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SOME REMARKS ON ALGEBRAIC EQUIVALENCE OF CYCLES

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Let $F \subseteq \mathbf{P}^4$ be a 3-fold with one ordinary double point p, and let F' be the proper transform of F under the blowing up of $\mathbf{P}^4$ at p. If $H \subseteq F'$ is the preimage of p on F', we prove that for F general the algebraic 1-cycle given by the difference of the two generators of the smooth quadric surface H, is not algebraically equivalent to zero on F'. Griffiths has shown this cycle to be homologically equivalent to zero. Also, we show that on a general quintic 3-fold X there are no non-trivial algebraic equivalence relations between the lines of X.

One of the most remarkable results of Griffiths' paper on rational integrals [**3**] is the proof that homological equivalence does not imply algebraic equivalence for algebraic cycles. The argument is essentially based on two theorems, the so-called inversion theorem and the theorem of §14, stating properties of primitive cycles.

Our purpose here is to show that the inversion theorem alone implies, quite directly, that on a general threefold of degree 5 in $\mathbf{P}^4$ two lines are not algebraically equivalent, although they are homologically equivalent because of Lefschetz' theorem. Strengthening the inversion theorem a little bit we can also answer a natural question which may occur to a reader of [**3**] which we explain now. Let $F \subseteq \mathbf{P}^4$ be a threefold with exactly one singular point p, which is a node (ordinary double point) and let F' be the proper transform of F under the blowing up of $\mathbf{P}^4$ at the node. F' is non-singular and the inverse image of p is a smooth quadric surface H. If L, M, are two lines on H belonging to the two different rulings then L is homologically equivalent to M on F', loc. cit. §15. The question is whether L and M are algebraically equivalent:

THEOREM. *If* $\deg F \geq 5$ *and* F *is general then* L *and* M *are not algebraically equivalent on* F'.

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I. The inversion theorem in the nodal case.

(1.1) THEOREM. *If $X \subseteq \mathbf{P}^4$ is a generic threefold with one node, of degree $d \geq 5$ and if b: $X' \to X$ is the desingularization of X obtained by blowing up the node, then every* 1*-cycle algebraically equivalent to* 0 *is contained in the kernel of the Abel-Jacobi map of X'.*

We recall that on a non-singular threefold Y a class $\alpha \in H_3(Y, \mathbf{Z})$ is said to be of rank 2 if there is a surface W and an inclusion g: $W \to Y$ such that $\alpha = g_*(\beta)$, $\beta \in H_3(W, \mathbf{Z})$. Proposition 13.3 of [**3**] says that if there are no non-zero classes of rank 2 on Y then the Abel-Jacobi map sends to 0 every cycle which is algebraically equivalent to 0. To prove (1.1) it is enough to show

(1.2) If X is as in (1.1) then X' contains no non-zero classes of rank 2.

REMARK. We stress that generic means that the set of threefolds which have non-zero classes of rank 2 is contained in a countable union of proper analytic subvarieties of the family T of threefolds of degree d with one node. In particular this implies that if in a pencil of threefolds one element has no non-zero classes of rank 2 then at most a countable number of elements in the pencil have non-zero classes of rank 2.

Proof of (1.1). For simplicity we set $d = 5$. Let $\mathbf{P}^N$ be the projective space parametrizing the hypersurfaces of degree 5 in $\mathbf{P}^4$ and let T be the subset of hypersurfaces with one node. Let $z_0 \in T$ represent X and let D_0 be its equation. Then:

(1.3) Locally at z_0 T is a non-singular hypersurface in $\mathbf{P}^N$. The tangent space to T at z_0 is the hyperplane in $\mathbf{P}^N$ given by the lines through z_0 which correspond to pencils $D_0 + \lambda E$, where E is a polynomial of degree d which satisfies the adjoint condition, namely E passes through the node of D_0.

The proof of this fact is elementary and we omit it.

Now, fix $\alpha \in H_3(X', \mathbf{Z})$ and suppose that there is a neighborhood U of z_0 in T containing a dense subset U^* with the property that for $z \in U^*$ the cycle α_z is of rank 2 on X'_z. By α_z, we mean the cycle class obtained by the following process. Choose a representative cycle for α on X which does not pass through the node, (such a representative exists by (15.9) of [**3**]), and transport it to nearby X_z by taking a solid tube over this representative and intersecting it with X_z, see loc. cit. §3. If X_z is sufficiently close to X, then the transported cycle does not meet the node on X_z. The lifting of this cycle to X'_z is what we call α_z. Note that α_z of rank 2 implies that there exists a surface W_z and an inclusion map g_z: $W_z \to X'_z$ such that $\alpha_z = (g_z)_*(\beta_z)$. Following the notation of [3], set $\eta_z = \Omega/D_z$, where D_z is the equation of X_z, and let b_z: $(\mathbf{P}^4)' \to \mathbf{P}^4$ be the blowing up of $\mathbf{P}^4$ at the node of X_z. By loc. cit. §16, the residue $R(b_z^*\eta_z)$ induces a cohomology class in $H^{3,0}(X'_z)$. Then

$$(1.4) \qquad \int_{\alpha_z} R(b_z^*\eta_z) = \int_{\beta_z} g_z^* R(b_z^*\eta_z) = 0$$

because on a surface every (3, 0) form is 0.

Also, since the integral (1.4) is an analytic function of z vanishing on U^*, it is identically zero on U.

Let $D(\lambda) = \Sigma_{k\geq 0} D_k(x_0,\ldots,x_4)\lambda^k$ be an analytic curve on T centered at D_0, and suppose that for small $|\lambda|$ $D(\lambda)$ is the equation of a threefold X_λ with one single node so that the corresponding point $z(\lambda)$ is in U. By (1.3) the polynomial D_1 satisfies the adjoint condition, and conversely every polynomial which satisfies the adjoint condition can be given as D_1 in the power series expansion of some $D(\lambda)$.

The integral (1.4) is then a function of λ which is identically 0 because $z(\lambda) \in U$. Differentiating (1.4) at $\lambda = 0$ gives (see [3], pg. 508)

$$(1.5) \quad 0 = \int_\alpha R\big(b_o^*(-D_1\Omega/D_0^2)\big) \quad \text{where } Rb^*\big(D_1\Omega/D_0^2\big) \in F^2H^3(X').$$

Thus $\int_\alpha \omega = 0$ for every $\omega \in F^2H^3(X')$, because this vector space is generated by residues of type (1.5) (see loc. cit. §16). Since α is a real homology class, $0 = \int_\alpha \omega = \int_\alpha \bar{\omega}$, hence by Hodge's theorems $\alpha = 0$.

We have therefore proved that if $0 \neq \alpha \in H_3(X', \mathbf{Z})$ then there is a neighborhood U of z_0 in T such that the set of points $z \in U$ for which α_z is of rank 2 is a proper analytic subvariety of U. The statement in (1.2) follows by varying α in $H_3(X', \mathbf{Z})$.

II. The nodal case. Let $G = Q(x_0,\ldots,x_3)x_4^{d-2} + K(x_0,\ldots,x_3)$ be the homogeneous equation of a threefold V_0 in $\mathbf{P}^4$ of degree d. We assume that K and Q are the equations of two non-singular surfaces in $\mathbf{P}^3$ which intersect transversally along a curve C. Then V_0 is non-singular, but for the node at $p = (0, 0, 0, 0, 1)$. Blowing up V_0 at p yields V'_0, and the linear projection from p induces a morphism g: $V'_0 \to B_C(\mathbf{P}^3)$, where $B_C(\mathbf{P}^3)$ is the blowing up of $\mathbf{P}^3$ along C. The map g is a finite covering of degree $(d-2)$, and the exceptional divisor H on V_0' is mapped isomorphically onto Q', the proper transform on $B_C(\mathbf{P}^3)$ of the quadric Q.

Since the Jacobian variety of C and the intermediate Jacobian of $B_C(\mathbf{P}^3)$ are isomorphic **[2]**, the morphism of intermediate Jacobians g_*: $J(V_0') \to J(B_C(\mathbf{P}^3))$ induces a morphism g_*: $J(V_0') \to J(C)$. A straight computation gives

$$(2.1) \qquad g_*(\varphi(L-M)) = -i^*(L-M) \quad \text{in } J(C),$$

where φ is the Abel-Jacobi map, i: $C \to Q$ is the inclusion, i^*: $\mathrm{Pic}(Q) \to \mathrm{Pic}^0(C) \simeq J(C)$, and L, M are lines representing the two different rulings of $Q \simeq Q' \simeq H$.

(2.2) LEMMA. *$\varphi(L-M) \neq 0$ in $J(V_0')$.*

Proof. It suffices to show that $i^*(L-M)$ is not trivial in $\mathrm{Pic}(C)$. If it were, the first ruling on Q would cut on C a linear system which would not be complete, because $i^*(M)$ is not cut by the first ruling. On the other hand, it is easy to show that the first ruling cuts on C a complete linear system.

Now, let V be a generic threefold of degree d with one single node. Without restriction we may assume that the equation of V is $F(x_0,\ldots,x_4) = Qx^{d-2} + \cdots = 0$, i.e., V is singular at p and it has the same tangent cone as V_0. Define a fourfold $\mathcal{V} \subseteq \mathbf{P}^1 \times \mathbf{P}^4$ by the equation $sF + tG = 0$ and blow it up along $\mathbf{P}^1 \times \{p\}$ to obtain a family π: $\mathcal{V}' \to \mathbf{P}^1$, $\pi^{-1}((0,1)) = V_0$. Note that the exceptional divisor on $\mathcal{V}'$ is isomorphic to $\mathbf{P}^1 \times H$, so that on $V_s' = \pi^{-1}((s,1))$ the exceptional divisor is identified with H, the exceptional divisor on V_0'.

Set $B = \{z \in \mathbf{P}^1\colon V_z' \text{ is non-singular}\}$, and let $\mathcal{J} \to B$ be the associated family of intermediate Jacobians.

Fixing L and M on H, the family of cycles $(\mathbf{P}^1 \times L - \mathbf{P}^1 \times M)$ gives a section σ: $B \to \mathcal{J}$, defined by means of the Abel-Jacobi maps, i.e., $\sigma(z) = \varphi_z(L-M) \in J(V_z')$.

(2.3) LEMMA. *σ is not identically zero.*

Proof. By (2.2) $\sigma(0) \neq 0$ and σ is analytic.

By our choice of $V_\infty' = V'$ and the remark after (1.2), for generic $z \in B$ the kernel of the Abel-Jacobi map φ_z contains all the cycles which are algebraically equivalent to zero on V_z'; on the other hand, by the lemma $0 \neq \sigma(z) = \varphi_z(L - M)$, hence

(2.4) For generic z the lines L and M are not algebraically equivalent on V_z'.

III. Lines on a quintic threefold. We recall that on a generic non-singular threefold of degree 5 there are at least two lines which do not intersect [**4**]; we shall show that they are not algebraically equivalent. Since the method of the proof is the same as in the nodal case, we only construct the analogue of V_0' and leave further details to the reader.

Our purpose is to produce a smooth quintic threefold W containing two lines, l_a and l_b, which do not intersect and such that the cycle $(l_a - l_b)$ does not belong to the kernel of the Abel-Jacobi map φ. For this we consider the threefold W defined by the equation $x_0 x_4^4 + K(x_0, \ldots, x_3) = 0$, where: (i) K is a non-singular surface in $\mathbf{P}^3$ of degree 5, (ii) K contains two lines l_a^* and l_b^* which do not intersect and do not lie on the plane H: $\{x_0 = 0\}$, (iii) in $\mathbf{P}^3$, H and K intersect transversally along a curve C. Then, on W, the lines l_a and l_b are the lines l_a^*, l_b^* contained in the hyperplane section $x_4 = 0 = K$.

Blowing up W at $p = (0,0,0,0,1)$ gives W', and we have $J(W) \simeq J(W')$ and $\varphi(l_a - l_b) = \varphi(l_a' - l_b')$, where l' denotes the proper transform of l on W'.

As in (2.1) we get a morphism $g_*\colon J(W') \to J(C)$ and $g_*(\varphi(l_a - l_b)) = -\mathrm{class}(z_a - z_b)$, where z_a, $z_b \in C$ are the points of intersections of l_a and l_b with H.

(3.1) Lemma. $\varphi(l_a - l_b) \neq 0$.

Proof. It suffices to show that on C z_a and z_b are not linearly equivalent. This is true since C is a plane curve of degree > 3, hence not hyperelliptic.

Arguing as in the nodal case one has

(3.2) Two lines on a general quintic threefold are not algebraically equivalent.

Also we thank the referee for suggesting to us this more general statement.

(3.3) If X is a general quintic threefold, $l_1, \ldots, l_{2875}$ the lines on X, then *no* linear combination

$$\sum a_i l_i$$

of the l_i is algebraically equivalent to zero.

The reason is that if we have a relation $\sum a_i l_i \sim 0$, it would follow that $\sum a_i l_{\sigma(i)} \sim 0$ for any σ in the monodromy group M of the 2875 lines. Since $M = S_{2875}$, then, any relation at all would imply that $l_i \sim l_j$ for all i, j.

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