PACIFIC JOURNAL OF MATHEMATICS
Vol. 108, No. 1, 1983

A PROOF OF THE BENDER-KNUTH CONJECTURE

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Let $b_r(n \mid I_m)$ denote the number of r-rowed partitions of n whose parts lie in the set $I_m = \{1, 2, \ldots, m\}$ and decrease strictly along each row. It is shown that

$$\sum_{n=0}^{\infty} b_r(n \mid I_m) x^n = \prod_{i=1}^{m} \prod_{j=1}^{i} (1 - x^{r+i+j-1})/(1 - x^{i+j-1}).$$

1. Introduction. For any given set S of positive integers, let $b_r(n \mid S)$ denote the number of r-rowed partitions of n whose parts lie in S and decrease strictly along each row. Put $b(n \mid S) = \lim_{r \to \infty} b_r(n \mid S)$. Bender and Knuth [3] have proved the remarkable formula

$$\begin{aligned} B(x \mid S) &= \sum_{n=0}^{\infty} b(n \mid S) x^n \\ &= \prod_{i \in S} (1 - x^i)^{-1} \prod_{\substack{j,k \in S \\ j<k}} (1 - x^{j+k})^{-1}, \end{aligned}$$

valid for $|x| < 1$. As yet no such simple expression has been found for $B_r(x \mid S) = \sum_{n=0}^{\infty} b_r(n \mid S) x^n$. However there are two situations where a "product" formula for $B_r(x \mid S)$ can indeed be given, namely when $S = I_m = \{1, 2, \ldots, m\}$ or $S = J_m = \{1, 3, 5, \ldots, 2m - 1\}$. The formulas are:

$$B_r(x \mid I_m) = \prod_{i=1}^{m} \prod_{j=1}^{i} \frac{1 - x^{r+i+j-1}}{1 - x^{i+j-1}}, \tag{1}$$

$$B_r(x \mid J_m) = \prod_{i=1}^{m} \frac{1 - x^{r+2i-1}}{1 - x^{2i-1}} \prod_{j=i+1}^{m} \frac{1 - x^{2(r+i+j-1)}}{1 - x^{2(i+j-1)}}. \tag{2}$$

Equation (2) was conjectured by MacMahon [5], while (1) was conjectured by Bender and Knuth [2]. Some years earlier the author had already found a proof of (1), but published only the limiting case $m \to \infty$. Then Andrews [1] proved (2), and also showed in [3] that (1) and (2) are equivalent. This of course gave another proof of (1). Over the years, a number of people have expressed a desire to see the original direct proof of (1) in print. It will therefore be belatedly presented here.

2. Notation. If n is a positive integer, while a and y are indeterminates, we write

$$(a;\, y)_n = (1-a)(1-ay)(1-ay^2)\cdots(1-ay^{n-1}).$$

By convention, $(a;\, y)_0 = 1$. If $0 \le k \le n$, put

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(x;\, x)_n}{(x;\, x)_k (x;\, x)_{n-k}}.$$

If $k < 0$ or $k > n$, define $[{n \atop k}] = 0$.

3. Proof of (1). Let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be a sequence of integers satisfying $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_r \ge 0$. We consider r-rowed partitions of the type enumerated by $b_r(n \mid S)$, but where there are exactly λ_i non-zero parts in the ith row ($i = 1, \ldots, r$). Let $b_r(n;\, \lambda_1, \ldots, \lambda_r \mid S)$ be the number of such partitions of n, and put

$$B_r(x;\, \lambda_1, \ldots, \lambda_r \mid S) = \sum_{n=0}^{\infty} b_r(n;\, \lambda_1, \ldots, \lambda_r \mid S) x^n.$$

Clearly $B_r(x \mid S) = \Sigma_{(\lambda_i)} B_r(x;\, \lambda_1, \ldots, \lambda_r \mid S)$, where the sum is extended over all sequences (λ_i) with $\lambda_1 \ge \cdots \ge \lambda_r \ge 0$. We now obtain an expression for $B_r(x;\, \lambda_1, \ldots, \lambda_r \mid I_m)$ as a determinant.

THEOREM 1.

$$B_r(x;\, \lambda_1, \ldots, \lambda_r \mid I_m) = \det_{1 \le i, j \le r} \left(x^{\binom{i-j+\lambda_j+1}{2}} \begin{bmatrix} m \\ i-j+\lambda_j \end{bmatrix} \right).$$

Proof. Given a partition π of n of the type enumerated by $b_r(n;\, \lambda_1, \ldots, \lambda_r \mid I_m)$, we subtract 1 from each of its parts. If $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_r$, this gives a partition of $n - \lambda$ of the type enumerated by $b_r(n-\lambda;\, \lambda_1 - \varepsilon_1, \ldots, \lambda_r - \varepsilon_r \mid I_{m-1})$, where $\varepsilon_j = 0$ or 1 according as the last part of the jth row of π is greater than 1 or equal to 1. Moreover, every such partition of $n - \lambda$ is the image of exactly one π under this map. It follows readily that

$$B_r(x;\, \lambda_1, \ldots, \lambda_r \mid I_m) = x^{\lambda} \sum_{\varepsilon_j = 0}^{1} B_r(x;\, \lambda_1 - \varepsilon_1, \ldots, \lambda_r - \varepsilon_r \mid I_{m-1}),$$

where we make the convention that $B_r(x;\, \lambda_1 - \varepsilon_1, \ldots, \lambda_r - \varepsilon_r \mid I_{m-1}) = 0$ if the inequalities $\lambda_1 - \varepsilon_1 \ge \lambda_2 - \varepsilon_2 \ge \cdots \ge \lambda_r - \varepsilon_r \ge 0$ are not satisfied. If $\lambda_1 = \cdots = \lambda_r = 0$, we have $B_r(x;\, 0, \ldots, 0 \mid I_m) = 1$.

Now let $C_r(x; \lambda_1,\ldots,\lambda_r \mid I_m)$ be the determinant in the statement of the theorem. Then

$$C_r(x; 0,\ldots,0 \mid I_m) = \det_{1\le i,j\le r}\left(x^{\binom{i-j+1}{2}}\begin{bmatrix} m \\ i-j\end{bmatrix}\right) = 1,$$

since the diagonal entries are all equal to 1, while those above the diagonal vanish. Thus $B_r(x; 0,\ldots,0 \mid I_m) = C_r(x; 0,\ldots,0 \mid I_m)$. The proof can now be completed by induction by showing that

$$C_r(x; \lambda_1,\ldots,\lambda_r \mid I_m) = x^{\lambda} \sum_{\varepsilon_j=0}^{1} C_r(x; \lambda_1 - \varepsilon_1,\ldots,\lambda_r - \varepsilon_r \mid I_{m-1}).$$

To do this, we recall the well-known identity

$$\begin{bmatrix} a \\ b\end{bmatrix} = \begin{bmatrix} a-1 \\ b-1\end{bmatrix} + x^b \begin{bmatrix} a-1 \\ b\end{bmatrix}.$$

Using this identity, we can write the general term of the determinant $C_r(x; \lambda_1,\ldots,\lambda_r \mid I_m)$ in the form

$$x^{\binom{i-j+\lambda_j+1}{2}}\begin{bmatrix} m-1 \\ i-j+\lambda_j-1\end{bmatrix} + x^{\binom{i-j+\lambda_j+1}{2}+i-j+\lambda_j}\begin{bmatrix} m-1 \\ i-j+\lambda_j\end{bmatrix}.$$

Now

$$\binom{i-j+\lambda_j+1}{2} = \binom{i-j+\lambda_j}{2} + i - j + \lambda_j,$$

so the above term can be written as

$$x^{i-j+\lambda_j}\left(x^{\binom{i-j+\lambda_j}{2}}\begin{bmatrix} m-1 \\ i-j+\lambda_j-1\end{bmatrix} + x^{\binom{i-j+\lambda_j+1}{2}}\begin{bmatrix} m-1 \\ i-j+\lambda_j\end{bmatrix}\right).$$

We can now remove a common factor of x^i from the ith row, and a common factor of $x^{-j+\lambda_j}$ from the jth column. The product of these factors is $x^{\lambda_1+\cdots+\lambda_r} = x^{\lambda}$. Hence

$$(3)\quad C_r(x; \lambda_1,\ldots,\lambda_r \mid I_m)$$
$$= x^{\lambda}\det\left(x^{\binom{i-j+\lambda_j}{2}}\begin{bmatrix} m-1 \\ i-j+\lambda_j-1\end{bmatrix} + x^{\binom{i-j+\lambda_j+1}{2}}\begin{bmatrix} m-1 \\ i-j+\lambda_j\end{bmatrix}\right).$$

The general term of the determinant on the right side of (3) can be written in the form

$$\sum_{\varepsilon_j=0}^{1} x^{\binom{i-j+\lambda_j+1-\varepsilon}{2}}\begin{bmatrix} m-1 \\ i-j+\lambda_j-\varepsilon\end{bmatrix}.$$

Hence, since the determinant is a linear function of each column vector, we have

$$C_r(x; \lambda_1,\ldots,\lambda_r \mid I_m) = x^{\lambda} \sum_{\varepsilon_j=0}^{1} \det\left(x^{\binom{i-j+\lambda_j+1-\varepsilon_j}{2}} \begin{bmatrix} m-1 \\ i-j+\lambda_j-\varepsilon_j \end{bmatrix}\right)$$

$$= x^{\lambda} \sum_{\varepsilon_j=0}^{1} C_r(x; \lambda_1-\varepsilon_1,\ldots,\lambda_r-\varepsilon_r \mid I_m).$$

This completes the proof of Theorem 1.

Now let

$$a_\nu = x^{\binom{\nu+1}{2}}\begin{bmatrix} m \\ \nu \end{bmatrix}, \quad g_i = i, \quad h_j = \lambda_j - j.$$

Then the result of Theorem 1 can be written in the form

$$B_r(x; \lambda_1,\ldots,\lambda_r \mid I_m) = \det(a_{g_i+h_j}).$$

The requirement $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$ is equivalent to $h_1 > h_2 > \cdots > h_r$, and therefore

$$B_r(x \mid I_m) = \sum_{h_1>\cdots>h_r} \det(a_{g_i+h_j}).$$

Set $s = \sum_{n=0}^{\infty} a_n$, $c_\nu = \sum_{n=0}^{\infty} a_n a_{n+\nu}$, $d_\nu = c_0 + 2(c_1 + \cdots + c_{\nu-1}) + c_\nu$ $(\nu > 0)$, $d_0 = 0$, and $d_{-\nu} = -d_\nu$. Then by Lemma 1 of [4], we see that for even r, $B_r(x \mid I_m)$ is the Pfaffian of the skew-symmetric $r \times r$ matrix $D_r = (d_{j-i})$, while for odd r, $B_r(x \mid I_m)$ is the Pfaffian of the $(r+1) \times (r+1)$ matrix

$$D_r' = \begin{bmatrix} D_r & -s \\ s & 0 \end{bmatrix}$$

obtained by bordering D_r with a row of s's, a column of $-s$'s, and a zero.

We next proceed to evaluate the quantities s and $c_{\nu-1} + c_\nu$.

THEOREM 2. $s = \prod_{i=1}^{m}(1 + x^i)$, *and*

$$c_{\nu-1} + c_\nu = x^{\binom{\nu}{2}}\begin{bmatrix} 2m+1 \\ m+\nu \end{bmatrix}.$$

Proof. The quantity $a_\nu = x^{\binom{\nu+1}{2}}\left[\begin{smallmatrix} m \\ \nu \end{smallmatrix}\right]$ can be interpreted combinatorially as the generating function of ordinary (i.e. one-rowed) partitions into exactly ν distinct parts, all $\leq m$. This observation follows from Theorem 1 with $r = 1$, but is also easy to see directly. On the other hand, if the product $\prod_{i=1}^{m}(1 + x^i y)$ is expanded as a power series in x and y, the coefficient of y^ν is also the generating function of such partitions. Therefore

$$(4) \qquad \prod_{i=1}^{m} (1 + x^i y) = \sum_{\nu=0}^{\infty} a_\nu y^\nu.$$

Putting $y = 1$ in (4), we get $\prod_{i=1}^{m}(1 + x^i) = \sum_{\nu=0}^{\infty} a_\nu = s$. Now let $f_m(x, y) = \prod_{i=1}^{m}(1 + x^i y)(1 + x^i y^{-1})$. Then from (4) it follows that

$$f_m(x, y) = \sum_{\lambda=0}^{m} \sum_{\mu=0}^{m} a_\lambda a_\mu y^{\lambda-\mu} = \sum_{\nu=-m}^{m} \Big(\sum_{\lambda-\mu=\nu} a_\lambda a_\mu \Big) y^\nu.$$

For $\nu \geq 0$, we have $\sum_{\lambda-\mu=\nu} a_\lambda a_\mu = \sum_{n=0}^{m} a_n a_{n+\nu} = c_\nu$. Hence if we put $c_{-\nu} = c_\nu$, we have $f_m(x, y) = \sum_{\nu=-\infty}^{\infty} c_\nu(m) y^\nu$, where we have written $c_\nu(m)$ instead of c_ν to emphasize the dependence on m. Now

$$f(x, y) = f_{m-1}(x, y)(1 + x^m y)(1 + x^m y^{-1}),$$

and hence

$$\sum_{-\infty}^{\infty} c_\nu(m) y^\nu = (1 + x^m y)(1 + x^m y^{-1}) \sum_{-\infty}^{\infty} c_\nu(m - 1) y^\nu.$$

Equating powers of y^ν in these two Laurent series, we find that

$$(5) \quad c_\nu(m) = (1 + x^{2m}) c_\nu(m - 1) + x^m c_{\nu-1}(m - 1) + x^m c_{\nu+1}(m - 1).$$

For convenience of notation, put $\gamma_\nu(m) = c_{\nu-1}(m) + c_\nu(m)$. If in equation (5) we replace ν by $\nu - 1$, and then add the result to (5), we obtain

$$(6) \quad \gamma_\nu(m) = (1 + x^{2m}) \gamma_\nu(m - 1) + x^m \gamma_{\nu-1}(m - 1) + x^m \gamma_{\nu+1}(m - 1).$$

When $m = 0$, we have $a_0 = 1$ and $a_\nu = 0$ for all $\nu > 0$. Hence $c_0(0) = 1$, while $c_\nu(0) = 0$ for all $\nu \neq 1$. This gives $\gamma_0(0) = \gamma_1(0) = 1$, and $\gamma_\nu(0) = 0$ for all $\nu \neq 0, 1$. On the other hand, when $m = 0$ we have

$$x^{\binom{\nu}{2}} \begin{bmatrix} 2m + 1 \\ m + 1 - \nu \end{bmatrix} = x^{\binom{\nu}{2}} \begin{bmatrix} 1 \\ 1 - \nu \end{bmatrix},$$

which is also equal to 1 when $\nu = 0$ or 1, and is equal to 0 otherwise. This proves the theorem in the case $m = 0$.

Now suppose that $m > 0$, and that the theorm has already been proved for $m - 1$. Then from equation (6) we have

$$(7) \quad \gamma_\nu(m) = (1 + x^{2m}) x^{\binom{\nu}{2}} \begin{bmatrix} 2m - 1 \\ m - 1 + \nu \end{bmatrix} + x^m x^{\binom{\nu-1}{2}} \begin{bmatrix} 2m - 1 \\ m - 2 + \nu \end{bmatrix}$$
$$+ x^m x^{\binom{\nu+1}{2}} \begin{bmatrix} 2m - 1 \\ m + \nu \end{bmatrix}.$$

It is a straightforward matter to check that the expression on the right side of (7) is equal to

$$x^{\binom{\nu}{2}}\begin{bmatrix} 2m+1 \\ m+\nu \end{bmatrix}.$$

This proves Theorem 2 by induction on m.

If $\nu > 0$, we have $d_\nu = \sum_{\mu=1}^{\nu}(c_{\mu-1} + c_\mu)$, and therefore by Theorem 2,

$$d_\nu = \sum_{\mu=1}^{\nu} x^{\binom{\mu}{2}}\begin{bmatrix} 2m+1 \\ m+\mu \end{bmatrix}.$$

Consider now the determinant det D_r, where r is even. Subtracting each row from the previous row, we obtain

$$\det D_r = \begin{vmatrix} \gamma_1 & \gamma_1 & \gamma_2 & \gamma_3 & \cdots & \gamma_{r-1} \\ \gamma_2 & \gamma_1 & \gamma_1 & \gamma_2 & \cdots & \gamma_{r-2} \\ \vdots & & & & & \\ \gamma_{r-1} & \gamma_{r-2} & \gamma_{r-3} & \gamma_{r-4} & \cdots & \gamma_1 \\ -d_{r-1} & -d_{r-2} & -d_{r-3} & -d_{r-4} & \cdots & 0 \end{vmatrix}.$$

Adding all the rows to the last row, we get

$$\det D_r = \begin{vmatrix} \gamma_1 & \gamma_1 & \gamma_2 & \gamma_3 & \cdots & \gamma_{r-1} \\ \gamma_2 & \gamma_1 & \gamma_1 & \gamma_2 & \cdots & \gamma_{r-2} \\ \vdots & & & & & \\ \gamma_{r-1} & \gamma_{r-2} & \gamma_{r-3} & \gamma_{r-4} & \cdots & \gamma_1 \\ 0 & d_1 & d_2 & d_3 & \cdots & d_{r-1} \end{vmatrix}.$$

It is now convenient to extend the definition of γ_ν to $\nu \le 0$, by putting $\gamma_\nu = \gamma_{1-\nu}$ in that case. Thus for $\nu \le 0$, we have

$$\gamma_\nu = x^{\binom{1-\nu}{2}}\begin{bmatrix} 2m+1 \\ m+1-\nu \end{bmatrix} = x^{\frac{\nu(\nu-1)}{2}}\begin{bmatrix} 2m+1 \\ m+\nu \end{bmatrix}.$$

If we make the convention that $\binom{\nu}{2} = \nu(\nu-1)/2$ for all $\nu \in \mathbf{Z}$, then we have

$$\gamma_\nu = x^{\binom{\nu}{2}}\begin{bmatrix} 2m+1 \\ m+\nu \end{bmatrix}$$

for all ν. The above expression for det D_r can be written as

(8)
$$\det D_r = \begin{vmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_{r-1} \\ \gamma_{-1} & \gamma_0 & \gamma_1 & \cdots & \gamma_{r-2} \\ \vdots & & & & \\ \gamma_{2-r} & \gamma_{3-r} & \gamma_{4-r} & \cdots & \gamma_1 \\ 0 & d_1 & d_2 & \cdots & d_{r-1} \end{vmatrix}.$$

In exactly the same way we find that for r odd,

$$\det D_r' = \begin{vmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_{r-1} & 0 \\ \gamma_{-1} & \gamma_0 & \gamma_1 & \cdots & \gamma_{r-2} & 0 \\ \vdots & & & & & \\ \gamma_{2-r} & \gamma_{3-r} & \gamma_{4-r} & \cdots & \gamma_1 & 0 \\ 0 & d_1 & d_2 & \cdots & d_{r-1} & -s \\ s & s & s & \cdots & s & 0 \end{vmatrix},$$

and hence

(9)
$$\det D_r' = s^2 \begin{vmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_{r-1} \\ \gamma_{-1} & \gamma_0 & \cdots & \gamma_{r-2} \\ \vdots & & & \\ \gamma_{2-r} & \gamma_{3-r} & \cdots & \gamma_1 \\ 1 & 1 & \cdots & 1 \end{vmatrix}.$$

The idea is now to put the determinants (8) and (9) into superdiagonal form by elementary row operations. All but the last of these operations are the same for both determinants; we describe these in terms of an arbitrary matrix (a_{ij}), $1 \le i, j \le r$. For the moment we leave the bottom row unchanged. For each i in the range $2 \le i \le r-1$, we multiply the $(i-1)$th row by $a_{i,1}/a_{i-1,1}$ and subtract the result from the ith row. This gives a matrix (b_{ij}) with $b_{i1} = 0$ for $2 \le i \le r-1$. Next, for each i in the range $3 \le i \le r-1$, we multiply the $(i-1)$th row by $b_{i,2}/b_{i-1,2}$ and subtract the result from the ith row. This gives a matrix (c_{ij}) with $c_{i1} = 0$ for $2 \le i \le r-1$ and $c_{i2} = 0$ for $3 \le i \le r-1$. We proceed in this manner until we obtain a matrix which, except for its bottom row, is in supertriangular form.

In the present case, if we temporarily ignore the factor s^2 in (9) we have, for both (8) and (9).

$$a_{i,j} = \gamma_{j-i} = \begin{bmatrix} 2m+1 \\ m+j-i \end{bmatrix} x^{\binom{j-i}{2}} \quad \text{for } 1 \leq i \leq r-1.$$

When the above procedure is applied, the first step yields

$$b_{ij} = \frac{1-x^{j-1}}{1-x^{m+i}} \begin{bmatrix} 2m+2 \\ m+j-i+1 \end{bmatrix} x^{\binom{j-i}{2}} \quad \text{for } 2 \leq i \leq r-1,$$

the second step yields

$$c_{ij} = \frac{(1-x^{j-1})(1-x^{j-2})}{(1-x^{m+i})(1-x^{m+i+1})} \begin{bmatrix} 2m+3 \\ m+j-i+2 \end{bmatrix} x^{\binom{j-i}{2}} \quad \text{for } 3 \leq i \leq r-1,$$

etc. In general if $a_{ij}^{(p)}$ is the matrix obtained after p steps of the procedure, we have

$$(10) \quad a_{ij}^{(p)} = \frac{\begin{bmatrix} j-1 \\ p \end{bmatrix}}{\begin{bmatrix} m+i \\ p \end{bmatrix}} \begin{bmatrix} 2m+p+1 \\ m+j-i+p \end{bmatrix} x^{\binom{j-i}{2}} \quad \text{for } p+1 \leq i \leq r-1.$$

The proof is by a straightforward induction on p, which we omit here. Since the ith row $(1 \leq i \leq r-1)$ remains constant after $i-1$ steps, the final determinant obtained from (8) is

$$(11) \quad \begin{vmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1,r-1} & A_{1,r} \\ 0 & A_{22} & A_{23} & \cdots & A_{2,r-1} & A_{2,r} \\ 0 & 0 & A_{33} & \cdots & A_{3,r-1} & A_{3,r} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_{r-1,r-1} & A_{r-1,r} \\ 0 & d_1 & d_2 & \cdots & d_{r-2} & d_{r-1} \end{vmatrix},$$

where

$$(12) \quad A_{ij} = \frac{\begin{bmatrix} j-1 \\ i-1 \end{bmatrix}}{\begin{bmatrix} m+i \\ i-1 \end{bmatrix}} \begin{bmatrix} 2m+i \\ m+j-1 \end{bmatrix} x^{\binom{j-i}{2}}.$$

(This is obtained by putting $p = i - 1$ in (10).) Similarly, the result of performing these operations on (9) is the determinant

$$
(13) \qquad s^2 \begin{vmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1,r-1} & A_{1,r} \\ 0 & A_{22} & A_{23} & \cdots & A_{2,r-1} & A_{2,r} \\ 0 & 0 & A_{33} & \cdots & A_{3,r-1} & A_{3,r} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_{r-1,r-1} & A_{r-1,r} \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{vmatrix}.
$$

The next task is to clear out the bottom rows of (11) and (13), except for their rightmost entries. To do this for (13), we multiply the $(2k + 1)$th row $(0 \leq k \leq (r - 4)/2)$ by

$$
(14) \qquad B_k = \frac{1}{\begin{bmatrix} 2m+1 \\ m \end{bmatrix}} \frac{(x; x^2)_k}{(x^{2m+3}; x^2)_k},
$$

and subtract the result from the bottom row. To clear out (11), we multiply its $(2k + 2)$th row $(0 \leq k \leq (r - 3)/2)$ by

$$
(15) \qquad C_k = \frac{(x^2; x^2)_k}{(x^{2m+4}; x^2)_k (1 + x^{m+1})},
$$

and subtract the result from the bottom row. To show that these operations do indeed clear out the bottom rows of (13) and (11), we must evidently prove that

$$
\sum_{k=0}^{(r-4)/2} B_k A_{2k,j} = 0 \qquad (1 \leq j \leq r - 1)
$$

and

$$
\sum_{k=0}^{(r-3)/2} C_k A_{2k+1,j} = 0 \qquad (1 \leq j \leq r - 1).
$$

In view of (12), (14) and (15), this is tantamount to showing that

$$
(16) \qquad \sum_{k=0}^{(r-4)/2} \frac{\begin{bmatrix} j-1 \\ 2k \end{bmatrix} \begin{bmatrix} 2m+2k+1 \\ m+j-1 \end{bmatrix}}{\begin{bmatrix} 2m+1 \\ m \end{bmatrix}} \frac{(x^2; x^2)_k}{(x^{2m+3}; x^2)_k} x^{\binom{j-1-2k}{2}} = 1
$$

$$
(1 \leq j \leq r - 1),
$$

and

$$\sum_{k=0}^{(r-3)/2} \begin{bmatrix} j-1 \\ 2k+1 \end{bmatrix} \begin{bmatrix} 2m+2k+2 \\ m+j-1 \end{bmatrix} \frac{(x^2;x^2)_k}{(x^{2m+4};x^2)_k} \frac{x^{\binom{j-2k-2}{2}}}{1+x^{m+1}} = d_{j-1} \qquad (1 \le j \le r-1). \tag{17}$$

To simplify the notation a little, we put $j-1=n$. Moreover we note that because of the inequality $j \le r-1$, the summations in (16) and (17) can be extended from $k=0$ to ∞ without affecting the left sides. Indeed outside the indicated k-ranges, we have $[{}^{j-1}_{2k}]=0$ in (16), and $[{}^{j-1}_{2k+1}]=0$ in (17). The restriction $j \le r-1$ then becomes irrelevant. Thus we wish to prove that

$$\sum_{k\ge 0} \frac{\begin{bmatrix} n \\ 2k \end{bmatrix} \begin{bmatrix} 2m+2k+1 \\ m+n \end{bmatrix}}{\begin{bmatrix} 2m+1 \\ n \end{bmatrix}} \frac{(x;x^2)_k}{(x^{2m+3};x^2)_k} x^{\binom{n-2k}{2}} = 1 \tag{18}$$

and

$$\sum_{k\ge 0} \begin{bmatrix} n \\ 2k+1 \end{bmatrix} \begin{bmatrix} 2m+2k+2 \\ m+n \end{bmatrix} \frac{(x^2;x^2)_k x^{\binom{n-2k-1}{2}}}{(x^{2m+4};x^2)_k(1+x^{m+1})} = d_n = \sum_{\nu=1}^{n} \begin{bmatrix} 2m+1 \\ m+\nu \end{bmatrix} x^{\binom{\nu}{2}} \tag{19}$$

for all $n \ge 0$. Professor Andrews has pointed out to me that (18) and (19) can be derived from Saalschütz's summation of basic hypergeometric series [**5**, p. 247]. To keep the presentation self-contained, however, we will give direct proofs. Let $F(m,n)$ and $G(m,n)$ denote the left sides of (18) and (19) respectively. When $n=0$ or 1, the only non-vanishing term of the series in (18) is the one with $k=0$. Hence

$$F(m,0) = \begin{bmatrix} 2m+1 \\ m \end{bmatrix} \Big/ \begin{bmatrix} 2m+1 \\ m \end{bmatrix} = 1$$

and

$$F(m,1) = \begin{bmatrix} 2m+1 \\ m+1 \end{bmatrix} \Big/ \begin{bmatrix} 2m+1 \\ m \end{bmatrix} = 1.$$

Thus to complete the proof of (18) it suffices to show that $F(m,n) = F(m,n+2)$ for all $n \ge 0$. It is easy to verify that

$$\begin{bmatrix} 2m+2k+1 \\ m+n \end{bmatrix} = x^{2n-2k+1} \begin{bmatrix} 2m+2k+1 \\ m+n+2 \end{bmatrix} + \frac{1-x^{2n-2k+1}}{1-x^{2m+2k+3}} \begin{bmatrix} 2m+2k+3 \\ m+n+2 \end{bmatrix}. \tag{20}$$

Since

$$\binom{n-2k}{2} + 2n - 2k + 1 = \binom{n+2-2k}{2} + 2k,$$

it follows from (20) that

$$\begin{bmatrix} 2m+1 \\ m \end{bmatrix} F(m,n) = \sum_{k\geq 0} \begin{bmatrix} n \\ 2k \end{bmatrix} \begin{bmatrix} 2m+2k+1 \\ m+n+2 \end{bmatrix} \frac{(x;x^2)_k}{(x^{2m+3};x^2)_k} x^{\binom{n+2-2k}{2}+2k}$$

$$+ \sum_{k\geq 0} \begin{bmatrix} n \\ 2k \end{bmatrix} \begin{bmatrix} 2m+2k+3 \\ m+n+2 \end{bmatrix} \frac{(x;x^2)_k (1-x^{2n-2k+1})}{(x^{2m+3};x^2)_{k+1}} x^{\binom{n-2k}{2}}.$$

In the second sum we replace k by $k-1$, thus obtaining

$$\begin{bmatrix} 2m+1 \\ m \end{bmatrix} F(m,n) = \sum_{k\geq 0} \begin{bmatrix} n \\ 2k \end{bmatrix} \begin{bmatrix} 2m+2k+1 \\ m+n+2 \end{bmatrix} \frac{(x;x^2)_k}{(x^{2m+3};x^2)_k} x^{\binom{n+2-2k}{2}}$$

$$+ \sum_{k\geq 1} \begin{bmatrix} n \\ 2k-2 \end{bmatrix} \begin{bmatrix} 2m+2k+1 \\ m+n+2 \end{bmatrix} \frac{(x;x^2)_{k-1} (1-x^{2n-2k+3})}{(x^{2m+3};x^2)_k} x^{\binom{n+2-2k}{2}}$$

$$= \sum_{k\geq 0} \left\{ \begin{bmatrix} n \\ 2k \end{bmatrix} x^{2k} + \begin{bmatrix} n \\ 2k-2 \end{bmatrix} \frac{1-x^{2n-2k+3}}{1-x^{2k-1}} \right\}$$

$$\cdot \begin{bmatrix} 2m+2k+1 \\ m+n+2 \end{bmatrix} \frac{(x;x^2)_k}{(x^{2m+3};x^2)_k} x^{\binom{n+2-2k}{2}}.$$

Since

$$\begin{bmatrix} n \\ 2k \end{bmatrix} x^{2k} + \begin{bmatrix} n \\ 2k-2 \end{bmatrix} \frac{1-x^{2n-2k+3}}{1-x^{2k}} = \begin{bmatrix} n+2 \\ 2k \end{bmatrix}, \tag{21}$$

we have

$$\begin{bmatrix} 2m+1 \\ m \end{bmatrix} F(m,n) = \sum_{k\geq 0} \begin{bmatrix} n+2 \\ 2k \end{bmatrix} \begin{bmatrix} 2m+2k+1 \\ m+n+2 \end{bmatrix} \frac{(x;x^2)_k}{(x^{2m+3};x^2)_k} x^{\binom{n+2-2k}{2}}$$

$$= \begin{bmatrix} 2m+1 \\ m \end{bmatrix} F(m, n+2),$$

completing the proof.

The treatment of (19) is similar. First of all, when $n = 0$ the terms of (19) all vanish, so $G(m,0) = 0$. When $n = 1$, only the term $k = 0$ of (19) is non-zero, so

$$(1+x^m)G(m,1) = \begin{bmatrix} 2m+2 \\ m+1 \end{bmatrix} = \frac{1-x^{2m+2}}{1-x^{m+1}} \begin{bmatrix} 2m+1 \\ m \end{bmatrix} = (1+x^{m+1})d_1.$$

Hence it suffices to show that

$$G(m,n+2)=G(m,n)+\begin{bmatrix}2m+1\\m+n+1\end{bmatrix}x^{\binom{n+1}{2}}+\begin{bmatrix}2m+1\\m+n+2\end{bmatrix}x^{\binom{n+2}{2}}$$

for all $n>0$. The relevant analogue of (20) is

$$\begin{bmatrix}2m+2k+2\\m+n\end{bmatrix}=x^{2n-2k}\begin{bmatrix}2m+2k+2\\m+n+2\end{bmatrix}+\frac{1-x^{2n-2k}}{1-x^{2m+2k+4}}\begin{bmatrix}2m+2k+4\\m+n+2\end{bmatrix}.$$

Using this, we split the series for $G(m,n)$ into a sum of two series, and replace k by $k-1$ in the second of these. This yields

$$G(m,n)=\sum_{k\geq 0}\left\{\begin{bmatrix}n\\2k+1\end{bmatrix}x^{2k+1}+\begin{bmatrix}n\\2k-1\end{bmatrix}\frac{1-x^{2n-2k+2}}{1-x^{2k}}\right\}\cdot\begin{bmatrix}2m+2k+2\\m+n+2\end{bmatrix}\frac{(x^2;x^2)_k x^{\binom{n+1-2k}{2}}}{(x^{2m+4};x^2)_k(1+x^{m+1})},$$

where the second term in the curly bracket is to be interpreted as 0 when $k=0$. In analogy with (21) we have

$$\begin{bmatrix}n\\2k+1\end{bmatrix}x^{2k+1}+\begin{bmatrix}n\\2k-1\end{bmatrix}\frac{1-x^{2n-2k+2}}{1-x^{2k}}=\begin{cases}\begin{bmatrix}n+2\\2k+1\end{bmatrix} & \text{if } k>0,\\ \begin{bmatrix}n\\1\end{bmatrix} & \text{if } k=0.\end{cases}$$

Therefore

$$(22)\quad G(m,n)=\sum_{k\geq 0}\begin{bmatrix}n+2\\2k+1\end{bmatrix}\begin{bmatrix}2m+2k+2\\m+n+2\end{bmatrix}\frac{(x^2;x^2)_k x^{\binom{n+1-2k}{2}}}{(x^{2m+3};x^2)_k(1+x^{m+1})}+\left(\begin{bmatrix}n\\1\end{bmatrix}x-\begin{bmatrix}n+2\\1\end{bmatrix}\right)\begin{bmatrix}2m+2\\m+n+2\end{bmatrix}\frac{x^{\binom{n+1}{2}}}{1+x^{m+1}}.$$

It is easily checked that

$$\left(\begin{bmatrix}n\\1\end{bmatrix}x-\begin{bmatrix}n+2\\1\end{bmatrix}\right)\begin{bmatrix}2m+2\\m+n+2\end{bmatrix}\frac{x^{\binom{n+1}{2}}}{1+x^{m+1}}=-\begin{bmatrix}2m+1\\m+n+1\end{bmatrix}x^{\binom{n+1}{2}}-\begin{bmatrix}2m+1\\m+n+2\end{bmatrix}x^{\binom{n+2}{2}}.$$

Hence from (22) we get

$$G(m,n) = G(m,n+2) - \begin{bmatrix} 2m+1 \\ m+n+1 \end{bmatrix} x^{\binom{m+1}{2}} - \begin{bmatrix} 2m+1 \\ m+n+2 \end{bmatrix} x^{\binom{n+2}{2}},$$

completing the proof of (19).

Identities (18) and (19) also enable us to determine the entries which appear in the lower right corners when the clearing process is applied to (11) and (13). In the case of (11), the process subtracts from d_{r-1} all the non-zero terms of the series on the left of (19) with $n = r - 1$, except for the term with $k = (r-2)/2$. In view of (19), the resulting entry in the lower right corner is the term with $n = r - 1$, $k = (r-2)/2$, viz.

$$q\begin{bmatrix} r-1 \\ r-1 \end{bmatrix}\begin{bmatrix} 2m+r \\ m+r-1 \end{bmatrix} \frac{(x^2;x^2)_{(r-2)/2} x^{\binom{0}{2}}}{(x^{2m+4};x^2)_{(r-2)/2}(1+x^{m+1})}$$

$$= \begin{bmatrix} 2m+r \\ m+r-1 \end{bmatrix} \frac{(x^2;x^2)_{(r-2)/2}}{(x^{2m+4};x^2)_{(r-2)/2}(1+x^{m+1})}.$$

Hence for even r we have

$$\det D_r = \prod_{i=1}^{r-1} A_{ii} \begin{bmatrix} 2m+r \\ m+r-1 \end{bmatrix} \frac{(x^2;x^2)_{(r-2)/2}}{(x^{2m+4};x^2)_{(r-2)/2}(1+x^{m+1})}$$

$$= \prod_{i=1}^{r-1} \frac{\begin{bmatrix} 2m+i \\ m+i-1 \end{bmatrix}}{\begin{bmatrix} m+i \\ i-1 \end{bmatrix}} \begin{bmatrix} 2m+r \\ m+r-1 \end{bmatrix} \frac{(x^2;x^2)_{(r-2)/2}}{(x^{2m+4};x^2)_{(r-2)/2}(1+x^{m+1})}$$

by (12). Denote this last expression by $f(r)$. For even r, $B_r(x \mid I_m)$ is the Pfaffian of D_r, and hence $B_r(x \mid I_m) = \sqrt{f(r)}$. Now

$$f(2) = \begin{bmatrix} 2m+1 \\ m \end{bmatrix}\begin{bmatrix} 2m+2 \\ m+1 \end{bmatrix} \frac{1-x^{m+1}}{1-x^{2m+2}} = \begin{bmatrix} 2m+1 \\ m \end{bmatrix}^2.$$

Hence $B_2(x \mid I_m) = \sqrt{f(2)} = [{}^{2m+1}_{\ \ m}]$. On the other hand, when $r = 2$ the right hand side of equation (1) telescopes to

$$\frac{(1-x^{m+2})(1-x^{m+3})\cdots(1-x^{2m+1})}{(1-x)(1-x^2)\cdots(1-x^m)} = \begin{bmatrix} 2m+1 \\ m \end{bmatrix}.$$

This proves the Bender-Knuth formula for $r = 2$. We proceed by induction to prove it for all even r. Clearly

$$\frac{f(r+2)}{f(r)} = \frac{\begin{bmatrix} 2m+r+1 \\ m+r \end{bmatrix}}{\begin{bmatrix} m+r \\ r-1 \end{bmatrix}} \frac{\begin{bmatrix} 2m+r+2 \\ m+r+1 \end{bmatrix}}{\begin{bmatrix} m+r+1 \\ r \end{bmatrix}} \frac{1-x^r}{1-x^{2m+r+2}}.$$

After some straightforward cancellation, the right side reduces to

$$\prod_{\nu=1}^{m} \frac{(1-x^{m+r+\nu+1})^2}{(1-x^{r+\nu})^2},$$

from which we conclude that

$$\frac{B_{r+2}(x \mid I_m)}{B_r(x \mid I_m)} = \prod_{\nu=1}^{m} \frac{1-x^{m+r+\nu+1}}{1-x^{r+\nu}}. \tag{23}$$

On the other hand, if the right side of (1) is denoted by $h(r)$, then

$$\frac{h(r+2)}{h(r)} = \prod_{i=1}^{m} \prod_{j=1}^{m} \frac{1-x^{r+i+j+1}}{1-x^{r+i+j-1}}.$$

Here we have essentially the same telescope as the one mentioned above. The surviving factors are

$$\frac{(1-x^{r+m+2})(1-x^{r+m+3}) \cdots (1-x^{r+2m+1})}{(1-x^{r+1})(1-x^{r+2}) \cdots (1-x^{r+m})},$$

which is the right side of (23). This completes the induction (through even values of r).

We can deal similarly with the case of odd r. When the clearing process is applied to (13), it subtracts from the 1 in the lower right corner all the non-zero terms of the series (18) with $n = r - 1$ except the term with $k = (r-1)/2$. Hence the resulting entry in the lower right corner is just this missing term, viz.

$$\frac{\begin{bmatrix} 2m+r \\ m+r-1 \end{bmatrix}}{\begin{bmatrix} 2m+1 \\ m \end{bmatrix}} \frac{(x; x^2)_{(r-1)/2}}{(x^{2m+3}; x^2)_{(r-1)/2}}.$$

Thus for odd r we have

$$\det D'_r = s^2 \prod_{i=1}^{r-1} \frac{\begin{bmatrix} 2m+i \\ m+i-1 \end{bmatrix}}{\begin{bmatrix} m+i \\ i-1 \end{bmatrix}} \frac{\begin{bmatrix} 2m+r \\ m+r-1 \end{bmatrix}}{\begin{bmatrix} 2m+r \\ m \end{bmatrix}} \frac{(x; x^2)_{(r-1)/2}}{(x^{2m+3}; x^2)_{(r-1)/2}}.$$

If we temporarily denote the right hand side of this equation by $g(r)$, we have

$$B_r(x \mid I_m) = \sqrt{g(r)} \quad \text{for odd } r.$$

The proof that $\sqrt{g(r)}$ is equal to the right side of (1) is completely analogous to the one given above for $\sqrt{f(r)}$, so can be omitted here.

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Received October 30, 1981

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