PACIFIC JOURNAL OF MATHEMATICS
Vol. 111, No. 1, 1984

CAUCHY SPACES WITH REGULAR COMPLETIONS

D. C. KENT AND G. D. RICHARDSON

A T_3 Cauchy space which has a regular completion is shown to have a T_3 completion, but an example shows that such Cauchy spaces need not have strict T_3 completions. Various conditions are found for the existence of T_3 completions and strict T_3 completions; for instance, every Cauchy-separated, locally compact, T_3 Cauchy space has a T_3 completion. Convergence spaces and topological spaces which have a coarsest compatible Cauchy structure with a strict T_3 completion are characterized, as are those spaces for which every compatible T_3 Cauchy structure has a T_3 completion.

Introduction. Cauchy spaces were first defined in their present form by H. Keller [3] in 1968. Most subsequent work in this area has dealt with Cauchy space completions; the study of regular completions was initiated by J. Ramaley and O. Wyler [9] in 1970. Cauchy spaces are finding applications in various areas; they were applied in the study of C^*-algebras by K. McKennon [8], and recently some very nice results on Cauchy completions of lattice ordered groups were obtained by R. Ball [1].

At the present time, the two most important unsolved problems in the study of Cauchy spaces are:

(1) Find a completion functor for Cauchy groups (or prove that none exists).

(2) Find internal (and usable) characterizations of Cauchy spaces which have regular completions.

This paper attacks the second problem by expanding on ideas introduced in [5] and [7].

The first section is concerned with "C_3 Cauchy spaces," which is the name we give to Cauchy spaces that have T_3 completions. An early result is that a T_3 Cauchy space which has a regular (non-T_2) completion also has a T_3 completion. This result is surprising in view of the very different behavior of regular and T_3 compactifications studied in [6]. We also obtain useful criteria for locally compact Cauchy spaces to be C_3.

In the second section, we give an example of a C_3 Cauchy space which has no strict T_3 completion. Thus Problem (2) splits into two problems, the second being to characterize those Cauchy spaces which have strict T_3 completions (we call these SC_3 Cauchy spaces). The solution which we give to the latter problem (see Proposition 2.3) is useful, but not entirely satisfying.

In §3 we shift our attention to T_3 convergence spaces and pose two questions:

(1) When does a T_3 convergence space have a coarsest compatible C_3 (or SC_3) Cauchy structure?

(2) When are all compatible T_3 Cauchy structures C_3 (or SC_3) Cauchy structures?

In answering these questions we introduce two new convergence space notions which we call "*r*-boundedness" and "*s*-boundedness." The former is closely related to local compactness, the latter bears some resemblance to countable compactness. The *r*-bounded convergence spaces provide the answer to Question (1); spaces which have both properties (we call these *rs*-bounded spaces) are the answer to Question (2).

1. C_3 Cauchy spaces. For basic definitions and terminology pertaining to Cauchy spaces and convergence spaces, the reader is asked to refer to **[5]**. A few changes in the notation of **[5]** will be made; most notably, we will use "cl_q" instead of "Γ_q" for the closure operator of a convergence space (X, q), and we shall denote the set of Cauchy equivalence classes of a Cauchy space $(X, \mathcal{C})$ by X^*. The set of all filters on a set X will be denoted by $F(X)$.

A convergence space (X, q) is said to be T_2 if each filter converges to at most one point; (X, q) is *regular* if $\mathrm{cl}_q \mathcal{F} \to x$ whenever $\mathcal{F} \to x$. A regular T_2 convergence space is said to be T_3. A regular convergence space is defined to be *symmetric* if $\mathcal{F} \to x$ whenever $\mathcal{F} \to y$ and $\dot{y} \to x$; T_3 spaces are obviously symmetric. A convergence space is *locally compact* if every convergent filter contains a compact set.

A Cauchy space $(X, \mathcal{C})$ is defined to be T_2 or *locally compact* if the induced convergence structure $q_{\mathcal{C}}$ has the same property. On the other hand, a Cauchy space $(X, \mathcal{C})$ is *regular* if $\mathrm{cl}_{q_{\mathcal{C}}} \mathcal{F} \in \mathcal{C}$ whenever $\mathcal{F} \in \mathcal{C}$; if $(X, \mathcal{C})$ is regular then so is $(X, q_{\mathcal{C}})$, but the converse is false. Like a convergence space, a Cauchy space is said to be T_3 if it is both regular and T_2. This paper is devoted to the study of Cauchy spaces which have T_3 completions; we shall call these *C_3 Cauchy spaces.*

PROPOSITION 1.1. *If $(X, \mathcal{C})$ is a regular Cauchy space, then the induced convergence space $(X, q_{\mathcal{C}})$ is symmetric.*

Let $(X, \mathcal{C})$ be a T_2 Cauchy space. If $\mathcal{F} \in \mathcal{C}$ let $[\mathcal{F}] = \{\mathcal{G} \in \mathcal{C}\colon \mathcal{F} \cap \mathcal{G} \in \mathcal{C}\}$ denote the Cauchy equivalence class containing $\mathcal{F}$. Let $X^* = \{[\mathcal{F}]\colon \mathcal{F} \in \mathcal{C}\}$, and let $j\colon X \to X^*$ be the natural injection $j(x) = [\dot{x}]$, all $x \in X$.

A completion of $(X, \mathcal{C})$ which has X^* as its underlying set and j as its embedding map such that $j\mathcal{F} \to [\mathcal{F}]$ for each $\mathcal{F} \in \mathcal{C}$ is said to be in *standard form*. Reed **[10]** has shown that every T_2 completion of $(X, \mathcal{C})$ is equivalent to one in standard form.

A complete Cauchy structure $\mathcal{C}^*$ on X^* is given by: $\mathcal{C}^* = \{\mathcal{A} \in F(X^*)$: $\mathcal{A} \geq j(\mathcal{F}) \cap [\dot{\mathcal{F}}]$, for some $\mathcal{F} \in \mathcal{C}\}$. If $(X, \mathcal{C})$ is T_2, then $(X^*, \mathcal{C}^*)$ is also T_2, and $((X^*, \mathcal{C}^*), j)$ is called *Wyler's completion* of $(X, \mathcal{C})$. It is easy to see that if $((X^*, \mathcal{D}), j)$ is any other T_2 completion of $(X, \mathcal{C})$ in standard form, then $\mathcal{C}^* \subseteq \mathcal{D}$; in other words, Wyler's completion is (up to equivalence) the finest completion of any T_2 Cauchy space.

For any Cauchy space $(X, \mathcal{C})$ let $\mathcal{C}_r$ be the finest regular Cauchy structure coarser than $\mathcal{C}$. It is easy to verify that $\mathcal{C}_r$ is complete whenever $\mathcal{C}$ is complete. If $q = q_{\mathcal{C}}$ is the convergence structure induced by $\mathcal{C}$, then it follows easily from Proposition 1.1 that the convergence structure on X induced by $\mathcal{C}_r$ is σq, the finest symmetric convergence structure coarser than q. In what follows we shall denote by q^* (respectively, σq^*) the convergence structure on X^* induced by $\mathcal{C}^*$ (respectively, $\mathcal{C}_r^*$).

PROPOSITION 1.2. *If a T_2 Cauchy space $(X, \mathcal{C})$ has a regular completion, then $((X^*, \mathcal{C}_r^*), j)$ is a regular completion of $(X, \mathcal{C})$. Furthermore, in this case, $((X^*, \mathcal{C}_r^*), j)$ is a T_3 completion of $(X, \mathcal{C})$.*

Proof. Let $((X^*, \mathcal{D}), j)$ be a regular completion of $(X, \mathcal{C})$ in standard form. Then $\mathcal{D} \leq \mathcal{C}_r^* \leq \mathcal{C}^*$, and it follows immediately that j: $(X, \mathcal{C}) \to (X^*, \mathcal{C}_r^*)$ is a Cauchy embedding, which proves the first assertion. To prove the second, note that $[\dot{\mathcal{F}}] \cap [\dot{\mathcal{G}}] \in \mathcal{C}_r^*$ implies $(\mathrm{cl}_{\sigma q^*} j\mathcal{F}) \vee (\mathrm{cl}_{\sigma q^*} j\mathcal{G}) \neq \varnothing$, so

$$(\mathrm{cl}_{\sigma q^*} j\mathcal{F}) \cap (\mathrm{cl}_{\sigma q^*} j\mathcal{G}) = \mathrm{cl}_{\sigma q^*} j(\mathcal{F} \cap \mathcal{G}) \in \mathcal{C}_r^*.$$

Hence, $\mathcal{F} \cap \mathcal{G} \in \mathcal{C}$, and $[\mathcal{F}] = [\mathcal{G}]$. It follows that $(X^*, \mathcal{C}_r^*)$ is T_2 and, hence, T_3. □

COROLLARY 1.3. *The following statements about a T_3 Cauchy space are equivalent.*

(1) $(X, \mathcal{C})$ *is* C_3.

(2) $(X, \mathcal{C})$ *has a regular completion.*

(3) $((X^*, \mathcal{C}_r^*), j)$ *is a T_3 completion of* $(X, \mathcal{C})$.

When one compares regular versus T_3 compactifications of T_3 convergence spaces, the results differ significantly from those obtained for

Cauchy completions in Corollary 1.3. In a recent paper [**6**], we showed that the T_3 spaces having regular compactifications are precisely the ω-regular convergence spaces, whereas those having T_3 compactifications constitute the proper subclass of completely regular spaces. The apparent discrepancy is due to the fact that regular compactifications, unlike regular Cauchy completions, need not be symmetric. Convergence spaces which are ω-regular, but not completely regular, have non-symmetric regular compactifications which cannot be constructed via Cauchy completions.

LEMMA 1.4. *If* $(X, \mathcal{C})$ *is locally compact and* T_3, $(X^*, \mathcal{C}_r^*)$ *is* T_2, *and* $A \subseteq X^*$, *then* $j^{-1}(\text{cl}_{\sigma q^*} A) = \text{cl}_q\, j^{-1}(A)$.

Proof. Let $q = q_{\mathcal{C}}$ be the convergence structure induced by $\mathcal{C}$. The continuity of j implies $\text{cl}_q\, j^{-1}(A) \subseteq j^{-1}(\text{cl}_{\sigma q^*} A)$. If $x \in j^{-1}(\text{cl}_{\sigma q^*} A)$, then there is a filter $\mathcal{Q} \to [\dot{x}]$ in $(X^*, \sigma q^*)$ such that $A \in \mathcal{Q}$. It follows that $\mathcal{Q} \geq \text{cl}^n_{\sigma q^*}(j(\mathcal{F}) \cap [\dot{\mathcal{F}}])$ for some $n \in N$ and $\mathcal{F} \in \mathcal{C}$. Since $(X^*, \mathcal{C}_r^*)$ is T_2, $[\mathcal{F}] = [\dot{x}]$, so $\mathcal{Q} \geq \text{cl}^n_{\sigma q^*}\, j(\mathcal{F} \cap \dot{x})$, where $\mathcal{F} \cap \dot{x} \in \mathcal{C}$. Since $(X, \mathcal{C})$ is locally compact and T_3, $\text{cl}_q(\mathcal{F} \cap \dot{x})$ has a base of compact sets. Using the fact that $(X^*, \mathcal{C}_r^*)$ is also T_3, it follows that $\text{cl}^n_{\sigma q^*}\, j(\mathcal{F} \cap \dot{x}) = j(\text{cl}_q(\mathcal{F} \cap \dot{x}))$, so $\mathcal{Q} \geq j(\text{cl}_q(\mathcal{F} \cap \dot{x}))$. The filter $\text{cl}_q(\mathcal{F} \cap \dot{x})$ thus has a trace $\mathcal{G}$ on $j^{-1}(A)$ which q-converges to x, and we conclude that $x \in \text{cl}_q\, j^{-1}(A)$. □

THEOREM 1.5. *A locally compact,* T_3 *Cauchy space* $(X, \mathcal{C})$ *is* C_3 *iff* $(X^*, \mathcal{C}_r^*)$ *is* T_2.

Proof. It is obvious that j: $(X, \mathcal{C}) \to (X^*, \mathcal{C}^*)$ is Cauchy-continuous and one-to-one. If $(X^*, \mathcal{C}_r^*)$ is T_2, then, by the preceding lemma, $j^{-1}(\text{cl}^n_{\sigma q^*}(j\mathcal{G})) = \text{cl}^n_q\, \mathcal{G}$ for all $n \in N$ and $\mathcal{G} \in \mathcal{C}$; this is precisely what is needed to show that j^{-1}: $(jX, \mathcal{C}_r^* |_{jX}) \to (X, \mathcal{C})$ is Cauchy-continuous, and it follows that $(X^*, \mathcal{C}_r^*)$ is a T_3 completion of $(X, \mathcal{C})$. The converse follows immediately from Proposition 1.2. □

Given a T_2 Cauchy space $(X, \mathcal{C})$, let $\hat{\mathcal{C}}(X, \mathcal{C})$ be the set of all Cauchy-continuous functions from $(X, \mathcal{C})$ into the set R of real numbers with the usual (complete) Cauchy structure. A T_2 Cauchy space $(X, \mathcal{C})$ is said to be *Cauchy separated* if the set $(X, \mathcal{C})$ separates Cauchy equivalence classes; i.e., if $\mathcal{F}, \mathcal{G} \in \mathcal{C}$ and $\mathcal{F} \cap \mathcal{G} \notin \mathcal{C}$ implies there is $f \in \hat{\mathcal{C}}(X, \mathcal{C})$ such that $f(\mathcal{F})$ and $f(\mathcal{G})$ converge to distinct points in R.

LEMMA 1.6. *If* $(X, \mathcal{C})$ *is Cauchy-separated, then* $(X^*, \mathcal{C}_r^*)$ *is* T_2.

Proof. If $[\mathcal{F}]$, $[\mathcal{G}]$ are distinct elements of X^*, then $\mathcal{F} \cap \mathcal{G} \notin \mathcal{C}$, so there is $f \in \hat{\mathcal{C}}(X, \mathcal{C})$ such that $\lim f(\mathcal{F}) \neq \lim f(\mathcal{G})$ in R. By Proposition 2.1, [2], f has a continuous extension $\bar{f}$: $(X^*, \mathcal{C}^*) \to R$, and, since R is regular, $\bar{f}$: $(X^*, \mathcal{C}_r^*) \to R$ is also continuous. But $\bar{f}([\mathcal{F}]) = \lim f(\mathcal{F}) \neq \lim f(\mathcal{G}) = \bar{f}([\mathcal{G}])$, and the conclusion is established. □

An immediate consequence of Lemma 1.6 and Theorem 1.5 is the following.

THEOREM 1.7. *A regular, Cauchy separated, locally compact Cauchy space is* C_3.

THEOREM 1.8. *A totally bounded* T_3 *Cauchy space* $(X, \mathcal{C})$ *is* C_3 *iff* $(X, \mathcal{C})$ *is Cauchy separated.*

Proof. If $(X, \mathcal{C})$ is C_3, then $(X^*, \mathcal{C}_r^*)$ is a T_3 compactification of X, which means that $(X^*, \mathcal{C}_r^*)$ has the same ultrafilter convergence as a Tychonoff topological space. Since each continuous real-valued function on $(X^*, \mathcal{C}_r^*)$, when restricted to the image of $(X, \mathcal{C})$, is Cauchy-continuous, it follows that $(X, \mathcal{C})$ is Cauchy-separated. The converse follows from Theorem 2.2 [5]. □

Can the assumptions made in the above theorems be weakened? We have an example (omitted for the sake of brevity) of a T_3 Cauchy space $(X, \mathcal{C})$ such that $(X^*, \mathcal{C}_r^*)$ is T_2 but is not a completion of $(X, \mathcal{C})$; thus the assumption of local compactness in Theorem 1.5 cannot be entirely dispensed with. We also have an example of a locally compact, T_3 Cauchy space which is not C_3, so "Cauchy separated" cannot be replaced by "T_2" in Theorem 1.7. On the other hand, a locally compact C_3 Cauchy space need not be Cauchy-separated. For it should be noted that any T_3 convergence space is a (complete) C_3 Cauchy space. In Example 2.10, [4], a locally compact, T_3 convergence space is constructed whose topological modification is not T_2; this space, when regarded as a complete Cauchy space, is clearly not Cauchy separated.

2. SC_3 Cauchy spaces. A completion $((Y, \mathcal{D}), h)$ of a Cauchy space $(X, \mathcal{C})$ is defined to be *strict* if, whenever $\mathcal{A} \in \mathcal{D}$, there is $\mathcal{F} \in \mathcal{C}$ such that $\mathcal{A} \geq \mathrm{cl}_{q_{\mathcal{D}}} h(\mathcal{F})$. This terminology was introduced in [5], where it was shown that every T_2 Cauchy space has a coarsest strict (not necessarily regular or T_2) completion in standard form.

Topological completions are necessarily strict; also, if $(X, \mathcal{C})$ is a T_2 Cauchy space such that $X^* - j(X)$ is finite, then any completion of $(X, \mathcal{C})$ is strict. On the other hand, T_3 completions need not be strict. When they exist, however, strict T_3 completions are unique up to equivalence (see [5]). One of the main results of this section is an example of a locally compact Cauchy space which has a T_3 completion but no strict T_3 completion. Thus the Cauchy spaces having strict T_3 completions form a proper subclass of the C_3 Cauchy spaces; a member of the former class will be called an SC_3 Cauchy space.

To facilitate our study of SC_3 Cauchy spaces, we shall make use of the "Σ operator" defined in [5] for a T_2 Cauchy space $(X, \mathcal{C})$ as follows. If $A \subseteq X$, let $\Sigma A = \{[\mathcal{F}] \in X^*: A \in \mathcal{G}$ for some $\mathcal{G} \in [\mathcal{F}]\}$. If $\mathcal{F} \in F(X)$, let $\Sigma\mathcal{F}$ be the filter on X^* generated by $\{\Sigma F: F \in \mathcal{F}\}$. Let $\mathcal{C}_1^*$ be the Cauchy structure on X^* generated by $\{\Sigma\mathcal{F}: \mathcal{F} \in \mathcal{C}\}$, and let q_1^* be the induced convergence structure. (In [5] q_1^* was denoted by p and $\mathcal{C}_1^*$ by $\mathcal{C}_p$.) From the construction, it is clear that $\mathcal{C}_r^* \leq \mathcal{C}_1^* \leq \mathcal{C}^*$.

PROPOSITION 2.1. *Let $(X, \mathcal{C})$ be a T_3 Cauchy space. Then $(X, \mathcal{C})$ is SC_3 iff $\mathcal{C}_1^* = \mathcal{C}_r^*$.*

Proof. Corollary 1.6, [5], asserts that $(X^*, \mathcal{C}_1^*)$ is the only possible strict T_3 completion of $(X, \mathcal{C})$ in standard form. If $(X, \mathcal{C})$ is SC_3, then $\mathcal{C}_1^* = \mathcal{C}_r^*$. Conversely, if $\mathcal{C}_1^* = \mathcal{C}_r^*$, then $((X^*, \mathcal{C}_r^*), j)$ is a strict completion of $(X, \mathcal{C})$ by the results of [5]; it then follows from Proposition 1.2 that $((X^*, \mathcal{C}_r^*), j)$ is a strict T_3 completion of $(X, \mathcal{C})$. $\square$

Theorem 1.5 shows that every locally compact T_3 Cauchy space for which $(X^*, \mathcal{C}_r^*)$ is T_2 is a C_3 Cauchy space. The next example shows that such spaces need not be SC_3.

EXAMPLE 2.2. Let X be an infinite set partitioned into infinite subsets $\{X_n: n = 0, 1, 2, \ldots\}$. Let $\mathcal{F}$ be a free filter on X which contains X_0 and has a nested filter base $\{F_n: n \in N\}$ such that $F_n - F_{n+1}$ is an infinite set for $n \geq 1$. Furthermore, let each set $F_n - F_{n+1}$ be partitioned into infinite subsets $\{H_{n,k}: k \in N\}$, and let $\mathcal{H}_{n,k}$ be a free filter on X which contains $H_{n,k}$ for all $n, k \in N$.

Next, for each $n \in N$, let $\mathcal{G}_n$ be a free filter on X which contains X_n and which has a nested base $\{G_{n,k}: k \geq 1\}$ such that $G_{n,k} \subseteq X_n$ and $G_{n,k} - G_{n,k-1}$ is an infinite set for all $n \geq 1$. Furthermore, for all n, $k \in N$, let $\mathcal{L}_{n,k}$ be a free filter on X which contains $G_{n,k} - G_{n,k+1}$.

Let $\mathcal{C} = \{\dot{x}, \mathcal{F}, \mathcal{G}_n, \mathcal{K}_{n,k}\colon x \in X, \mathcal{K}_{n,k} \geq \mathcal{K}_{n,k} \cap \mathcal{L}_{n,k}, n \in N, k \in N\}$. Then $q_{\mathcal{C}}$ is discrete, and it follows easily that $(X, \mathcal{C})$ is locally compact and T_3. One can also show that $(X^*, \mathcal{C}_r^*)$ is T_2, so by Theorem 1.5 $(X, \mathcal{C})$ is a C_3 Cauchy space. To show that $(X, \mathcal{C})$ has no strict T_3 completion it is sufficient, by Proposition 2.1, to show that $\mathcal{C}_1^* \neq \mathcal{C}_r^*$. This can be accomplished by showing that $\mathrm{cl}_{q_1^*} j(\mathcal{F}) \neq \mathrm{cl}_{\sigma q^*} j(\mathcal{F})$. Indeed, each member of the latter filter contains elements of the form $[\mathcal{G}_n]$, and this is not true of the former. □

In order to characterize SC_3 Cauchy spaces, we shall extend the Σ operator. Given a T_2 Cauchy space $(X, \mathcal{C})$ and $A \subseteq X$, define $\Sigma^2 A = \{[\mathcal{F}] \in X^*\colon$ there is $\mathcal{G} \in [\mathcal{F}]$ such that $(\Sigma G) \cap (\Sigma A) \neq \varnothing$ for all $G \in \mathcal{G}\}$. If $\mathcal{F} \in F(X)$, define $\Sigma^2 \mathcal{F}$ to be the filter on X^* generated by $\{\Sigma^2 F\colon F \in \mathcal{F}\}$.

PROPOSITION 2.3. *Let $(X, \mathcal{C})$ be a T_2 Cauchy space. Then $(X, \mathcal{C})$ is SC_3 iff both of the following conditions are satisfied*: (1) *If $\mathcal{F}$, $\mathcal{G} \in \mathcal{C}$ and $(\Sigma\mathcal{F}) \vee (\Sigma\mathcal{G}) \neq \varnothing$, then $[\mathcal{F}] = [\mathcal{G}]$*; (2) *If $\mathcal{F} \in \mathcal{C}$, there is $\mathcal{G} \in \mathcal{C}$ such that $\Sigma^2\mathcal{F} \geq \Sigma\mathcal{G}$.*

Proof. Assume the two conditions. The first guarantees that the space $(X^*, \mathcal{C}_1^*)$ is T_2. Since Σ is the closure operator for the convergence structure q_1^*, the second condition guarantees that $(X^*, \mathcal{C}_1^*)$ is regular. Thus $\mathcal{C}_1^* = \mathcal{C}_r^*$, and $(X, \mathcal{C})$ is SC_3 by Proposition 2.1. Conversely, the two conditions follow directly from the assumption that $\mathcal{C}_1^*$ is T_3. □

Although we have seen that locally compact C_3 Cauchy spaces are not necessarily SC_3 spaces, one can obtain the following partial result.

PROPOSITION 2.4. *If $(X, \mathcal{C})$ is a locally compact C_3 Cauchy space, and $\mathcal{Q} \to [\dot{x}]$ in $(X^*, \mathcal{C}_r^*)$, where $x \in X$, then there is $\mathcal{F} \in \mathcal{C}$ such that $\mathcal{Q} \geq \mathrm{cl}_{\sigma q^*} j(\mathcal{F})$.*

Proof. This result is implicit in the proof of Lemma 1.4. □

The preceding proposition asserts that for locally compact C_3 spaces, "strictness" can fail only at "new" points which are added in the completion process.

We conclude this section by mentioning a class of locally compact T_3 Cauchy spaces which are a subclass of the SC_3 Cauchy spaces. A Cauchy space $(X, \mathcal{C})$ is called a *sequential Cauchy space* if every Cauchy filter contains a Cauchy filter which is generated by a sequence. A T_2 sequential Cauchy space is locally compact and T_3; furthermore, it is easy to verify that Wyler's completion preserves both of these properties. Thus if $(X, \mathcal{C})$ is a T_2 sequential Cauchy space, then $(X^*, \mathcal{C}^*) = (X^*, \mathcal{C}_1^*) = (X^*, \mathcal{C}_r^*)$; the next proposition is an immediate consequence.

PROPOSITION 2.5. *A T_2 sequential Cauchy space is SC_3.*

3. Coarse C_3 Cauchy spaces. Let (X, q) be a T_3 convergence space, and let $[q]$ denote the set of all Cauchy structures on X compatible with q. It is shown in [7] that $[q]$ always contains a finest T_3 member, denoted by $\mathcal{C}^q$, and a coarsest T_3 member, denoted by $\mathfrak{D}^q$. $\mathcal{C}^q$ is complete, and is therefore the finest C_3 member of $[q]$. On the other hand, $\mathfrak{D}^q$ is not C_3 in general; necessary and sufficient conditions for the existence of a coarsest C_3 member of $[q]$ are obtained below. We also characterize those T_3 spaces (X, q) such that each T_3 member of $[q]$ is C_3.

Starting with a T_3 convergence space (X, q) let $\mathcal{C}^q$ be the set of all q-convergent filters on X. Let $\Delta_q = \{\mathfrak{F} \in F(X): \mathfrak{G} \vee (\mathrm{cl}_q^n \mathfrak{F}) = \varnothing$ for all $\mathfrak{G} \in \mathcal{C}^q$ and $n \in N\}$. Let $\mathfrak{D}_q = \mathcal{C}^q \cup \Delta_q$, and let $\mathfrak{M}_q = \bigcap \Delta_q$. A T_3 convergence space (X, q) is defined to be *r-bounded* if, for each $\mathfrak{F} \in \mathcal{C}^q$, $\mathfrak{F} \vee \mathfrak{M}_q = \varnothing$.

LEMMA 3.1. *If $(X, \mathfrak{D}_q)$ has a T_3 completion, then (X, q) is r-bounded.*

Proof. Let (X, q) be a T_3 space which is not r-bounded. First note that Δ_q consists of a single Cauchy equivalence class of non-convergent filters; thus any completion of $(X, \mathfrak{D}_q)$ is necessarily a one-point completion. If $((X^*, p), j)$ is such a completion in standard form, let $\alpha = [\mathfrak{F}]$, where $\mathfrak{F} \in \Delta_q$; then $X^* = \{[\dot{x}]: x \in X\} \cup \{\alpha\}$, and if $\mathfrak{G} \in \Delta_q$ then $j(\mathfrak{G}) \to \alpha$ in (X^*, p). Since (X, q) is not r-bounded, there is $\mathfrak{F} \to x$ in (X, q) such that $\mathfrak{F} \vee \mathfrak{M}_q \neq \varnothing$; this implies that each $F \in \mathfrak{F}$ belongs to some ultrafilter $\mathfrak{G}_F$ in Δ_q. Consequently, $\dot{\alpha} \geq \mathrm{cl}_p\, j(\mathfrak{F})$. Since $j(\mathfrak{F}) \to [\dot{x}]$ in (X^*, p), the assumption that (X^*, p) is T_3 is contradicted. □

THEOREM 3.2. *The following statements about a T_3 convergence space (X, q) are equivalent.*

(a) *(X, q) is r-bounded.*

(b) *$(X, \mathfrak{D}_q)$ has T_3 completion.*

(c) *$[q]$ contains a coarsest C_3 member.*

(d) *$[q]$ contains a coarsest SC_3 member.*

Proof. (b) $\Rightarrow$ (a). This is Lemma 3.1.

(b) $\Rightarrow$ (c) and (d). This follows from the fact that $\mathcal{D}_q$ is the coarsest T_3 member of $[q]$ (Proposition 2.1, [7]), and any completion of $(X, \mathcal{D}_q)$ is necessarily strict.

(a) $\Rightarrow$ (b). Let $Y = X \cup \{a\}$ and let $i\colon X \to Y$ be the identity injection. Let p be the finest convergence structure on Y such that (X, q) is a subspace of (Y, p) and $i(\mathcal{G}) \to a$ for all $\mathcal{G} \in \Delta_q$. If $\mathcal{G} \in \Delta_q$, then $\mathrm{cl}_q\, i(\mathcal{G}) = i(\mathrm{cl}_q\, \mathcal{G}) \cap \dot{a} \to a$ in (Y, p). Also, the assumption of r-boundedness guarantees that $\mathrm{cl}_p\, i(\mathcal{F}) = i(\mathrm{cl}_q\, \mathcal{F})$ for all $\mathcal{F} \in \mathcal{C}^q$. Considering p as a complete Cauchy structure on Y, it follows that $((Y, p), i)$ is a T_3 completion of $(X, \mathcal{D}_q)$.

(c) $\Rightarrow$ (b). If $\Delta_q = \varnothing$, then $\mathcal{D}_q = \mathcal{C}^q$ is the coarsest C_3 member of $[q]$. If $\Delta_q \neq \varnothing$, let $\mathcal{F} \in \Delta_q$. Define $\mathcal{C}_{\mathcal{F}} = \mathcal{C}^q \cup \{\mathcal{G} \in F(X)\colon$ there is $n \in N$ such that $\mathcal{G} \geq \mathrm{cl}_q^n\, \mathcal{F}\}$. Then $\mathcal{C}_{\mathcal{F}} \in [q]$ and $\mathcal{C}_{\mathcal{F}}$ is easily seen to be a T_3 Cauchy structure. Next we construct a T_3 one-point completion of $(X, \mathcal{C}_{\mathcal{F}})$. Let $Y = X \cup \{a\}$, and let p be the finest convergence structure on Y such that (X, q) is a subspace of (Y, p) and $i(\mathcal{G}) \to a$ for all $\mathcal{G} \geq \mathrm{cl}_q^n\, \mathcal{F}$ for some $n \in N$. It is easy to verify that $((Y, p), i)$ is a T_3 completion of $(X, \mathcal{C}_{\mathcal{F}})$. Since $\mathcal{D}_q$ is the infimum in $[q]$ of $\{\mathcal{C}_{\mathcal{F}}\colon \mathcal{F} \in \Delta_q\}$, it follows that $\mathcal{D}_q$ is the coarsest C_3 member of $[q]$, and (b) is established.

(d) $\Rightarrow$ (b). The preceding proof is applicable here, since the T_3 completion constructed for each $(X, \mathcal{C}_{\mathcal{F}})$ is strict. □

A Tychonoff topological space has a coarsest compatible uniformity iff the space is locally compact. Thus one would expect r-boundedness to be closely related to local compactness. This relationship is described in the next two propositions.

PROPOSITION 3.3. *A locally compact T_3 convergence space (X, q) is r-bounded.*

Proof. Let $\mathcal{F} \to x$ in (X, q), and let A be a compact set in $\mathcal{F}$. It follows easily that each $\mathcal{G} \in \Delta_q$ contains a set $G_{\mathcal{G}}$ which is disjoint from A. If $M = \bigcup\{G_{\mathcal{G}}\colon \mathcal{G} \in \Delta_q\}$, then $M \cap A = \varnothing$, and therefore $\mathcal{F} \vee \mathcal{M}_q = \varnothing$, which establishes that (X, q) is r-bounded.

As in [5] and [7], we define a convergence space to be *almost topological* if it has the same ultrafilter convergence as a topological space.

PROPOSITION 3.4. *If (X, q) is a T_3, almost topological space, then (X, q) is locally compact iff (X, q) is r-bounded.*

Proof. If (X, q) is not locally compact, then there is $\mathcal{F} \to x$ such that each $F \in \mathcal{F}$ is a member of a non-convergent ultrafilter $\mathcal{G}_F$. If $\mathrm{cl}_q(\mathcal{G}_F) \vee \mathcal{H} \neq \varnothing$ for some filter $\mathcal{H} \to y$, then y is an adherent point of $\mathrm{cl}_q(\mathcal{G}_F)$; since (X, q) is assumed to be almost topological, $\mathcal{G}_F \to y$, contrary to assumption. Thus $\mathcal{G}_F \in \Delta_q$ for all $F \in \mathcal{F}$, and so $\mathcal{F} \vee \mathfrak{M}_q \neq \varnothing$. It follows that (X, q) is not r-bounded. □

We next define a T_3 convergence space (X, q) to be *s-bounded* if, for all $\mathcal{G} \in \Delta_q$, $\mathcal{G} \vee (\bigcap \mathcal{A}_{\mathcal{G}}) = \varnothing$, where $\mathcal{A}_{\mathcal{G}} = \{\mathcal{F} \in \Delta_q$: for all $n \geq 1$, $(\mathrm{cl}_q^n \mathcal{F}) \vee (\mathrm{cl}_q^n \mathcal{G}) = \varnothing\}$.

PROPOSITION 3.5. *If a T_3 space (X, q) contains a closed, infinite, discrete subset A, then (X, q) is not s-bounded.*

Proof. Any ultrafilter $\mathcal{G}$ on X which contains A is a closed member of Δ_q. $\mathcal{A}_{\mathcal{G}}$ contains all free ultrafilters on X which contain A and are distinct from $\mathcal{G}$. Thus $\mathcal{G} \vee (\bigcap \mathcal{A}_{\mathcal{G}}) \neq \varnothing$, so (X, q) is not s-bounded. □

THEOREM 3.6. *For a T_3 convergence space (X, q), the following are equivalent.*

(a) *(X, q) is both r-bounded and s-bounded.*

(b) *Each T_3 member of $[q]$ is C_3.*

(c) *Each T_3 member of $[q]$ is SC_3.*

Proof. (b) $\Rightarrow$ (a). (X, q) is r-bounded by Lemma 3.1. Let $\mathcal{G} \in \Delta_q$ and define $\mathcal{B}_{\mathcal{G}}$ to be the set of all filters on X which are finer than some finite intersection of members of $\mathcal{A}_{\mathcal{G}}$. Let $\mathcal{D}_{\mathcal{G}} = \mathcal{C}^q \cup \mathcal{C}_{\mathcal{G}} \cup \mathcal{B}_{\mathcal{G}}$, where $\mathcal{C}_{\mathcal{G}} = \{\mathcal{F} \in F(X)$: there is $n \in N$ such that $\mathcal{F} \geq \mathrm{cl}_q^n \mathcal{G}\}$. It can be shown by direct arguments that $\mathcal{D}_{\mathcal{G}}$ is a T_3 member of $[q]$. If $\mathcal{F} \in \mathcal{A}_{\mathcal{G}}$, then $[\mathcal{F}] = \mathcal{A}_{\mathcal{G}}$. Since $[\mathcal{G}] \neq [\mathcal{F}]$, it must be true that $\mathcal{G} \vee (\bigcap \mathcal{A}_{\mathcal{G}}) = \varnothing$; otherwise $[\dot{\mathcal{F}}] \geq \Sigma\mathcal{G}$, so $[\dot{\mathcal{F}}] \to [\mathcal{G}]$ in $(X^*, \mathcal{C}_r^*)$, contrary to Proposition 1.2. Thus (X, q) is s-bounded.

(c) $\Rightarrow$ (b). Obvious.

(a) $\Rightarrow$ (c). Assume (X, q) is both r-bounded and s-bounded, and let $\mathcal{C} \in [q]$ be T_3. By Proposition 2.1 it is sufficient to show that (X^*, q_Γ^*) is T_3; recall that $\mathcal{H} \to [\mathcal{F}]$ in q_Γ^* iff there is $\mathcal{G} \in [\mathcal{F}]$ such that $\mathcal{H} \geq \Sigma\mathcal{G}$.

If $\Sigma\mathcal{F} \to [\dot{x}]$ in (X^*, q_Γ^*), then $\mathcal{F} \to x$ in (X, q) and also $\mathrm{cl}_q \mathcal{F} \to x$. Using r-boundedness, one can show that $\mathrm{cl}_{q_\Gamma^*}(\Sigma\mathcal{F}) = \Sigma(\mathrm{cl}_q \mathcal{F})$, and the latter q_Γ^*-converges to $[\dot{x}]$. If $\mathcal{G} \in \mathcal{C} - \mathcal{C}^q$, then $\mathcal{G} \in \Delta_q$ and the two types of boundedness can be used to show that $\mathrm{cl}_{q_\Gamma^*} \Sigma\mathcal{G} = \Sigma(\mathrm{cl}_X \mathcal{G}) \cap [\dot{\mathcal{G}}]$; again, the latter filter q_Γ^*-converges to $[\mathcal{G}]$. Thus (X^*, q_Γ^*) is regular. The T_3 property is established by showing, in addition, that $\mathrm{cl}_{q_\Gamma^*}\{[\mathcal{H}]\} = \{[\mathcal{H}]\}$ for all $\mathcal{H} \in \mathcal{C}$; this again follows direct from r-boundedness and s-boundedness. □

To simplfy the terminology, we shall say that a T_3 convergence space satisfying any of the three equivalent conditions of Theorem 3.6 is *rs-bounded*. The next corollary is an immediate consequence of the preceding theorem, Propositions 3.4 and 3.5, and the fact that a T_3 topological space which is not countably compact contains an infinite, closed, discrete subset.

COROLLARY 3.7. *An rs-bounded T_3 topological space is both locally compact and countably compact.*

COROLLARY 3.8. *A metrizable space is rs-bounded iff it is compact.*

Proof. If (X, q) is compact, then $[q] = \{\mathcal{C}^q\}$, and $\mathcal{C}^q$ is obviously C_3. The converse follows from Corollary 3.7. □

REFERENCES

[1] R. N. Ball, *Convergence and Cauchy structures on lattice ordered groups*, Trans. Amer. Math. Soc., **259** (1980), 357–392.

[2] R. Fric and D. C. Kent, *Completion functors for Cauchy spaces*, Internat. J. Math. & Math. Sci., **4** (1979), 589–604.

[3] H. H. Keller, *Die Limes-Uniformisierbarkeit der Limesräume*, Math. Ann., **176** (1968), 334–341.

[4] D. C. Kent and G. D. Richardson, *The decomposition series of a convergence space*, Czech. Math. J., **23** (1973), 437–446.

[5] ______, *Regular completions of Cauchy spaces*, Pacific J. Math., **51** (1974), 483–490.

[6] ______, *Completely regular and ω-regular spaces*, Proc. Amer. Math. Soc., **82** (1981), 649–652.

[7] D. C. Kent, *A note on regular Cauchy spaces*, Pacific J. Math., **94** (1981), 333–339.

[8] K. McKennon, *The strict topology and the Cauchy structure of the spectrum of a C^*-algebra*, General Topology *Appl.*, **5** (1975), 249–262.

[9] J. Ramaley and O. Wyler, *Cauchy spaces II. Regular completions and compactifications*, Math. Ann., **187** (1970), 187–199.

[10] E. E. Reed, *Completions of uniform convergence spaces*, Math. Ann., **194** (1971), 83–108.

Received March 15, 1982.

WASHINGTON STATE UNIVERSITY
PULLMAN, WA 99164

AND

EAST CAROLINA UNIVERSITY
GREENVILLE, NC 27834