. *Holds without changes: note that, corresponding to the three choices* (12.1) *for* Θ_n *we have* $\mathfrak{L}\mathfrak{C}_n(\lambda)^{-1} = e^{2\lambda\tau_n}R(\tau_n^{-2}(e^{\lambda\tau_n} - 1)^2; A)$, $e^{\lambda\tau_n}R(\tau_n^{-2}(e^{\lambda\tau_n/2} - e^{-\lambda\tau_n/2})^2; A)$, $R(\tau_n^{-2}(1 - e^{-\lambda\tau_n})^2; A)$.

LEMMA 11.6. *Holds without changes.*

Results of the type of Theorem 11.7 can be formulated in terms of approximations of the form

$$\mathfrak{F}_n = \sum_{j=0}^{\infty} \xi_n(\hat{t} - j\tau_n) \otimes C_{nj} \tag{12.4}$$

where the C_{nj} satisfy functional equations analogous to the equation $S_{n,j+k} = S_{nj}S_{nk}$ encountered in the case $P = \delta' \otimes I - \delta \otimes A$. To avoid complicated algebraic computations we state the following result only for $\mathfrak{S}_n = \mathfrak{C}_n^{*-1}$, where $\mathfrak{S}_n$ is any of the three distributions in (12.1).

12.1 THEOREM. *Let* $P \in \delta' \otimes I - \delta \otimes A \in \mathfrak{D}_0'((E; X))^{-1}$. *Let* $\eta \leq 1$ *and assume that for some integer* $r \geq 0$ *the sequence* $\{X_n^{*r} * \mathfrak{S}_n\}$ *is* η*-uniformly bounded in* $0 \leq t \leq T$ *and there exists a subspace* $D \subseteq X$ *such that, for every* $u \in D$ *there is a sequence* $\{u_n\}$, $u_n \in D(A_n)$ *with*

$$u_n \Rightarrow u, \qquad A_n u_n \Rightarrow Au. \tag{12.5}$$

Then, if $p > \max(r - 1, \eta + \alpha T + m + 1)$, $\{\chi^p * \mathfrak{S}_n\}$ *is* η*-strongly uniformly convergent to* $S = (\delta'' \otimes I - \delta \otimes A)^{*-1}$ *in* $0 \leq t \leq T$.

Once again, substantial simplifications occur when $\{\mathfrak{C}_n; p\}$ is assumed to be equi-invertible in a half plane Ξ with $m = 0$, especially if $\eta = 1$ (which corresponds to approximation of the distribution cosine function $C = S'$. Observe first that $\mathfrak{F}_n = \chi_n^{*2} * (Y_{-1} * \mathfrak{S}_n) = \chi_n^{*2} * \mathfrak{S}_n' = \chi_n^{*2} * \Sigma\, \delta'(t - j\tau_n) \otimes S_{nj}$ is a function of the form (12.4). If $\mathfrak{F}_n$ is uniformly bounded in $0 \leq t \leq T$ we may apply Theorem 12.1 with $p = 3$, thus the conclusion is strong uniform convergence of $\chi_n^{*3} * \mathfrak{S}_n' = \chi_n * \mathfrak{F}_n$. We conclude from Remark 9.4 that $\mathfrak{F}_n$ itself is strongly uniformly convergent to C in $0 \leq t \leq T$ (note that if $E_n = E$, $A_n = I$, uniform boundedness of $\{\mathfrak{S}_n\}$ in $0 \leq t \leq T$ implies exponential growth at infinity, hence the equi-boundedness assumption above is automatic). The preceding particular case of Theorem 12.1 includes results of Piskarev **[24]** on approximation of strongly continuous cosine functions by functions of the form (12.4).

13. Application. $P = \delta'(\hat{t}) \otimes I - \delta(\hat{t}) \otimes A - \delta(\hat{t} - h) \otimes B$. We can use here the explicit approximation corresponding to $\Theta_n = \delta(\hat{t} - \tau_n) \otimes A_n - \delta(\hat{t} - h) \otimes B_n$ or the implicit one where $\Theta_n = \delta(\hat{t}) \otimes A_n - \delta(\hat{t} - h) \otimes B_n$; if one takes, say, $\tau_n = h/n$ and A_n, B_n bounded the inverse

$\mathfrak{S}_n = \mathfrak{C}_n^{*-1}$ is given by an expression of the form (9.9) with $S_{0,j} = I$; in the implicit case, $I_n - \tau_n A_n$ is assumed invertible and $(I_n - \tau_n A_n)^{-1} B_n$ bounded; again $\mathfrak{S}_n$ is given by (9.9) this time with $S_{n,0} = (I_n - \tau_n A_n)^{-1}$. Theorem 8.1 and the remarks following it have an obvious analogue here. We limit ourselves to the case $\eta < 0$, with $\mathfrak{C}_n$ being any of the approximating distributions above.

13.1 THEOREM. *Let $(\mathfrak{C}_n, P)$ be equi-invertible in a half-plane Ξ with $m = 0$ in* (3.10). *Assume $\chi_n * \mathfrak{S}_n$ is η-uniformly bounded in $0 \le t \le T$ with $\eta < 0$ and that there exists a subspace $D \subseteq X$ dense in X such that for each $u \in D$ there is a sequence $\{u_n\}$, $u_n \in X_n$ with*

$$u_n \Rightarrow u, \quad A_n u_n \Rightarrow Au, \quad B_n u_n \Rightarrow Bu. \tag{13.1}$$

Then $\chi_n * \mathfrak{S}_n$ is η-strongly uniformly convergent to

$$S = \left(\delta'(\hat{t}) \otimes I - \delta(\hat{t}) \otimes A - \delta(\hat{t} - h) \otimes B\right)^{*-1} \quad \textit{in } 0 \le t \le T.$$

The more interesting case $\eta = 0$ is handled in the same way as in the comments following Theorem 8.1; direct application of Theorem 9.2 produces convergence of $\chi_n^{*2} * \mathfrak{S}_n$, and convergence of $\chi_n * \mathfrak{S}_n$ follows from Remark 9.4.

14. Extensions. The methods and results in the previous sections can be made to work (with obvious modifications and omissions) in more general situations, for instance when E is a locally convex space. We mention, however, that of the diverse characterizations of invertible distributions in §2 the only one that survives is the particular case of Theorem 2.1 where $S = P^{*-1}$ grows exponentially at infinity.

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