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DIVERGENCE OF COMPLEX RATIONAL APPROXIMATIONS

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General rational interpolations, orthogonal-Padé approximations and best rational real approximations are shown to diverge as badly as classical Padé approximants. The examples also show known convergence results to be best possible in a strong sense.

1. Introduction. In [3], the author used extensions of Wallin's methods [**10**] to show that the well known Nuttall-Pommerenke theorem on convergence in capacity of Padé sequences is substantially best possible. One might expect that general rational interpolations with free poles, should fare better than classic Padé approximants, at least inside the closure of the interpolation points. Surprisingly they do not.

In this note, a new method is used to establish counterexamples to extension of known convergence results for (i) rational interpolants (ii) Padé-orthogonal approximations (iii) best rational real approximations. More specifically, it is shown that diagonal and non-diagonal rational sequences formed from entire functions may diverge in the limit on given σ-compact sets of capacity zero, and that diagonal sequences formed from functions with finite radius of analyticity may diverge in the limit on sets whose intersection with every open ball has positive area. Even in the classic Padé case, the latter example is more complete than Theorem 3 in [3]. It also settles conclusively a problem posed by Goncar[1].

2. Notation. (i) Throughout L, L_i, M, M_i, N, N_i denote positive integers and

$$T(i) = L_i + M_i + 1. \tag{2.1}$$

Further I is a bounded real interval and for any function $f\colon I \to \mathbf{C}$, let $\|f\| = \sup\{|f(t)| : t \in I\}$. Also let $\|I\| = \sup\{|t| : t \in I\}$.

(ii) Given any integer $n \geq 1$, $\mathcal{P}_n$ is the class of polynomials of degree n with 1 as (leading) coefficient of z^n. Also $\mathcal{P}_0 = \{1\}$.

[1] A. A. Goncar, *On the convergence of generalized Padé approximants to meromorphic functions*, Math. USSR Sbornik, **27** (1975), 503–514. On page 504: "If $D(f)$ is a disc of finite radius..."

(iii) For any Borel set $\mathcal{E} \subset \mathbf{C}$, $\text{cap}(\mathcal{E})$ denotes the logarithmic capacity of $\mathcal{E}$.

(iv) A rational function $R = P/Q$ is of type (L/M) if $\deg(P) \leq L$; $\deg(Q) \leq M$; $Q \not\equiv 0$. Further R is real if P, Q have real coefficients.

DEFINITION 2.1. Let $\beta = \{\beta_{LMj}\}$ be complex number s.t. β_{LMj} is given for each $1 \leq j \leq L + M + 1$, all L, M. We assume for some $\Gamma \geq 1$

$$|\beta_{LMj}| \leq \Gamma \quad \text{all } L, M, j \tag{2.2}$$

and set

$$\zeta(L/M)(z) = \prod_{j=1}^{L+M+1} (z - \beta_{LMj}) \quad \text{all } L, M. \tag{2.3}$$

If f: $\beta \to \mathbf{C}$ is analytic at the zeroes of $\zeta(L/M)$ then $\mathcal{I}(L/M) = P/Q$ is the rational function of type (L/M) s.t. $(fQ - P)/\zeta(L/M)$ is analytic at the zeroes of $\zeta(L/M)$.

See Wallin [9] for the convergence results for the rational interpolating functions $\mathcal{I}(L/M)$.

DEFINITION 2.2. Let α: $I \to R$ be non-decreasing with infinitely many points of increase. Let $\Phi_j \in \mathcal{P}_j$ (all $j \geq 0$) satisfy $\int_I \Phi_i \Phi_j \, d\alpha = 0$ all $i \neq j$.

Then given continuous f: $I \to \mathbf{C}$, the linear α-Padé approximant $\langle [L/M] \rangle = P/Q$ and non-linear α-Padé approximant $\langle\langle L/M \rangle\rangle$ are rational functions of type (L/M) s.t.

$$\int_I (fQ - P)\Phi_j \, d\alpha = 0 \quad \text{all } 0 \leq j \leq L + M,$$

$$\int_I (f - \langle\langle L/M \rangle\rangle)\Phi_j \, d\alpha = 0 \quad \text{all } 0 \leq j \leq L + M.$$

See [7] and Suetin [8] for convergence results for $\langle [L/M] \rangle$, $\langle\langle L/M \rangle\rangle$.

DEFINITION 2.3. Let f: $I \to \mathbf{R}$ be continuous in I. The best rational real approximation $\mathcal{R}(L/M)$ to f is a real rational function of type (L/M) s.t. $\|f - \mathcal{R}(L/M)\| = \min \|f - R\|$, the minimum being taken over all real rational functions R of type (L/M).

See [5] for some "overconvergence" results for $\mathcal{R}(L/M)$.

3. Construction of the functions. The examples are based on:

LEMMA 3.1. *Let M, $N > 0$ and $L \geq N$. Let $C \neq 0$ and $Q \in \mathcal{P}_N$; $U \in \mathcal{P}_{L-N+1}$; $W \in \mathcal{P}_{L+M+1}$. Let*

$$b = \max\{|z|: (UQW)(z) = 0\} \tag{3.1}$$

and

$$r > 0 \quad \text{and} \quad s \geq \max\{1, 2b, 2r\}. \tag{3.2}$$

Then there exist polynomials P^* of degree $\leq L$, and Q^* of degree M s.t.

$$UQ^*Q = P^* + CW \tag{3.3}$$

and

$$|UQ^*|(z) \leq (3s)^{L+M+1}|C| \quad \text{all } |z| \leq r. \tag{3.4}$$

If C, U, W, Q are real, so are P^*, Q^*.

Proof. Let P^* be the polynomial of degree $\leq L$ that interpolates to $-CW$ at the $(L + 1)$ zeroes of UQ. Then $Q^* = (P^* + CW)/(UQ)$ has degree M and (3.3) follows. Further for $| z | < s$,

$$\begin{aligned} Q^*(z) &= (2\pi i)^{-1}\int_{|t|=s} (P^* + CW)(t)/[(UQ)(t)(t - z)]\,dt \\ &= (2\pi i)^{-1}\int_{|t|=s} CW(t)/[(UQ)(t)(t - z)]\,dt \end{aligned} \tag{3.5}$$

(as $P^*(t)/[(UQ)(t)(t - z)]$ is analytic in t for $| t | > s$ and is $O(| t |^{-2})$ as $| t | \to \infty$). Then for $| z | \leq r$, (3.1), (3.5) give

$$|Q^*U|(z) \leq s|C|(s + b)^{L+M+1}(s - b)^{-(L+1)}(s - r)^{-1}(r + b)^{L-N+1}$$

and (3.2) then gives (3.4). □

Following is the construction for the interpolation examples:

LEMMA 3.2. *Let $\{L_i\}$, $\{M_i\}$, $\{N_i\}$ satisfy $M_i \geq N_i > 0$ and*

$$L_i - N_i + 1 \geq \sum_{j=1}^{i-1} T(j) \quad \text{all } i > 1 \tag{3.6}$$

(where $T(j)$ is given by (2.1)). Let $Q_i \in \mathcal{P}_{N_i}$ have no zeroes in β ($i \geq 1$). Let $\{s_i\}$ be a monotone increasing sequence s.t. $s_i \geq 2\Gamma$ and

$$s_i \geq 2\max\{|z| : Q_i(z) = 0\} \quad \text{all } i \geq 1 \tag{3.7}$$

and

$$r = \lim_i s_i/2. \tag{3.8}$$

Then there is a function f analytic in $|z| < r$ s.t.

$$f - \mathcal{G}(L_i/M_i) = E_i + (6s_i)^{-T(i)}\zeta(L_i/M_i)/Q_i \tag{3.9}$$

where

$$\limsup_i |E_i(z)|^{1/T(i)} \leq 1/2 \quad \text{all } |z| < r \tag{3.10}$$

the upper bound holding uniformly in compact subsets for large i.

Proof. Let $W_i = \zeta(L_i/M_i)$; $U_i = z^{l(i)}\prod_{j=1}^{i-1}\zeta(L_j/M_j)$ all $i > 1$, where $l(i)$ is chosen s.t. $\deg(U_i) = L_i - N_i + 1$ (possible by (3.6)). Let

$$C_i = (6s_i)^{-T(i)} \quad \text{all } i \geq 1. \tag{3.11}$$

Then for each $i > 1$, Lemma 3.1 with $Q = Q_i$, $U = U_i$, $W = W_i$, $C = C_i$ shows that there exists P_i^* of degree at most L_i, Q_i^* of degree M_i s.t.

$$U_i Q_i^* Q_i = P_i^* + C_i W_i \tag{3.12}$$

and

$$|U_i Q_i^*|(z) \leq 2^{-T(i)} \quad \text{all } |z| \leq s_i/2 \tag{3.13}$$

(by (3.4), (3.11)). Let

$$f(z) = \sum_{j=2}^{\infty} U_j Q_j^* \tag{3.14}$$

and for each $i > 1$, let

$$E_i = \sum_{j=i+1}^{\infty} U_j Q_j^* \tag{3.15}$$

and

$$P_i = Q_i \sum_{j=2}^{i-1} U_j Q_j^* + P_i^*. \tag{3.16}$$

Now we see that (3.6) implies $T(i) > i$, so (3.10) follows from (3.13) and it also follows that f is analytic in $|z| < r$. Using (3.12), (3.14), (3.15), (3.16), we see

$$fQ_i - P_i = E_iQ_i + C_iW_i. \tag{3.17}$$

It is easily seen from the definition of U_j, W_i that the right member of (3.17) has zeroes when $\zeta(L/M)$ does; and (3.6) shows that $\deg(P_i) \le L_i$, while $\deg(Q_i) \le M_i$ so $\zeta(L_i/M_i) = P_i/Q_i$ by uniqueness. Finally (3.11), (3.17) $\Rightarrow$ (3.9). □

Note that in the Newton-Padé case (all $\beta_{LMj} = \beta_j$) we could just take $L_i - N_i + 1 \ge T(i-1)$ and $U_i = z^{l(i)}\zeta(L_i/M_i)$. Following is the construction for the Padé-orthogonal examples:

LEMMA 3.3. *Let* $\{L_i\}, \{M_i\}, \{N_i\}$ *satisfy* $M_i \ge N_i$ *and*

$$L_i - M_i - N_i \ge L_{i-1} + M_{i-1} + N_{i-1} \quad \text{all } i \ge 1. \tag{3.18}$$

Let $Q_i \in \mathcal{P}_{N_i}$ *be real all* $i \ge 1$. *Let* $\{s_i\}$ *be monotone increasing and satisfy* $s_i \ge 2\|I\|$ *and* (3.7). *Let r be given by* (3.8).

(a) *There is a function f analytic in* $|z| < r$ *and real in* $(-r, r)$ *s.t.*

$$f - \langle [L_i/M_i] \rangle = E_i + (6s_i)^{-T(i)}\Phi_{T(i)}/Q_i \tag{3.19}$$

all $|z| < r$, $i > 1$, *where* (3.10) *holds.*

(b) *If the* $\{Q_i\}$ *have no zeroes in I, then there is a function f analytic in* $|z| < r$ *and real in* $(-r, r)$ *s.t.*

$$f - \langle\langle L_i/M_i \rangle\rangle = E_i + (6s_i)^{-T(i)}W_i/Q_i \tag{3.20}$$

all $|z| < r$, $i > 1$, *where* (3.10) *holds and* $W_i \in \mathcal{P}_{T(i)}$ *has all its zeroes in I.*

Proof. (a) Let $U_i = \Phi_{L_i - N_i + 1}$, $W_i = \Phi_{T(i)}$ and C_i be given by (3.11) all $i \ge 1$. Lemma 3.1 with $U = U_i$, $W = W_i$, $Q = Q_i$, $C = C_i$, show that there exist P_i^*, Q_i^* s.t. both (3.12), (3.13) hold. Let f, E_i, P_i be given by (3.14), (3.15), (3.16) respectively. We see that (3.10), (3.17) hold and so

$$\begin{aligned}\int_I (fQ_i - P_i)\Phi_k\,d\alpha &= \sum_{j=i+1}^{\infty} \int_I U_jQ_j^*Q_i\Phi_k\,d\alpha + C_i\int_I \Phi_{T(i)}\Phi_k\,d\alpha \\ &= 0 \quad \text{for } 0 \le k < T(i)\end{aligned}$$

(by (3.18) and choice of U_j).

(b) Let U_i, C_i be as in (a) but define W_i differently: Since $d\alpha(x)/Q_i(x)$ is of one sign in I with infinitely many points of change, there exists $W_i \in \mathcal{P}_{T(i)}$ s.t. $\int_I W_i P d\alpha/Q_i = 0$ for all polynomials P s.t. $\deg(P) < T(i)$. Then all W_i's zeroes lie in I. Further with f, E_i, P_i given by (3.14), (3.15), (3.16) respectively we see

$$\int_I (f - P_i/Q_i)\Phi_k d\alpha = \sum_{j=i+1}^{\infty} \int_I U_j Q_j^* \Phi_k d\alpha + C_i \int_I W_i \Phi_k d\alpha/Q_i$$
$$= 0 \quad \text{all } 0 \le k < T(i). \qquad \square$$

Following is the construction for the best approximation examples.

LEMMA 3.4. *Let $\{L_i\}$, $\{M_i\}$, $\{N_i\}$ satisfy $M_i \ge N_i$ and*

$$L_i - N_i - 1 \ge 2 \sum_{j=1}^{i-1} \left(L_j + M_j + N_j\right) \quad \text{all } i > 1. \tag{3.21}$$

Let $Q_i \in \mathcal{P}_{N_i}$ be real with no zeroes in $I (i \ge 1)$. Let monotone increasing $\{s_i\}$ satisfy

$$s_i \ge 24 \max\left\{1, \|I\|, \max\left\{|z|: Q_i(z) = 0\right\}\right\} \tag{3.22}$$

and let r be given by (3.8).

Then there is a function f analytic in $|z| < r$ and real in $(-r, r)$ s.t.

$$f - \mathcal{R}(L_i/M_i) = E_i + (3s_i)^{-2T(i)} W_i/Q_i \tag{3.23}$$

in $|z| < r$ where (3.10) *holds and $W_i \in \mathcal{P}_{T(i)}$ has all its zeroes in I $(i \ge 1)$.*

Proof. Let $C_i = (3s_i)^{-2T(i)}$ all $i \ge 1$. Let $W_i(z) = z^{T(i)} + P_i^{\#}(z)$ where $\deg(P_i^{\#}) \le T(i) - 1$ and $\|W_i/Q_i\| = \min\|(z^{T(i)} + P)/Q_i\|$ the minimum being taken over all polynomials P s.t. $\deg(P) \le T(i) - 1$. Then Achieser [**1**, p. 55] shows that W_i/Q_i equioscillates at $T(i) + 1$ points in I. Further then all W_i's zeroes lie in I. Thus (3.22) and $T(i) > 2N_i$ and Theorem 4 in Walsh [**11**, p. 104] show that all zeroes of the derivative $(W_i/Q_i)'$ lie in the region $\{z: |z| \le s_i/8\}$. So

$$(W_i/Q_i)' = (T(i) - N_i)H_i/Q_i^2 \tag{3.24}$$

where $H_i \in \mathcal{P}_{L_i+M_i+N_i}$ is real and has at least $T(i) + 1$ zeroes at the points of equioscillation of W_i/Q_i in I, the remaining at most $N_i - 2$ zeroes lying in $|z| \le s_i/8$. Then $|z| = s_i/4$ implies

$$|(W_i/Q_i)'|(z) > (5s_i/24)^{T(i)+1}(s_i/8)^{N_i-2}(7s_i/24)^{2N_i} > 1 \tag{3.25}$$

(as $s_i \geq 24$ and $T(i) + 1 \geq 2N_i + 2$). Next let

$$U_i(z) = z^{l(i)}\left(\prod_{j=1}^{i-1} H_j^2(z)\right)(z - \inf I)(z - \sup I) \tag{3.26}$$

where $l(i)$ is determined so that $\deg(U_i) = L_i - N_i + 1$ (possible by (3.21)). Then for each $i \geq 1$, Lemma 3.1 with $Q = Q_i$, $U = U_i$, $W = W_i$, $C = C_i$ gives polynomials P_i^*, Q_i^* s.t. (3.12) holds and s.t.

$$\max\{|U_iQ_i^*|(z)\colon |z| \leq s_i/2\} \leq (3s_i)^{-T(i)} \quad \text{all } i \geq 1. \tag{3.27}$$

Let f, E_i, P_i be given by (3.14), (3.15), (3.16) respectively. It remains to show $P_i/Q_i = \mathfrak{R}(L_i/M_i)$. Cauchy's integral formula for E_i' applied to the contour $\{t\colon |t| = s_i/2\}$ gives for $|z| \leq s_i/4$,

$$\begin{aligned}|E_i'(z)| &\leq (s_i/2) \sum_{j=i+1}^{\infty} \max\{|U_jQ_j^*|(t)\colon |t| = s_i/2\}(s_i/4)^{-2}\\ &\leq (1/3) \sum_{j=i+1}^{\infty} (3s_j)^{-T(j)}\\ &\qquad\qquad \big(\text{by monotonicity of } \{s_j\} \text{ and by (3.27)}\big)\\ &< C_i/3 \quad (\text{by choice of } C_i \text{ and as } T(j) > 2T(j-1))\end{aligned}$$

So for $|z| = s_i/4$, $|E_i'(z)| < |(C_iW_i/Q_i)'(z)|$ (by (3.25)). Then Rouche's Theorem and analyticity of E_i shows that $(E_i + C_iW_i/Q_i)'$ has the same number of zeroes as $(C_iW_i/Q_i)'$ in $\{z\colon |z| < s_i/4\}$. But E_i and E_i' vanish at all zeroes of $(C_iW_i/Q_i)'$ (by (3.15), (3.24), (3.26)) while E_i vanishes at I's endpoints. We deduce that $f - P_i/Q_i = E_i + C_iW_i/Q_i$ equioscillates at the at least $T(i) + 1$ points of equioscillation of C_iW_i/Q_i in I. Hence $P_i/Q_i = \mathfrak{R}(L_i/M_i)$. □

Finally, we define some polynomials.

LEMMA 3.5. *Let* $\{L_i\}$, $\{M_i\}$, $\{N_i\}$ *satisfy for some* $c > 0$, $\eta > 0$

$$N_i \geq cT(i), \tag{3.28}$$

$$L_i \geq (1 + \eta)T(i - 1). \tag{3.29}$$

Let $a > 1$ *and* $0 < \varepsilon < 1 < r$. *Let* $\mathcal{V}$ *be a subset of* $\{z\colon |z| < r\}$ *with empty interior and* $\mathcal{U} = \{z\colon |z| < r\}\setminus\overline{\mathcal{V}}$ *be non-empty. Let* $W_i \in \mathcal{P}_{T(i)}$ *have all its zeroes in* $\overline{\mathcal{V}}$ $(i \geq 1)$.

Then there exist real polynomials $Q_i \in \mathcal{P}_{N_i}$ *with zeroes in* $\{z: |z| < r\} \setminus \mathcal{V}$ *s.t. if*

$$\mathcal{E}_i = \{z: |W_i/Q_i|(z) > a^{T(i)}\} \quad \text{all } i \geq 1 \tag{3.30}$$

then

(a) *If* B *is a ball s.t.* $B \cap \mathcal{U}$ *is non-empty, then there is a ball* B' *that is non-empty s.t.* $B' \subset \mathcal{E}_i \cap B$ *for all large* i.

(b) *If* B *is a ball in* $\{z: |z| < r\}$ *s.t.* $d(B) \geq \varepsilon$ *then*

$$\liminf_i \text{meas}(\mathcal{E}_i \cap B) > 0$$

where meas *denotes planar Lebesgue measure.*

Proof. We set $Q_i = Q_{1i}Q_{2i}$ where Q_{1i} is used to give (a) and Q_{2i} is used to give (b). Construction of $\{Q_{1i}\}$.

Choose $\{x_i\}$ dense in, and contained in $\mathcal{U}$. Set $\rho(i) = (\eta/4)(1+\eta)^{-i}$ all i so

$$\sum_{i=1}^{\infty} \rho(i) = 1/4. \tag{3.31}$$

Choose positive $\{r_i\}$ s.t. for all i

$$|x_i| + r_i < r \quad \text{and} \quad r_i^{\rho(i)/2} < \left(d(x_i, \overline{\mathcal{V}})/(2a)\right)^{1/c}(2r)^{-2} \tag{3.32}$$

where $d(x_i, \overline{\mathcal{V}})$ denotes the distance from x_i to $\overline{\mathcal{V}}$. Let $n(i; j) =$ greatest integer $\leq \rho(j)N_i$ all $1 \leq j \leq i$. Using (3.28), (3.29) and (3.31) we see

$$n(i; j) \geq \rho(j)N_i/2 \quad \text{all } 1 \leq j \leq i/2 \tag{3.33}$$

(for large i). Now let

$$Q_{1i}(z) = (z-r)^{m(i)} \prod_{j=1}^{i} \left((z-x_j)(z-\bar{x}_j)\right)^{n(i;j)}$$

where $m(i)$ is chosen s.t. $\deg(Q_{1i}) =$ greatest integer $\leq N_i/2$, this being possible by (3.31). Then Q_{1i} is a real polynomial with zeroes in $\mathcal{U}$. Let B_i be the open ball of radius r_i, center x_i, all $i \geq 1$. Then $1 \leq j \leq i/2$ and $z \in B_j$ implies

$$\begin{aligned} |Q_{1i(z)}| &\leq r_j^{n(i;j)}(2r)^{N_i/2} \\ &< \left(d(x_j, \overline{\mathcal{V}})/(2a)\right)^{N_i/c}(2r)^{-N_i} \quad \text{(by (3.32), (3.33))} \\ &< |W_i(z)|a^{-T(i)}(2r)^{-N_i} \end{aligned}$$

(by (3.28) and as if u is a zero of W then $u \in \overline{\mathcal{V}}$ so $|z-u| \geq |u-x_j| - |x_j - z| \geq d(x_j, \overline{\mathcal{V}}) - r_j \geq d(x_j, \overline{\mathcal{V}})/2$). Then as $\deg(Q_{2i}) < N_i$ and its zeroes will lie in $|z| < r$, we deduce from (3.30) that $\bigcup_{j=1}^{i/2} B_j \subset \mathcal{E}_i$ for all large enough i. As the $\{x_i\}$ are dense in $\mathcal{U}$ and $r_i \to 0$, (a) follows.

Construction of $\{Q_{2i}\}$. Choose $0 < \varepsilon_0 < \varepsilon/30$ s.t.

$$n(10\varepsilon_0) = 2r \tag{3.34}$$

for some integer n and choose $\delta > 0$ so small that δ/ε_0 is a positive integer > 1 and

$$\varepsilon_0^{1/c}\delta^{-1/(8n^2)}(4r)^{-1} > a^{1/c}(2r). \tag{3.35}$$

Let $\mathcal{S} = \{z: |\mathrm{Re}(z)| \leq r;\ |\mathrm{Im}(z)| \leq r\}$ and divide $\mathcal{S}$ into n^2 squares $S(1) \cdots S(n^2)$ of side $10\varepsilon_0$. Let

$$\mathcal{L}_i = \{z: |W_i(z)| \leq \varepsilon_0^{T(i)}\} \quad \text{all } i \geq 1 \tag{3.36}$$

which implies $\mathrm{meas}(\mathcal{L}_i) \leq 4e\pi\varepsilon_0^2$ all $i \geq 1$ (by Cartan's Lemma [**2**, p. 194]). Next, for $i \geq 1$, $1 \leq j \leq n^2$, $\mathrm{meas}(S(j)) = 100\varepsilon_0^2 > 2\,\mathrm{meas}(\mathcal{L}_i)$ so we can find a square $S(i;j)$ of centre $b(i;j)$ side δ s.t.

$$S(i;j) \subset S(j) \quad \text{and} \quad \mathrm{meas}(S(i;j)\backslash\mathcal{L}_i) > \delta^2/2 \qquad (1 \leq j \leq n^2;\ i \geq 1). \tag{3.37}$$

By displacing the $b(i;j)$ slightly we can assume all $b(i;j) \notin \mathcal{V}$. Let

$$Q_{2i}(z) = (z-r)^{q(i)} \prod_{j=1}^{n^2} \{(z - b(i;j))(z - \bar{b}(i;j))\}^{p(i)}$$

where $p(i)$ = greatest integer $\leq N_i/(4n^2)$ and $q(i)$ is determined so that $\deg(Q_{1i}Q_{2i}) = N_i$. Then $z \in \bigcup_{j=1}^{n^2} S(i;j)/\mathcal{L}_i$ implies

$$|W_i/Q_{2i}|(z) > \varepsilon_0^{T(i)}/\{\delta^{p(i)}(4r)^{N_i}\} > a^{T(i)}(2r)^{N_i}$$

(as $p(i) \geq N_i/(8n^2)$ and by (3.28), (3.35)). As before this implies $z \in \mathcal{E}_i$. Finally if $B \subset \{z: |z| < r\}$ and $d(B) \geq \varepsilon$, then (3.34) implies that B must contain some $S(j)$. Then for all i, $\mathrm{meas}(B \cap \mathcal{E}_i) \geq \mathrm{meas}(S(j) \cap \mathcal{E}_i) \geq \mathrm{meas}(S(i;j)\backslash\mathcal{L}_i) > \delta^2/2$ (by (3.37)). □

4. Results. We show now that for functions f analytic in a large circle centre 0 (but with singularities of positive capacity in **C**) diagonal sequences of rational approximations will not in general converge in measure or in capacity in any open set within f's radius of analyticity. So the requirement of singularities of cap 0 in **C** in the various Nuttall-Pommerenke theorems **[5, 7, 9]** is essential. Even for the usual Padé approximants (where all $\beta_{LMj} = 0$) this provides a more complete counterexample than Theorem 3 in **[3]**, which did not exclude the possibility that the Padé approximants converged in measure in a neighbourhood of zero.

THEOREM 4.1. *Let* $0 < \varepsilon < 1 < r$. *Let* $\{L_k\}$, $\{M_k\}$ *satisfy for some* $\lambda \geq 1, \eta > 1$

(4.1) $$1/\lambda \leq M_k/L_k \leq \lambda \qquad \text{all } k \geq 1,$$

(4.2) $$L_k \geq (1+\eta)T(k-1) \quad \text{all } k > 1.$$

Then there is a function f *analytic in* $|z| < r$ *s.t. if* $R_k = \mathcal{G}(L_k/M_k)$ *all* $k \geq 1$, *and if*

(4.3) $$\mathcal{F}_k = \{z\colon |z| < r \text{ and } |f - R_k|(z) > 2^{T(k)}\} \quad \text{all } k \geq 1$$

then

(a) *for any ball* B *s.t.* $B \cap \{z\colon |z| < r\} \setminus \bar{\beta} \neq \varnothing$ *there exists a ball* $B' \neq \varnothing$ *s.t.* $B' \subset \mathcal{F}_k \cap B$ *for large* k.

(b) *for any ball* B *in* $\{z\colon |z| < r\}$ *s.t.* $d(B) \geq \varepsilon$, *we have*

$$\liminf_k \operatorname{meas}(\mathcal{F}_k \cap B) > 0.$$

Hence $\{R_k\}$ (*and all its subsequences*) *cannot converge in measure in any open set in* $\{z\colon |z| < r\} \setminus \bar{\beta}$ *nor in any ball* B *in* $\{z\colon |z| < r\}$ *s.t.* $d(B) \geq \varepsilon$.

Proof. Let

(4.4) $$N_k = \text{greatest integer} \leq \min\{M_k, (1 - 1/\eta)L_k\} \quad \text{all } k \geq 1.$$

Then (4.2) implies $T(j) < (1+\eta)^{-1}T(j+1)$, so

$$\sum_{j=1}^{i-1} T(j) < T(i-1)(1+\eta)/\eta \leq L_i/\eta < L_i - N_i + 1$$

(by (4.4)). Hence (3.6) holds. Let $W_i = \zeta(L_i/M_i)$ all $i \geq 1$ and $a = 25r$ and $\mathcal{V} = \beta$ in Lemma 3.5. Since (4.1), (4.2), and (4.4) hold, Lemma 3.5 gives real polynomials $Q_i \in \mathcal{P}_{N_i}$ satisfying Lemma 3.5(a), (b) and with all their zeroes in $\{z\colon |z| < r\} \setminus \beta$. We can clearly assume $r > \Gamma$ (given by (2.2)). Letting $s_i = 2r$ all i, we see (3.7) holds. Then with the $\{L_i\}$, $\{M_i\}$, $\{N_i\}$, $\{W_i\}$, $\{Q_i\}$ chosen above, Lemma 3.2 gives a function f analytic in

$|z| < r$ s.t. (3.9), (3.10) hold. Then for all $|z| \le r' < r$ s.t. $z \in \mathcal{E}_i$ (given by (3.30)) we have for large i

$$|f - \mathcal{G}(L_i/M_i)|(z) \ge (12r)^{-T(i)} |\zeta(L_i/M_i)/Q_i|(z) - 1 > 2^{T(i)}$$

(by (3.30) and as $a = 25r$). Thus $\mathcal{E}_i \cap \{z\colon |z| \le r'\} \subset \mathcal{F}_i$ for large i and Lemma 3.5(a), (b) give our result. □

REMARKS. (a) So surprisingly enough, there is not even convergence in measure in $\bar{\beta}$, the closure of the interpolation points, in general.

(b) One may modify the above example so that divergence of the above type occurs even when most of the poles of the $\{\mathcal{G}(L_k/M_k)\}$ are fixed in advance and are not determined by interpolatory conditions–see the technical report [6] (not intended for publication).

THEOREM 4.2. *Let $r > 1$. There is a function f analytic in $|z| < r$ and real in $(-r, r)$ s.t. if $\{R_k\} = \{\langle[L_k/M_k]\rangle\}$ (or $\{R_k\} = \{\langle\langle L_k/M_k\rangle\rangle\}$; or $\{R_k\} = \{\mathcal{R}(L_k/M_k)\}$) then for any ball B s.t. $B \cap \{z\colon |z| < r\} \neq \varnothing$ there exists a ball $B' \neq \varnothing$ s.t. $B' \subset \mathcal{F}_k \cap B$ for large k (where $\mathcal{F}_k$ is given by* (4.3)).

Hence $\{R_k\}$ (and all its subsequences) cannot converge in measure or in capacity to f in any subset of $\{z\colon |z| < r\}$ with non-empty interior.

Here (a) *If $\{R_k\} = \{\langle[L_k/M_k]\rangle\}$ or $\{R_k\} = \{\langle\langle L_k/M_k\rangle\rangle\}$ we must insist that for some $\eta > 0$, $\lambda \ge 1$,*

$$M_k/\lambda \le L_k - M_k \le \lambda M_k;\ L_k - M_k \ge (1 + \eta)T(k-1). \tag{4.5}$$

(b) *If $\{R_k\} = \{\mathcal{R}(L_k/M_k)\}$ we must insist that* (4.1), (4.2) *hold for some $\eta > 2$, $\lambda \ge 1$.*

Proof. Similar to Theorem 4.1, but using Lemmas 3.3, 3.4. □

REMARKS. (a) While (4.5) excludes the case $L_i = M_i$, it allows $L_i = (1 + \delta)M_i$ where $\delta > 0$ may be arbitrarily small.

(b) For the linear Padé approximants $\{\langle[L_k/M_k]\rangle\}$ the example can be modified to show that there is not even convergence in capacity in any segment of I with diameter $\ge \varepsilon$ (ε being fixed in advance).

(c) We next show that general diagonal rational sequences formed from entire functions can diverge in the limit on given σ-compact sets of cap 0. As in [3] (where this was shown for the usual Padé approximants) this implies that convergence in cap cannot be strengthened to convergence in some "thinner" set function.

THEOREM 4.3. *Let $\mathfrak{S}$ be σ-compact and* cap($\mathfrak{S}$) = 0. *Then there is an entire function f s.t.*

$$\lim_k |f - R_k|(z)^{1/T(k)} = \infty \quad \text{all } z \in \mathfrak{S}. \tag{4.6}$$

Here (a) *In the interpolation case, we assume $\mathfrak{S} \subset \mathbf{C}\setminus\bar{\beta}$ and $\{R_k\} = \{\mathcal{I}(L_k/M_k)\}$ where* (4.1), (4.2) *hold for some $\lambda > 1$, $\eta > 1$.*

(b) *In the orthogonal-Padé approximation case, we assume $\mathfrak{S} \subset \mathbf{C}\setminus I$ and $\{R_k\} = \{\langle[L_k/M_k]\rangle\}$ (or $\{R_k\} = \{\langle\langle L_k/M_k\rangle\rangle\}$) where* (4.5) *holds for some $\lambda \geq 1$, $\eta > 0$.*

(c) *In the best approximation case, we assume $\mathfrak{S} \subset \mathbf{C}\setminus I$ and $\{R_k\} = \{\mathcal{R}(L_k/M_k)\}$ where* (4.1), (4.2) *hold for some $\lambda \geq 1$, $\eta > 2$.*

Proof. (a) With $\{N_k\}$ given by (4.4), we see as before that (3.6) holds. Further there exists $c > 0$ s.t. (3.28) holds. Next the arguments in [**3**, Theorem 2] show that there exist $Q_k \in \mathcal{P}_{N_k}$ (all $k \geq 1$) and $\{\varepsilon_k\}$ s.t. $\lim_k \varepsilon_k = 0$, s.t.

$$Q_k\text{'s roots lie in } |z| \leq \varepsilon_k^{-c/3}/2 \text{ all large enough } k, \tag{4.7}$$

$$z \in \mathfrak{S} \text{ implies } |Q_k(z)| < \varepsilon_k^{N_k} \text{ all large enough } k. \tag{4.8}$$

By minute displacements to the zeroes of $\{Q_k\}$, we may assume that they lie outside β. Set $s_k = \varepsilon_k^{-c/3}$ all $k \geq 1$ in Lemma 3.2. Then (2.2) and (4.7) imply (3.7). Further $\lim_k s_k = \infty$, so f of Lemma 3.2 is entire. Further $z \in \mathfrak{S}$ and (3.9), (3.10), (3.28), (4.8) give for large k,

$$\begin{aligned}|f - \mathcal{I}(L_k/M_k)|(z)^{1/T(k)} &\geq \left\{(6s_k)^{-T(k)}|\zeta(L_k/M_k)/Q_k|(z) - 1\right\}^{1/T(k)} \\ &\geq 0.1\varepsilon_k^{c/3-c}|\zeta(L_k/M_k)|(z)^{1/T(k)} \to \infty \quad \text{as } k \to \infty\end{aligned}$$

as $\zeta(L_k/M_k)$'s zeroes lie in β and $z \notin \bar{\beta}$. (b), (c) are similar, though one replaces Q_k above by $Q_k = Q_k^*(z)\bar{Q}_k^*(\bar{z})$ where $Q_k^* \in \mathcal{P}_{N_k/2}$ to obtain real Q_k. □

REMARKS. (a) One can relax $\mathfrak{S} \subset \mathbf{C}\setminus\bar{\beta}$ slightly and also allow the zeroes of $\zeta(L_k/M_k)$ to tend to ∞ sufficiently slowly (the rate depending on $\mathfrak{S}$). Similarly for the linear Padé approximants, one can relax $\mathfrak{S} \subset \mathbf{C}\setminus I$.

(b) It is well known that non-diagonal sequences of various rational approximations converge in capacity–in fact, this is best possible:

THEOREM 4.4. *Let $\mathfrak{S}$ be a σ-compact set s.t.* cap($\mathfrak{S}$) = 0. *There exists an entire function f and $\{M_k\}$, $\{L_k\}$ s.t. $\lim_k L_k = \infty$; $\lim_k M_k/L_k = 0$ and $\lim_k |f - R_k|(z)^{1/T(k)} = \infty$ all $z \in \mathfrak{S}$.*

Here (a) *In the interpolation case, we assume* $\mathcal{E} \subset \mathbf{C}\backslash\overline{\beta}$ *and* $\{R_k\} = \{\mathcal{I}(L_k/M_k)\}$.

(b) *In the orthogonal-Padé approximation case, we assume* $\mathcal{E} \subset \mathbf{C}\backslash I$ *and* $\{R_k\} = \{\langle[L_k/M_k]\rangle\}$ (*or* $\{R_k\} = \{\langle\langle L_k/M_k\rangle\rangle\}$).

(c) *In the best approximation case, we assume* $\mathcal{E} \subset \mathbf{C}\backslash I$ *and* $\{R_k\} = \{\mathcal{R}(L_k/M_k)\}$.

Proof. This is similar that of Theorem 4.3—a full proof appears in **[6]**. □

Note finally that the results in **[4]** which characterize the thinness of exceptional sets for non-diagonal sequences of Padé approximants also hold for the above approximations–see **[6]**.

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