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MAXIMAL SUBALGEBRAS OF C^*-CROSSED PRODUCTS

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Let (A, G, α) be a C^*-dynamical system. Suppose G is discrete, archimedian-linearly ordered, and A is simple with unit. In this paper we prove that the subalgebra of analytic elements $A(G, \hat{\alpha}) \subset C^*(A, G, \alpha)$ is a maximal subalgebra of the crossed product $C^*(A, G, \alpha)$.

The same question is solved for a C^*-dynamical system associated with a von Neumann algebra with a homogeneous periodic state. Finally, if $G = \mathbf{Z}$ we prove the converse of this result.

1. Introduction. In [**12**] Wermer proved that the algebra of all continuous functions on the unit circle $|z| = 1$ which can be extended to the unit disk $|z| \leq 1$, so as to be analytic in the interior, is a maximal subalgebra of the Banach algebra $C(\mathbf{T})$ of all continuous, complex-valued functions on the unit circle.

In [**1**] Arens and Singer presented a generalisation of part of the theory of analytic functions in the unit disc, established by observing the role played in the classical theory by the group of integers, then replacing this group by a locally compact abelian group G possessing a suitably distinguished semigroup G_+.

Further, in [**6**] Hoffman and Singer extended to this context the maximality theorem of Wermer. In this paper we extend the same theorem to the context of C^*-crossed products. We mention that the analogous study for W^*-crossed products was successfully made in [**8**] for the case $G = \mathbf{Z}$. In §4 we solve the same question for a C^*-dynamical system associated with a von Neumann algebra with a homogeneous periodic state.

2. Preliminaries and notations.

2A. *Dynamical systems and spectra.* Let (A, G, α) be a C^*-dynamical system with G abelian, i.e. a C^*-algebra and an abelian locally compact group G of $*$-automorphisms of A with the property that for each $a \in A$, the function $g \mapsto \alpha_g(a)$ is continuous.

We define a representation $\alpha(\cdot)$ of $L^1(G)$ into the bounded operators on A by $\alpha(f)a = \int f(g)\alpha_g(a)\,dg$ $(a \in A)$, where $f \in L^1(G)$. For $f \in L^1(G)$ we put $Z(f) = \{p \in \hat{G} \mid \hat{f}(p) = 0\}$, where $\hat{G}$ is the dual of G and $\hat{f}$ is the Fourier transform of f.

Let Sp α be defined as $\bigcap\{Z(f) \mid f \in L^1(G), \alpha(f) = 0\}$. If $a \in A$ let $\mathrm{Sp}_\alpha(a) = \bigcap\{Z(f) \mid f \in L^1(G), \alpha(f)a = 0\}$.

We refer the reader to [3] for the elementary properties of spectra and spectral subspaces.

Throughout this paper we suppose G discrete and, hence, $\hat{G}$ compact. Suppose there exists a subsemigroup $G_+ \subset G$ with the following properties:

$$G_+ \cup (-G_+) = G,$$
$$G_+ \cap (-G_+) = (0).$$

Let $(B, \hat{G}, \beta)$ be a C^*-dynamical system. Denote $\mathcal{Q}(G, \beta) = \{b \in B \mid \mathrm{Sp}_\beta(b) \subset G_+\}$. By [3], $\mathcal{Q}(G, \beta)$ is a norm-closed non-selfadjoint subalgebra of B.

Now, for each $g \in G$, we consider the weak integration

$$\varepsilon_g(b) = \int_{\hat{G}} \langle g, p\rangle \beta_p(b)\, dp, \qquad b \in B,$$

where dp is the normalised Haar measure on b. Then ε_g is a bounded linear mapping from B onto $B_g = \{b \in B \mid \beta_p(b) = \overline{\langle g, p\rangle} b, p \in \hat{G}\}$. We also have the following properties:

$$\varepsilon_{g_1} \circ \varepsilon_{g_2} = \delta_{g_1 g_2} \varepsilon_{g_1}, \qquad g_1, g_2 \in G,$$

(here $\delta_{g_1 g_2}$ is the Kronecker symbol)

$$\varepsilon_g(a_1 b a_2) = a_1 \varepsilon_g(b) a_2, \qquad a_1, a_2 \in B_0, b \in B.$$

Clearly $B_0 = \mathcal{Q}(G, \beta) \cap \mathcal{Q}(G, \beta)^*$ is the algebra of all fixed points with respect to β, and ε_0 is a faithful, β-invariant projection of norm one from B onto B_0.

The following lemma is a slight generalisation of [9, Lemma 1].

2.1. LEMMA (i) *For any g_1, g_2, $B_{g_1} \cdot B_{g_2} = B_{g_1+g_2}$ and $B_{g_1}^* = B_{-g_1}$.*
(ii) *Let $b_1, b_2 \in B$. If $\varepsilon_g(b_1) = \varepsilon_g(b_2)$ for all $g \in G$ then $b_1 = b_2$.*
(iii) *For $b \in B$, we have $\mathrm{Sp}_\beta(b) = \{g \in G \mid \varepsilon_g(b) \neq 0\}$.*
(iv) *For $g \in G$, $B_g = \{b \in B \mid \mathrm{Sp}_\beta(b) = \{g\}\}$.*

The following lemma is well known and easy to prove:

2.2. LEMMA. *B is linearly spanned by $\bigcup_{g \in G} B_g$ in the norm topology.*

2.3. REMARK. If $b \in B$ is such that $\varepsilon_{g_0}(b) = 0$ for some $g_0 \in G$, then there exists a sequence $b_n = \Sigma_g b^n(g)$, $b^n(g) \in B_g$, which converges to b (in norm) such that $b^n(g_0) = 0$ for all n. Indeed, by Lemma 2.2 there exists a sequence $c_n = \Sigma_g c^n(g)$, $c^n(g) \in B_g$, which converges to b. Since $\varepsilon_g(c_n) = c^n(g)$ (by Lemma 1.1(iii)) and ε_g is bounded for all $g \in G$, we have that $\varepsilon_{g_0}(c_n)$ converges to $\varepsilon_{g_0}(b) = 0$. Therefore $b_n = c_n - \varepsilon_{g_0}(c_n)$ satisfies the desired property.

2B. *C^*-crossed products.* Let (A, G, α) be a C^*-dynamical system with G discrete, abelian. Assume $A \subset B(H)$ for some Hilbert space H. Let $\mathcal{P}(G, A)$ denote the set of "trigonometric polynomials":

$$\mathcal{P}(G, A) = \{f \colon G \mapsto A \,|\, f(g) = 0 \text{ for all but finitely many } g \in G\}.$$

Define a faithful representation of $\mathcal{P}(G, A)$ on $l^2(G, H)$ by

$$(1) \quad (y\xi)(g) = \sum_{s \in G} \alpha_{-g}(y(s))\xi(g - s), \qquad y \in \mathcal{P}(G, A), \xi \in l^2(G, H).$$

We identify $\mathcal{P}(G, A)$ with its image in $B(l^2(G, H))$ and denote by $C^*(G, \alpha, A)$ the C^*-algebra generated by $\mathcal{P}(G, A)$. It can be shown that $C^*(G, \alpha, A)$ does not depend no the representation of A on H. We say that $C^*(G, \alpha, A)$ is the crossed product of G with A.

The following element of $\mathcal{P}(G, A)$:

$$y(g) = 1, \quad y(s) = 0 \qquad \text{for all } s \neq g$$

will be denoted by λ_G.

Also, the element $y \in \mathcal{P}(G, A)$:

$$\begin{aligned} y(0) &= a \quad \text{for some } a \in A, \\ y(g) &= 0 \quad \text{for all } g \neq 0, \end{aligned}$$

will be denoted by a.

We denote by $(C^*(G, \alpha, A), \hat{G}, \hat{\alpha})$ the dual system of (A, G, α) **[10]**.

If G is ordered by a subsemigroup G_+ as in 2A, then we may apply the results of 2A to $(C^*(G, \alpha, A), \hat{G}, \hat{\alpha})$.

3. **The main results.** We say that a C^*-algebra A is simple if it has no nontrivial closed two-sided ideals. We say that A is G-simple if it has no nontrivial, closed, G-invariant two-sided ideals. If G is ordered by the subsemigroup G_+, we say that G is archimedean ordered if for any $g_1, g_2 \in G_+ \backslash \{0\}$ there exists $n \in \mathbf{N}$ such that $ng_1 > g_2$. Then it is well known that G is isomorphic with a subgroup of $\mathbf{R}$.

Let B be a C^*-algebra with unit. We say that a closed subalgebra $\mathcal{A} \subset B$ with the unit of B is a Dirichlet subalgebra if $\mathcal{A} + \mathcal{A}^*$ is norm dense in B. Let ε be a faithful projection of norm one in B. A Dirichlet subalgebra $\mathcal{A} \subset B$ is said to be C^*-subdiagonal if the pair $(\mathcal{A}, \varepsilon)$ satisfies the following conditions:

(i) ε is multiplicative on $\mathcal{A}$.

(ii) $\varepsilon(\mathcal{A}) = \mathcal{A} \cap \mathcal{A}^*$.

We call the C^*-subalgebra $\mathcal{A} \cap \mathcal{A}^*$ the diagonal of $\mathcal{A}$. Then using [7, Theorem 3.1], the proof of [7, Theorem 2.4] can be adapted to prove:

3.1. THEOREM. *Let A be a C^*-algebra with unit, and G a discrete, commutative group of automorphisms of A.*

Suppose G is archimedean-linearly ordered. The $\mathcal{A}(G, \hat{\alpha})$ is a maximal C^*-subdiagonal subalgebra of $C^*(G, \alpha, A)$ with respect to the projection ε_0 (§1).

The following theorem gives a sufficient condition for $\mathcal{A}(G, \hat{\alpha})$ to be a maximal subalgebra of $C^*(G, \alpha, A)$.

3.2. THEOREM. *Let A be a simple C^*-algebra with unit and G an automorphism group of A. Suppose G is discrete and archimedean-linearly ordered. Then $\mathcal{A}(G, \hat{\alpha})$ is a maximal subalgebra of $C^*(G, \alpha, A)$.*

Proof. The proof is inspired from Cohen's proof in the classical case ([5]). Let $\mathcal{B} \subset C^*(G, \alpha, A)$ be a subalgebra which contains $\mathcal{A}(G, \hat{\alpha})$. Then there exist $b \in \mathcal{B}$ and $t_0 \in G_+ \setminus \{0\}$ such that $\varepsilon_{-t_0}(b) \neq 0$. It is easy to see that the set $J = \{a \in A \mid (\exists) b \in \mathcal{B}, \lambda_{t_0}\varepsilon_{-t_0}(b) = a\}$ is a non-zero two-sided ideal in A. Since A is simple, it follows that $J = A$. Therefore, there exists $b_0 \in \mathcal{B}$ such that $\lambda_{t_0}\varepsilon_{-t_0}(b_0) = 1$.

By definition of the crossed product and Remark 2.3, there exist two "trigonometric polynomials" $p, q \in \mathcal{A}(G, \hat{\alpha})$ and $h \in C^*(G, \alpha, A)$ such that

$$\lambda_{t_0} b_0 = 1 + \lambda_{t_1} p + \lambda_{-t_2} q^* + h, \qquad \|h\| < \tfrac{1}{2},$$

for some $t_1, t_2 \in G_+ \setminus \{0\}$.

Let $s_0 \in G_+$ be $s_0 = \min\{t_0, t_1, t_2\}$. We also set $b_1 = \lambda_{t_0 - s_0} b_0 \in \mathcal{B}$, $p_1 = \lambda_{t_1 - s_0} p \in \mathcal{A}(G, \hat{\alpha})$, and $q_1 = q\lambda_{t_2 - s_0} \in \mathcal{A}(G, \hat{\alpha})$. Then

$$\lambda_{s_0} b_1 = 1 + \lambda_{s_0} p_1 + \lambda_{-s_0} q_1^* + h, \qquad \|h\| < \tfrac{1}{2}.$$

Let $M = \|q_1\lambda_{s_0} - \lambda_{-s_0}q_1^*\|$. Since $q_1\lambda_{s_0} - \lambda_{-s_0}q_1^* = i \cdot h$, where h is selfadjoint, we have, for every $\delta > 0$,

$$\|1 + \delta(q_1\lambda_{s_0} - \lambda_{-s_0}q_1^*)\| \leq 1 + \delta^2 M^2. \tag{1}$$

Further, we have

$$\begin{aligned} \delta \cdot \lambda_{-s_0}q_1^* &= \delta(\lambda_{s_0}b_1 - 1 - \lambda_{s_0}p_1) - \delta h \\ &= \lambda_{s_0}(\delta b_1 - p_1) - \delta h - \delta \\ &= \lambda_{s_0}g - \delta h - \delta, \end{aligned} \tag{2}$$

where $g = \delta b_1 - p_1 \in \mathfrak{B}$.

Let $q_1' \in \mathfrak{A}(G, \hat{\alpha})$ be such that $q_1\lambda_{s_0} = \lambda_{s_0}q_1'$. From (1), (2) and the fact that $\|h\| < \frac{1}{2}$, it follows that

$$\|1 + \delta + \lambda_{s_0}(-g + \delta q_1')\| \leq 1 + \delta^2 \cdot M^2 + \delta/2. \tag{3}$$

If $\delta < 1/2M^2$, then from (3) we obtain

$$\|1 + \delta + \lambda_{s_0}(-g + \delta q_1')\| < 1 + \delta. \tag{4}$$

Since, obviously, $\lambda_{s_0}(-g + \delta q_1') \in \mathfrak{B}$, from (4) it follows that this element has an inverse k in $\mathfrak{B}$. So $\lambda_{s_0}(g + \delta q_1')k = 1$. From this it follows that $\lambda_{-s_0} = (g + \delta q_1')k \in \mathfrak{B}$.

Let $s \in G_+$ be arbitrary. Since G is archimedean ordered, there exists $n \in \mathbf{N}$ such that $ns_0 > s$. Then

$$\lambda_{-s} = \lambda_{ns_0 - s} \cdot \lambda_{-ns_0} \in \mathfrak{B}.$$

It follows that $\mathfrak{B} = C^*(G, \alpha, A)$.

In what follows we discuss some partial converses of the preceding theorem.

3.3. Proposition. *Let A be a unital C^*-algebra, and α a $*$-automorphism of A. If $\mathfrak{A}(\mathbf{Z}, \hat{\alpha})$ is a maximal subalgebra of $C^*(\mathbf{Z}, \alpha, A)$, then A is simple.*

Proof. Suppose A is not simple and let $J \subset A$ be a non-trivial two-sided ideal. We show that $\mathfrak{A}(\mathbf{Z}, \hat{\alpha})$ is not maximal by producing a subspace $\mathfrak{M} \subset C^*(\mathbf{Z}, \alpha, A)$ with the following properties:

(i) $b\mathfrak{M} \subset \mathfrak{M}$ for every $b \in \mathfrak{A}(\mathbf{Z}, \hat{\alpha})$.

(ii) There exists $g \in C^*(\mathbf{Z}, \alpha, A) \setminus \mathfrak{A}(\mathbf{Z}, \hat{\alpha})$ such that $g\mathfrak{M} \subset \mathfrak{M}$.

(iii) $\mathfrak{M}$ is not a left ideal of $C^*(\mathbf{Z}, \alpha, A)$.

Let $\mathfrak{M}$ be the closure of the set of polynomials b: $\mathbf{Z} \mapsto A$ with the property that $b(n) \in \alpha^n(J)$ for all $n < 0$.

Let $p \in \mathbf{N}$ and $f \in \mathfrak{M}$ a polynomial. Then $(\lambda_p \cdot f)(n) = \alpha^p f(n-p)$. Hence, if $n < 0$ we have $(\lambda_p f)(n) \in \alpha^n(J)$. Also, if $a \in A$, we have $(a \cdot f)(n) = a \cdot f(n) \in \alpha^n(J)$. Hence $b\mathfrak{M} \subset \mathfrak{M}$ for all $b \in \mathfrak{A}(\mathbf{Z}, \hat{\alpha})$ and (i) is proved.

To prove (ii) let $x \in \alpha^{-1}(J)$. Then the element $x \cdot \lambda_{-1}$ satisfies (ii). Obviously $\lambda_{-1}\mathfrak{M} \not\subset \mathfrak{M}$, whence (iii).

Therefore $\mathfrak{A}(\mathbf{Z}, \hat{\alpha})$ is not maximal. It follows that A is simple.

3.4. PROPOSITION. *Let A be a unital C^*-algebra and G a discrete, commutative, linearly ordered group of $*$-automorphisms of A. If $\mathfrak{A}(G, \hat{\alpha})$ is a maximal subalgebra of $C^*(G, \alpha, A)$, then*:

(i) *The order on G is archimedean.*

(ii) *A is G-simple.*

Proof. Suppose the order on G is not archimedean. Then there exist $t_1, t_2 \in G_+ \setminus \{0\}$ such that $nt_1 < t_2$ for every $n \in \mathbf{N}$. Then the algebra $\mathfrak{B}$ generated by $\mathfrak{A}(G, \hat{\alpha})$ and λ_{-t_1} satisfies $\mathfrak{A}(G, \hat{\alpha}) \underset{\neq}{\subset} \mathfrak{B} \underset{\neq}{\subset} C^*(G, \alpha, A)$. Hence $\mathfrak{A}(G, \hat{\alpha})$ is not maximal, and (i) is proved.

Now we show that A is G-simple. Suppose A is not G-simple. Then there exists a non-trivial G-invariant two-sided ideal $J \subset A$. Set $\mathfrak{B} = \{b \in C^*(G, \alpha, A) \mid \varepsilon_t(b) \in J, \ t < 0\}$. Then $\mathfrak{B}$ is an algebra and $\mathfrak{A}(G, \hat{\alpha}) \underset{\neq}{\subset} \mathfrak{B} \underset{\neq}{\subset} C^*(G, \alpha, A)$. Therefore A is G-simple.

3.5. PROPOSITION. *Let A be a unital C^*-algebra and G a discrete, commutative, linearly ordered group of $*$-automorphisms of A. Suppose*:

(i) *A is primitive and postliminar.*

(ii) *$\mathfrak{A}(G, \hat{\alpha})$ is a maximal subalgebra of $C^*(G, \alpha, A)$.*

Then we have:

(iii) *G is archimedean ordered.*

(iv) *A is a finite-dimensional factor.*

Proof. (iii) follows from Proposition 3.4. Let us prove (iv). Let $\hat{A}$ be the space of irreducible representations of $\hat{A}$ and Prim(A) the space of primitive ideals of A with the Jacobson topology. Then by [**4**, Théorème 4.3.7] the mapping $\pi \mapsto \ker \pi$ is a bijection between $\hat{A}$ and Prim(A). By [**4**, Théorème 4.4.5] there exists a maximal open set $U \subset$ Prim(A) which is separated. Since A is primitive, it follows that $(0) \in$ Prim(A). Obviously,

(0) is dense in Prim(A). Therefore, $(0) \in U$. Since every open set $V \subset$ Prim(A) contains (0), it follows that $(0) = U$. Therefore $\mathsf{C}(0) = \{J \in \mathrm{Prim}(A) \mid J \neq (0)\}$ is closed in Prim(A). Thus $\bigcap \mathsf{C}(0) = J_0 \neq 0$. Now it is easy to see that J_0 is G-invariant. Since by Proposition 3.4, A is G-simple, it follows that $J_0 = (0)$. This contradiction shows that A is simple. Since A is unital, primitive and postliminar, it follows that A is a finite-dimensional factor.

4. Subalgebras of a von Neumann algebra with a homogeneous periodic state. Let M be a von Neumann algebra. Suppose M has a homogeneous periodic state φ in the sense that $G(\varphi) = \{\sigma \in \mathrm{Aut}(M) \mid \varphi \circ \sigma = \varphi\}$ acts ergodically on M and the modular automorphism group σ_t^φ of M associated with φ is a periodic flow. A penetrating study of such algebras was made by Takesaki **[11]**. Let $T > 0$ be the period of σ_t^φ. Put $\rho = e^{-2\pi/T}$, $0 < \rho < 1$. Set $M_n = \{x \in M \mid \sigma_t^\varphi(x) = \rho^{\mathrm{int}} x\}$, $n \in \mathbf{Z}$. For each $n \in \mathbf{Z}$, we consider the integration

$$\varepsilon_n(x) = \frac{1}{T}\int_0^T \rho^{-inT}\sigma_t^\varphi(x)\,dt, \qquad x \in M.$$

Then

$$\begin{aligned}
\varepsilon_n(M) &= M_n, && n \in \mathbf{Z},\\
\varepsilon_n \circ \varepsilon_m &= \delta_{nm}\varepsilon_n, && m, n \in \mathbf{Z},\\
\varepsilon_n(axb) &= a\varepsilon_n(x)b, && a, b \in M_0,\ x \in M.\\
M_n M_m &= M_{n+m}, && n, m \in \mathbf{Z}.\\
M_n^* &= M_{-n}, && n \in \mathbf{Z}.
\end{aligned}$$

We collect some results from **[11]** in the following

4.1. THEOREM. (i) *The subspace M_1 of M contains an isometry u such that for $n \geq 1$, $M_n = M_0 u^n$ and $M_{-n} = u^{*n} M_0$.*

(ii) *In the pre-Hilbert space structure induced by the state φ, M is decomposed into an orthogonal direct sum as follows*:

$$M = \cdots \oplus u^{*n}M_0 \oplus \cdots \oplus u^*M_0 \oplus M_0 \oplus M_0 u \oplus \cdots \oplus M_0 u^n \oplus \cdots .$$

(iii) M_0 *is of type* II_1.
(iv) M *is of type* III.

Let B denote the C^*-subalgebra of M generated by M_0 and u. Obviously B is σ_t^φ-invariant, $t \in \mathbf{R}$.

Moreover since the mapping $t \mapsto \sigma_t^\varphi(x)$ is norm-continuous for every $x \in M_n$, $n \in \mathbf{Z}$, it follows that it is norm-continuous for every $x \in B$. Therefore, we can consider the C^*-dynamical system $(B, \sigma_t^\varphi, \mathbf{R})$. By Lemma 2.1 and Theorem 4.1(i) it follows that $\mathrm{Sp}(\sigma^\varphi)$ is isomorphic with $\mathbf{Z}$.

As in §2 let $\mathcal{A}(\mathbf{Z}, \sigma^\varphi)$ denote the algebra of all elements of B with non-negative spectrum.

4.2. PROPOSITION. *$\mathcal{A}(\mathbf{Z}, \sigma^\varphi)$ is a maximal subalgebra of B if and only if M_0 is a factor.*

Proof. Suppose M_0 is a factor. We follow the proof of Theorem 3.2. Let $\mathfrak{B} \subset B$ be such that $\mathcal{A}(\mathbf{Z}, \sigma^\varphi) \subsetneq \mathfrak{B}$. Then there exist $b_0 \in \mathfrak{B}$ and $n \in \mathbf{N}$ such that $\varepsilon_{-n}(b_0) \neq 0$. We may assume $n = 1$. Let $K = \{x \in M_0 \mid (\exists) b \in \mathfrak{B}, \varepsilon_{-1}(b) = u^*x\}$. K is a linear subspace of M_0. If we put $e = uu^*$, it can be easily shown that the mapping $\mathrm{Ad}(u)(x) = uxu^*$ is an isomorphism of M_0 onto eM_0e.

We claim that eKe is a two-sided ideal of eM_0e. Indeed, if $b \in \mathfrak{B}$ is such that $\varepsilon_{-1}(b) = x \in eKe$ and $a \in eM_0e$, then

$$(u^*au)u^*x = u^*a \in x = u^* \in a \in x = u^*ax.$$

Therefore, since $u^*au \in M_0 \subset \mathcal{A}(\mathbf{Z}, \sigma^\varphi) \subset \mathfrak{B}$, we have

$$\varepsilon_{-1}(u^*au \cdot b) = u^*ax, \quad \text{so} \quad ax \in eKe.$$

Similarly

$$\varepsilon_{-1}(ba) = u^*xa, \quad \text{so} \quad xa \in eKe.$$

Since M_0 is a finite factor it follows that eM_0e is a finite factor. Therefore eM_0e is simple. Hence $eKe = eM_0e$. Then there exists $b_0 \in \mathfrak{B}$ such that $\varepsilon_{-1}(b_0) = u^*e = u^*$.

By definition of B, there exist two "polynomials" $p, q \in \mathcal{A}(\mathbf{Z}, \sigma^\varphi)$ and $h \in B$ such that

$$b_0u = 1 + pu + u^*q^* + h, \qquad \|h\| < 1/2.$$

The rest of the proof of the "if" part is the same as that of Theorem 3.2. Now suppose $\mathcal{A}(\mathbf{Z}, \sigma^\varphi)$ is maximal in B.

If M_0 is not a factor, let $\tilde{\theta}$ be the automorphisms of the center Z_0 of M_0 defined as follows:

$$uzu^* = \tilde{\theta}(z)e \qquad (\text{see } [\mathbf{11}, \text{Lemma } 1.20]).$$

There are the following possibilities:

I. $\tilde{\theta}$ is not ergodic.

In this case there exists a projection $q \in Z_0$ such that $\tilde{\theta}(q) = q$. Then q belongs to the center Z of M. Therefore, the algebra $\mathfrak{B}$ generated by $\mathfrak{A}(\mathbf{Z}, \sigma^\varphi)$ and qB is such that $\mathfrak{A}(\mathbf{Z}, \sigma^\varphi) \underset{\neq}{\subset} \mathfrak{B} \underset{\neq}{\subset} B$, a contradiction. Hence M_0 is a factor.

II. $\tilde{\theta}$ is ergodic.

In this case there exists $q \in Z_0$ such that $q\tilde{\theta}(q) = 0$. Then it can easily be verified that the algebra $\mathfrak{B}$ generated by $\mathfrak{A}(\mathbf{Z}, \sigma^\varphi)$ and the set $\{U^*\tilde{\theta}(q)y \,|\, y \in M_0\}$ is such that $\mathfrak{A}(\mathbf{Z}, \sigma^\varphi) \underset{\neq}{\subset} \mathfrak{B} \underset{\neq}{\subset} B$. This contradiction shows that M_0 is a factor.

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