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A DUAL GEOMETRIC CHARACTERIZATION OF BANACH SPACES NOT CONTAINING l_1

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It is shown that a Banach space E does not contain a copy of l_1 if and only if every bounded subset of E^* is w^*-dentable in $(E^*, \sigma(E^*, E^{**}))$. The notion of w^*-scalarly dentable sets in dual Banach space is introduced and it is proved that a Banach space E does not contain a copy of l_1 if and only if every bounded set in E^* is w^*-scalarly dentable. Finally, a point of continuity criterion that characterizes Asplund operators and those operators that factor through Banach spaces not containing copies of l_1, is given.

Introduction. In **[11]**, **[13]** Rosenthal and Odell showed that a separable Banach space E does not contain an isomorphic copy of l_1 if and only if every element $x^{**} \in E^{**}$ is Baire-1 when restricted to $(B_{E^*}, \sigma(E^*, E))$. Haydon **[6]** showed that a Banach space E (separable or not) does not contain a copy of l_1 if and only if every element $x^{**} \in E^{**}$ is universally measurable when restricted to $(B_{E^*}, \sigma(E^*, E))$.

In this paper we first show that if E is any Banach space such that for every $x^{**} \in E^{**}$ and for every w^*-compact subset M of B_{E^*}, the restriction of x^{**} to $(M, \sigma(E^*, E))$ has a point of continuity then E contains no copy of l_1.

In **[10]** Namioka and Phelps showed that a dual Banach space E^* has the Radon-Nikodym property if and only if every bounded subset of E^* is w^*-dentable in $(E^*, \|\ \|)$. Here we shall show that a Banach space E does not contain a copy of l_1 or equivalently E^* has the weak Radon-Nikodym property if and only if every bounded subset of E^* is w^*-dentable in $(E^*, \sigma(E^*, E^{**}))$.

To do this we show that a Banach space E does not contain a copy of l_1 if and only if for every $x^{**} \in E^{**}$ and for every w^*-compact convex subset C of E^*, the set of points of continuity of x^{**} restricted to $(C, \sigma(E^*, E))$, that are extreme points of C is a G_δ dense subset of $(\mathrm{Ext}(C), \sigma(E^*, E))$, where $\mathrm{Ext}(C)$ denotes the set of extreme points of C. On the way of proving that we show that a Banach space does not contain a copy of l_1 if and only if every bounded set in E^* is w^*-scalarly dentable.

Finally, we give a point of continuity criterion that characterizes Asplund operators and those operators that factor through a Banach space not containing copies of l_1.

Preliminaries. Let X be a topological Hausdorff space and f be a real valued function on X. If $A \subset X$, the oscillation of f on A is defined by $O(f, A) = \sup\{|f(y) - f(x)|, x \in A, y \in A\}$ and the oscillation of f at a point x is given by $O(f, x) = \inf\{O(f, U), U \text{ open}, x \in U\}$. It is clear that f is continuous at x if and only if $O(f, x)$ is equal to zero. The function is said to be Baire-1 if f is the pointwise limit of a sequence of continuous functions on X. A Banach space E is said to contain a copy of a Banach space F if F is isomorphic to a subspace of E, we also say that F embeds into E. The closed unit ball of a Banach space E is denoted by B_E. If A is a subset of the dual E^*, we denote by $w^* - \bar{A}$ the weak* closure of A and by norm $-\bar{A}$ its norm closure, the convex hull of A will be denoted by conv(A). The symbol (A, τ) will mean A endowed with the topology τ. If C is a convex set in a Banach space, the set of its extreme points will be denoted by Ext(C). If B is a bounded subset of a Banach space E, an open slice of B is a set of the form

$$S(A, F, \alpha) = \left\{x \in A; f(x) > \sup_A f - \alpha\right\},$$

for some $f \neq 0, f \in E^*$ and some $\alpha > 0$. If $E = F^*$ is a dual Banach space and $f \in F$, the slice is called a w^*-open slice. A closed convex bounded subset C of a Banach space E is said to have the Radon-Nikodym property "RNP" (resp., the weak Radon-Nikodym property "WRNP") if for any bounded linear operator $T\colon L^1[0, 1] \to E$ such that $T(1_A/\lambda(A)) \in C$ for any Lebesgue measurable set A whose Lebesgue measure $\lambda(A) \neq 0$, the operator T is represented by a Bochner kernel (resp., by a Pettis-kernel) f taking its values in C. We also call such a set C an RNP set (resp., a WRNP set). If the unit ball of E has the RNP (resp., the WRNP) we say that E has the RNP (resp., the WRNP). For more about RNP and WRNP we refer the reader to **[4]**, **[8]**, **[5]** and **[12]**.

The Banach spaces l_1, c_0, l_∞ will have their usual meaning. A sequence $(x_n)_{n \geq 1}$ in a Banach space is said to be equivalent to the usual l_1-basis if there is a $\delta > 0$, such that

$$\left\| \sum_{n=1}^{k} a_n x_n \right\| \geq \sum_{n=1}^{k} |a_n|,$$

for any $k \geq 1$ and any scalars $a_1, a_2, \ldots, a_k$. All Banach spaces considered are over the real field.

If T: $E \to F$ is a bounded linear operator, T^*: $F^* \to E^*$ will always denote the adjoint of T.

Let E be a Banach space, let E^* be its dual and E^{**} its bidual. Let us consider the following two properties of E.

(P1)—For every x^{**} in E^{**}, the restriction of x^{**} to $(B_{E^*}, \sigma(E^*, E))$ is a Baire-1 function on $(B_{E^*}, \sigma(E^*, E))$.

(P2)—For every x^{**} in E^{**} and for every closed subset M of $(B_{E^*}, \sigma(E^*, E))$, the restriction of x^{**} to M has a point of continuity.

It is easy to see that (P1) implies (P2), the converse is not true in general, an example will be provided later. In **[11]** Odell and Rosenthal showed that if E is any Banach space that does not contain a copy of l_1 then E satisfies (P2) and if E is in addition separable then $(B_{E^*}, \sigma(E^*, E))$ is metrizable and therefore (P2) implies (P1) by the Baire charaterization theorem **[1]**.

In what follows, we will show that if E is any Banach space that satisfies (P2), then E does not contain any copy of l_1. First, we need the following two propositions.

PROPOSITION 1. *Let X be a compact Hausdorff space and f be a real valued function on X such that for every $\varepsilon > 0$, and every closed subset A of X, there exists an open subset U of X such that $U \cap A \neq \varnothing$ and $O(f, U \cap A) \leq \varepsilon$. Then the set of points of continuity of f is a G_δ dense subset of X.*

Proof. Let $n \geq 1$ be an integer and consider the set

$$Z_n = \{x \in X, x \text{ has an open neighborhood } U \text{ such that } O(f, U) \leq 1/n\}.$$

It is clear that Z_n is open. We will show that Z_n is dense in X. To this end, let W be a non-empty open subset of X and let $A = \overline{W}$ be the closure of

W in X. Choose an open set U such that $U \cap A \neq \varnothing$ and $O(f, U \cap A) \leq 1/n$. It is clear that $U \cap W \neq \varnothing$ and any $v \in U \cap W$ belongs to Z_n. Hence $W \cap Z_n \neq \varnothing$. Therefore Z_n is dense in X. Apply the Baire Category Theorem to conclude that the set

$$Z = \bigcap_{n=1}^{\infty} Z_n$$

is a G_δ dense subset of X. The set Z is precisely the set of points of continuity of f.

PROPOSITION 2. *Let E be a Banach space that satisfies* (P2) *and H be any Banach space. If L: $H \to E$ is a bounded linear operator, then for every w^*-compact subset M of E^* and every $x^{**} \in H^{**}$, the restriction of x^{**} to $(L^*(M), \sigma(H^*, H))$ has a point of continuity.*

Proof. Let L^*: $E^* \to H^*$ and let M be a w^*-compact subset in E^* and $x^{**} \in H^{**}$. Let B be a w^*-compact subset of $L^*(M)$. By Proposition 1 it is enough to show that B contains a w^*-relatively open non-empty subset on which x^{**} has arbitrarily small oscillation. To this end, let $\varepsilon > 0$ and let $A = L^{*^{-1}}(B) \cap M$, then A is a w^*-compact subset of E^* satisfying $L^*(A) = B$. Let A_1 be a minimal (under inclusion) w^*-compact subset of E^* such that $L^*(A_1) = B$. The linear functional $x^{**}L^*$ belongs to E^{**}, therefore by hypothesis A_1 contains a w^*-relatively open set W such that $O(x^{**}L^*, W) \leq \varepsilon$. Let $B_1 = L^*(A_1 \backslash W)$, the set B_1 is a w^*-compact subset of H^* and $B_1 \neq B$ by the minimality of A_1. Let u and v be elements in $B \backslash B_1$, there exists u_1 and v_1 in W such that $u = L^*(u_1)$ and $v = L^*(v_1)$. Notice that

$$|x^{**}(u) - x^{**}(v)| = |x^{**}L^*(u_1) - x^{**}L^*(v_1)| \leq O(x^{**}L^*, W) \leq \varepsilon.$$

This shows that $O(x^{**}, B \backslash B_1) \leq \varepsilon$, and finishes the proof of the proposition.

With the help of the above proposition we obtain the following theorem.

THEOREM 3. *Let E be a Banach space. The following statements are equivalent*:

(i) *The space E does not contain any copy of l_1*;

(ii) *For every x^{**} in E^{**} and for every w^*-compact subset M of E^*, the restriction of x^{**} to M has a point of continuity when M is endowed with the relative w^*-topology $\sigma(E^*, E)$.*

Proof. All we have to show is (ii) $\to$ (i). Suppose that (ii) holds and E contains a copy of l_1. Let H: $l_1 \to E$ be the isomorphic embedding of l_1 into E. Then L^*: $E^* \to l_\infty$ is onto. Let $C = B_{l_\infty}$ denote the unit ball of l_∞. Let $h \in l_\infty^*$ and M be a w^*-compact subset of C. Proposition 2 implies that the restriction of h to $(M, \sigma(l_\infty, l_1))$ has a point of continuity. By the Baire Charaterization Theorem **[1]** (C is metrizable) h will be a Baire-1 function on C. But this shows that any $h \in l_\infty^*$ is Baire-1 on C, and this is a contradiction since any $h \in l_\infty^* \setminus l_1$ is not Baire-1 on C.

Example of a Banach space E that satisfies (P2) *but not* (P1). **[11]**.

Let $E = c_0(\Gamma)$ where Γ is uncountable. Because l_1 does not embed into E, E satisfies (P2). Let K be the unit ball of $l_1(\Gamma)$, and let $x^{**} = (u_\alpha)_{\alpha \in \Gamma} \in l_\infty(\Gamma)$ if x^{**} restricted to K is Baire-1, then x^{**} will be the w^*-limit of a sequence in $E = c_0(\Gamma)$, therefore x^{**} will be countably supported. This shows that E does not satisfy (P1).

DEFINITION 4. Let A be a bounded subset of E^*. We say that A is *w^*-scarlarly dentable* if for every $\varepsilon > 0$ and every $x^{**} \in E^{**}$ there exists a w^*-open slice S of A such that $O(x^{**}, S) \leq \varepsilon$.

The above definition should be compared to the following definition of w^*-dentability **[10]**.

DEFINITION 5. Let A be a bounded subset of E^*. We say that A is w^*-dentable, if for every $\varepsilon > 0$, there exists a w^*-open slice S of A such that the norm diameter of S is less than ε.

THEOREM 6. *Let E be a Banach space and let E^* be its dual. The following statements are equivalent*:

(i) *The space E does not contain a copy of l_1*;

(ii) *Every non-empty bounded subset of E^* is w^*-scalarly dentable*;

(iii) *Every non-empty w^*-compact convex subset of E^* is w^*-scalarly dentable*;

(iv) *For every non-empty bounded subset A of E^* and every x^{**} in E^{**}, A contains a non-empty w^*-relatively open subset of E^* on which x^{**} has arbitrarily small oscillation*;

(v) *For every non-empty w^*-compact subset A of E^* and every x^{**} in E^{**}, A contains a non-empty w^*-relatively open subset of E^* on which x^{**} has arbitrarily small oscillation*;

(vi) *For every non-empty w^*-compact subset A of E^* and every x^{**} in E^{**} the restriction of x^{**} to $(A, \sigma(E^*, E))$ has a point of continuity.*

(vii) *For every non-empty w^*-compact subset A of E^* and every x^{**} in E^{**} the set of points of continuity of x^{**} restricted to $(A, \sigma(E^*, E))$ is a w^*-dense G_δ subset of $(A, \sigma(E^*, E))$.*

Proof. (i) $\leftrightarrow$ (vi) is Theorem 3.

(ii) $\rightarrow$ (iii), (iv) $\rightarrow$ (v) and (vii) $\rightarrow$ (vi) are evident.

(iii) $\rightarrow$ (iv) by taking $C = w^*\text{-}\overline{\text{conv}}(A)$.

(v) $\rightarrow$ (vii) is Proposition 1. All that remains is to prove that (vi) $\rightarrow$ (ii).

Let $C = w^*\text{-}\overline{\text{conv}}(A)$, let $x^{**} \in E^{**}$ and let f be the restriction of x^{**} to C. Consider the set

$$Z = \{u \in C;\ O(f, u) \geq \varepsilon\}.$$

It is easy to see that Z is a w^*-closed subset of C. The set Z is also convex, for let u and v be two elements in Z and let $0 \leq \alpha \leq 1$. Consider W a w^*-open subset such that $\alpha u + (1 - \alpha)v \in W \cap C$. Choose U and V two w^*-open neighborhoods of u and v respectively such that $(\alpha U + (1 - \alpha)V) \cap C \subset W \cap C$. It follows that $O(f, U \cap C) \geq \varepsilon$ and $O(f, V \cap C) \geq \varepsilon$. Therefore $O(f, W \cap C) \geq \varepsilon$. Hence $\alpha u + (1 - \alpha)v \in Z$. If $\text{Ext}(C) \subset Z$, then by the Krein-Milman theorem $Z = C$ but $Z \neq C$ because f has a point of continuity. Let $e \in \text{Ext}(C)$ such that e does not belong to Z. This means $O(f, e) < \varepsilon$. Let U be a w^*-closed convex neighborhood of e such that $O(f, U \cap C) \leq \varepsilon$. By the extremality of e, there exists a w^*-open slice S of C ([**2**], Theorem 25.13) such that $e \in S \subset U \cap C$. It is easy to see that $S \cap A \neq \varnothing$. Hence $O(f, S \cap A) \leq \varepsilon$. In fact we have more, since

$$w^*\text{-}\overline{\text{conv}}(S \cap A) \subset U \cap C,$$

then

$$O(f, w^*\text{-}\overline{\text{conv}}(S \cap A)) \leq \varepsilon.$$

This completes the proof.

In the proof of (vi) $\rightarrow$ (ii) we showed the following fact that we state as a proposition.

PROPOSITION 7. *Let A be a bounded subset of E^*, x^{**} be an element of E^{**}, and let $C = w^*\text{-}\overline{\text{conv}}(A)$. If x^{**} restricted to $(C, \sigma(E^*, E))$ has a point of continuity, then for any $\varepsilon > 0$, there exists a w^*-open slice S of C such that $A \cap S \neq \varnothing$, and*

$$O(x^{**}, w^*\text{-}\overline{\text{conv}}(A \cap S)) \leq \varepsilon.$$

REMARK. Using Proposition 7 and a result of Haydon [6] we are going to give another proof of (ii) → (i) in Theorem 3. The argument goes as follows: If l_1 embeds in E, then there exists a w^*-compact convex subset C in E^* such that $C \neq \text{norm-}\overline{\text{conv}}(\text{Ext } C)$ [6]. From this fact, Haydon was able to find $x^{**} \in E^{**}$, $\varepsilon > 0$ and a bounded non-empty subset A of E^* satisfying $O(x^{**}, w^*\text{-}\overline{\text{conv}}(U \cap A)) \geq \varepsilon$ for any w^*-open subset U of E^* such that $U \cap A \neq \emptyset$. Apply Proposition 7 to find a contradiction.

Let E be a Banach space not containing l_1. Let C be a w^*-compact convex subset of E^* and let $x^{**} \in E^{**}$. By Theorem 6 we know that the set Z of the points of continuity of x^{**} restricted to $(C, \sigma(E^*, E))$ is a G_δ dense subset of $(C, (E^*, E))$. A question can be asked: Does Z contain any extreme point of C? In the next proposition we will give an affirmative answer to this question. In fact we have more.

The proof of the next proposition uses the idea of ([9] Theorem 2.2) and Proposition 7.

PROPOSITION 8. *With the above notations, the set* $Z \cap \text{Ext}(C)$ *is a* G_δ *dense subset of* $(\text{Ext}(C), \sigma(E^*, E))$ *and consequently* $C = w^*\text{-}\overline{\text{conv}}(Z \cap \text{Ext}(C))$.

Proof. Let $X = \text{Ext}(C)$ and $\varepsilon > 0$. Let $B_\varepsilon = \{u \in X;\ u$ has a w^*-open neighborhood V such that $O(x^{**}, C \cap V) \leq \varepsilon\}$. The set B_ε is open in $(X, \sigma(E^*, E))$. It is also dense in $(X, \sigma(E^*, E))$. For, let W be a w^*-open subset such that $W \cap X \neq \emptyset$. Let $D = w^* - \overline{X}$ and let $A = W \cap D$. By Proposition 7, there exists a w^*-open subset U such that $U \cap A \neq \emptyset$ and $O(x^{**}, w^*\text{-}\overline{\text{conv}}(U \cap A)) \leq \varepsilon/2$. Let $V = U \cap W$, then $\emptyset \neq V \cap D \subset W \cap D$ and $O(x^{**}, w^*\text{-}\overline{\text{conv}}(V \cap D)) \leq \varepsilon/2$. From now on the proof goes as in ([9], Theorem 2.2) with some obvious changes.

COROLLARY 9. *A Banach space E does not contain a copy of* l_1 *if and only if for every* $x^{**} \in E^{**}$ *and every* w^**-compact convex subset C in* E^**, the intersection* $Z \cap \text{Ext}(C)$ *of the set Z of the points of continuity of* x^{**} *restricted to* $(C, \sigma(E^*, E))$ *with the extreme points of C is a dense* G_δ *subset of* $(\text{Ext}(C), \sigma(E^*, E))$ *and* $C = w^*\text{-}\overline{\text{conv}}(Z \cap \text{Ext}(C))$.

If (X, τ) is a locally convex Hausdorff topological vector space, and A is a bounded subset of (X, τ), the set A is said to be dentable if for every zero-neighborhood V in X, there exists an open slice S of A such that

$S - S \subset V$. We say that (X, τ) is dentable if every bounded subset of (X, τ) is dentable. It is clear from this definition that a subspace (closed or not) of a dentable space is dentable.

If A is a bounded subset of E^*, the dual of a Banach space E, let us agree to say that A is w^*-dentable in $(E^*, \sigma(E^*, E^{**}))$, if for any $\sigma(E^*, E^{**})$ zero-neighborhood V in E^*, there exists a w^*-open slice S such that $S - S \subset V$, accordingly, the set A is w^*-dentable in $(E^*, \| \ \|)$ if A is w^*-dentable in the sense of Definition 5.

In [10] Namioka and Phelps showed that the dual E^* of a Banach space E has the RNP if and only if every non-empty bounded subset of E^* is w^*-dentable in $(E^*, \| \ \|)$. It turns out, as we shall soon show, that E^* has the WRNP if and only if every non-empty bounded subset of E^* is w^*-dentable in $(E^*, \sigma(E^*, E^{**}))$.

THEOREM 10. *For a Banach space E, the following statements are equivalent*:

(i) *The space E does not contain a copy of l_1*;

(ii) *Every non-empty bounded subset A in E^* is w^*-scalarly dentable*;

(iii) *Every non-empty bounded subset A in E^* is w^*-dentable in $(E^*, \sigma(E^*, E^{**}))$.*

Proof. All we have to show is (i) implies (iii). For this, let A be a bounded subset of E^* and V be a $\sigma(E^*, E^{**})$ zero-neighborhood in E^*, the set V has the form

$$V = \{x^* \in E^*; |x_i^{**}(x^*)| \leq \varepsilon, x_i^{**} \in E^{**}, i = 1, 2, \ldots, n\}.$$

Let $C = w^*\text{-}\overline{\text{conv}}(A)$, and let Z_i be the set of points of continuity of the restriction x_i^{**} to $(C, \sigma(E^*, E))$, $i = 1, 2, \ldots, n$, and let $T_i = Z_i \cap \text{Ext}(C)$. By Proposition 8, T_i is a G_δ dense subset of $(\text{Ext}(C), \sigma(E^*, E))$. Hence $T = \bigcap_{i=1}^n T_i$ is also a G_δ dense subset of $(\text{Ext}(C), \sigma(E^*, E))$ since this later is a Baire space by a theorem of Choquet [2]. Let $e \in T$ and choose U a w^*-neighborhood of e such that $O(x_i^{**}, U \cap C) \leq \varepsilon$ for $i = 1, 2, \ldots, n$. By the extremality of e in C, choose a w^*-open slice S of C such that $e \in S \subset U \cap C$. This means that $O(x_i^{**}, S) \leq \varepsilon$ for $i = 1, 2, \ldots, n$. Hence $S - S \subset V$. Therefore $S \cap A \neq \emptyset$ is a w^*-open slice of A and $S \cap A - S \cap A \subset V$. This completes the proof.

It is known that the dual E^* of a Banach space E has the WRNP if and only if E does not contain a copy of l_1 [7]. Combining this fact with Theorem 10 we get

THEOREM 11. *The following statements about a Banach space E are equivalent*:

(i) *The space E^* has* WRNP;

(ii) *Every non-empty bounded subset of E^* is w^*-scalarly dentable*;

(iii) *Every non-empty bounded subset of E^* is w^*-dentable in $(E^*, \sigma(E^*, E^{**}))$.*

REMARK. It is easy to see that for every locally convex Hausdorff space F, the space $(F, \sigma(F, F^*))$ is dentable, for $(F, \sigma(F, F^*))$ can be identified with a subspace of $\mathbf{R}^{F^*}$ by the map $h(x) = (x^*(x))_{x^* \in F^*}$. The space $\mathbf{R}^{F^*}$ is of course dentable. Hence $(F, \sigma(F, F^*))$ is also dentable, therefore one cannot replace the statement "w^*-dentable in $(E^*, \sigma(E^*, E^{**}))$" in (iii) of Theorem 11 by the statement "dentable in $(E^*, \sigma(E^*, E^{**}))$". This also shows that there is no connection whatsoever between the WRNP for a Banach space F and the dentability of $(F, \sigma(F, F^*))$, while the RNP for a Banach space F is equivalent to the dentability of $(F, \| \ \|)$ see ([4], p. 136).

In the following theorems we give a point of continuity criterion that characterizes Asplund operators and those operators that factor through a Banach space not containing l_1.

THEOREM 12. *Let H and F be two Banach spaces, and let T: $H \to F$ be a bounded linear operator, then the following statements are equivalent*:

(i) *The operator T factors through a Banach space not containing l_1*;

(ii) *For every w^*-compact convex subset M in F^* and every $x^{**} \in H^{**}$, the restriction of x^{**} to $(T^*(M), \sigma(H^*, H))$ has a point of continuity.*

Proof. To see that (i) implies (ii), let E be a Banach space not containing l_1 and such that T factors through E as follows

$$\begin{tikzcd} H \arrow[rr, "T"] & & F \\ & E \arrow[ul, "L"] \arrow[ur, "Q"'] & \end{tikzcd}$$

If M is a w^*-compact subset of F^*, then $T^*(M) = L^*(Q^*(M))$. An appeal to Proposition 2 and Theorem 3 finishes the proof of this implication.

Conversely, it is enough to show that $T(B_H)$ contains no copy of the l_1-basis and apply the construction of Davis, Figiel, Johnson and Pelczynski [3]. Suppose not and let $(x_n)_{n \geq 1}$ be a sequence in $T(B_H)$ equivalent to the l_1-basis. For every $n \geq 1$, choose $y_n \in H$ such that $T(y_n) = x_n$. It is easy to see that $(y_n)_{n\geq 1}$ is also equivalent to the l_1-basis. Let S: $l_1 \to H$ defined by $S(e_n) = y_n$ where $(e_n)_{n\geq 1}$ is the usual basis of l_1. The map $S^* \circ T^*$: $F^* \xrightarrow{T} H^* \to l_\infty$ is onto. To see this, let $z \in l_\infty$ and let R the closed linear span of $(x_n)_{n\geq 1}$. Define $\tilde{u} \in R^*$ by $\tilde{u}(x_n) = \langle e_n, z\rangle$. Let $u \in F^*$ be an extension of $\tilde{u}$. It is clear that $S^* \circ T^*(u) = z$. Hence every w^*-compact subset N of l_∞ can be written $N = S^*(T^*(M))$, where M is a w^*-compact subset of F^*. Now use (ii) and Proposition 2 to find a contradiction.

Definition 13. A Banach space G is called an Asplund space if G^* has the Radon-Nikodym property.

Theorem 12 has to be compared with the following:

Theorem 14. *Let H and F be two Banach spaces and let T: $H \to F$ be a bounded linear operator, then the following statements are equivalent*:

(i) *The operator T factors through an Asplund Banach space*;

(ii) *For every w^*-compact convex subset M of F^* the identity map*

$$(T^*(M), \sigma(H^*, H)) \to (T^*(M), \|\ \|)$$

has a point of continuity.

Proof. Consider the same diagram as in Theorem 12, and suppose that E is an Asplund space, then $T^*(M) = L^*(Q^*(M))$ and $Q^*(M)$ is an RNP set. Therefore $L^*(Q^*(M))$ is an RNP set [**14**]. Any w^*-strongly exposed point [**10**] of $T^*(M)$ is a point of continuity of $(T^*(M), \sigma(H^*, H)) \to (T^*(M), \|\ \|)$. Conversely (ii) implies that any w^*-compact convex subset C of $T^*(B_{F^*})$ contains w^*-relatively open subsets of arbitrarily small diameter and therefore by [**10**] $T^*(B_F)$ is an RNP set. Apply [**15**] to finish the proof.

An operator that satisfies one of the above equivalent conditions is called an Asplund operator.

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