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CHARACTERIZING GLOBAL PROPERTIES IN INVERSE LIMITS

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This paper presents necessary and sufficient conditions, in terms of properties of bonding maps and bonding spaces, on an inverse sequence $\underline{X} = \{X_i, p_{i\,i+1}\}$ of compact metric spaces in order that its inverse limit $X = \lim\limits_{\leftarrow} \underline{X}$ is either an approximate absolute neighborhood retract, an (internally) e-calm compactum, an absolute neighborhood retract, an LC^n compactum, or that X has (covering) dimension $\leq n$.

1. Introduction. Let X denote the inverse limit of an inverse sequence $\underline{X} = \{X_i, p_{i\,i+1}\}$ of compact metric spaces. The main purpose of this paper is to identify necessary and sufficient conditions which will insure that X is either an approximate absolute neighborhood retract (both in the sense of Clapp **[Cl]** ($\mathrm{AANR_C}$) and in the sense of Noguchi **[No]** ($\mathrm{AANR_N}$)), an (internally) e-calm compactum **[Č1]**, an absolute neighborhood retract (ANR), an LC^n compactum, or that X has dimension $\leq n$.

The problem of characterizing the dimension of the inverse limit of an inverse system was studied earlier by Pasynkov **[Pa]** and by Delinić and Mardešić **[DM]**. On the other hand, Fort and Segal **[FS**, Theorems 2 and 3] considered a surjective inverse sequence $\underline{X} = \{X_i, p_{i\,i+1}\}$ (i.e., an inverse sequence with all bonding maps $p_{i\,i+1}$ onto) of locally connected continua and discovered that each bonding space X_i can be embeddded as a subset X_i^* of the product $P = \prod_{i>0} X_i$ (see §3) in such a way that the inverse limit X (considered as a subset of P) is a locally connected continuum iff the sequence $X_1^*, X_2^*, \ldots$ converges 0-regularly to X **[Wh]**. Another characterization of local connectedness in inverse limits was given by Gordh and Mardešić **[GM]**. They introduced a notion of local connectedness for inverse systems and proved that the inverse limit X of a surjective inverse system $\underline{X} = \{X_\alpha, p_{\alpha\alpha'}, A\}$ of locally connected continua is locally connected iff $\underline{X}$ is locally connected.

Our approach is motivated by shape theory and represents an application of ideas from the author's recent papers **[Č1]** and **[Č2]** and his earlier studies of globally regular convergences [**Č3**]–[**Č7**]. It can also be regarded as a natural extension of techniques both from [**FS**] and [**GM**].

The following is a brief description of our method for the case of $AANR_C$'s.

First we observe that the Mardešić-Segal treatment of movability in [**MS2**] and the author's notion of movably regular convergence [**Č3**] provide the following characterization.

(1.1) For an inverse ANR-sequence $\underline{X} = \{X_i, p_{i\,i+1}\}$ the following are equivalent:

(i) $X = \varprojlim \underline{X}$ is movable [**B**].

(ii) $\underline{X}$ is movable [**MS2**].

(iii) The sequence $X_1^*, X_2^*, \ldots$ converges movably regularly to X [**Č3**].

Then we use Corollary (4.3) in [**Č3**] which shows that $AANR_C$'s agree with e-movable compacta and perform changes necessary to make

e-(1.1): For a surjective ANR-sequence $\underline{X} = \{X_i, p_{i\,i+1}\}$ the following are equivalent:

(i) $X = \varprojlim \underline{X}$ is e-movable.

(ii) $\underline{X}$ is e-movable.

(iii) The sequence $X_1^*, X_2^*, \ldots$ converges e-movably regularly to X [**Č6**].

a true statement (see Theorem (4.2)). This requires defining a notion of e-movability for inverse sequences which is straightforward if one recalls that (roughly speaking) the concept of an e-movable compactum is obtained from Borsuk's original concept of a movable compactum by replacing homotopies with ε-homotopies.

In order to get characterizations of ANR's, LC^n compacta, (internally) e-calm compacta, and dimension, we shall "rigidify" (using results from [**Č1**] and [**Č2**]) the following theorems for corresponding shape invariants of strong movability [**B**], strong n-movability [**Č8**], calmness [**Č9**], and fundamental dimension [**B**], respectively.

(1.2) Let $\underline{X} = \{X_i, p_{i\,i+1}\}$ be an inverse ANR-sequence. The following are equivalent.

(i) $X = \varprojlim \underline{X}$ is strongly movable (strongly n-movable).

(ii) $\underline{X}$ is strongly movable [**M1**] (strongly n-movable).

(iii) The sequence $X_1^*, X_2^*, \ldots$ converges strongly movably regularly (strongly n-movably regularly) to X.

The strongly movable regular convergence in (iii) is less restrictive than the weakly $\mathcal{P}_p$-movably regular convergence in [**Č5**] and is defined as follows. A sequence $\{A_i\}_{i=1}^{\infty}$ of compacta in a metric space Y converges *strongly movably regulary* (*strongly n-movably regularly*) to a compactum

A_0, $A_0 \subset Y$, provided in some, and hence in every, ANR M which contains Y, for every neighborhood U of A_0 in M there is a neighborhood V of A_0 in M, $V \subset U$, such that for every neighborhood W of A_0 in M there is an index i_W with the property that for every $i \geq i_W$ there is a neighborhood W_0^i of A_i in M, $W_0^i \subset V \cap W$, so that for every $\mathcal{P}_p$-map (for every $\mathcal{P}_p^n$-map) (see §2) f: $(K, K_0) \to (V, W_0^i)$ there is a homotopy f_t: $K \to U$, $0 \leq t \leq 1$, with $f_0 = f$, $f_1(K) \subset W$, and $f_1 | K_0 = f | K_0$.

(1.3) For an inverse ANR-sequence $\underline{X} = \{X_i, p_{i\,i+1}\}$ the following are equivalent.

(i) $X = \lim\limits_{\leftarrow} \underline{X}$ is calm.

(ii) $\underline{X}$ is calm (i.e. $\underline{X}$ satisfies (4.2)(vi) in **[Č9]**).

(iii) The sequence $X_1^*, X_2^*, \ldots$ converges calmly regularly to X.

The calmly regular convergence is weaker than $\mathcal{P}$-calmly regular convergence studied in **[Č4]**. Its definition is analogous to the above definition of the strongly movably regular convergence (see (5.5)).

(1.4) (Nowak **[N]** and Čerin **[Č10]**) The inverse limit X of an inverse ANR-sequence $\underline{X} = \{X_i, p_{i\,i+1}\}$ has fundamental dimension $\leq n$ iff $\underline{X}$ is n-tame (i.e., iff for every index i there is $j \geq i$, an at most n-dimensional finite polyhedron P, and maps α: $X_j \to P$ and β: $P \to X_i$ such that the diagram

$$\begin{array}{ccc} X_i & \overset{p_{ij}}{\leftarrow} & X_j \\ \beta \nwarrow & & \swarrow \alpha \\ & P & \end{array}$$

is homotopy commutative).

We thank the referee for helpful suggestions (especially for Remark (6.4)).

2. Preliminaries. Throughout the paper $\mathcal{P}$ will denote the class of all compact ANR's and $\mathcal{P}_p$ will denote the class of all pairs (K, K_0) where K and K_0 are compact ANR's and K_0 is a subset of K. By $\mathcal{P}^n$ ($\mathcal{P}_p^n$) we denote all $K \in \mathcal{P}$ ($(K, K_0) \in \mathcal{P}_p$) with $\dim K \leq n$.

A map f: $K \to Y$ is called a *$\mathcal{P}$-map* provided $K \in \mathcal{P}$. Similarly, a map of pairs f: $(K, K_0) \to (Y, Y_0)$ is a *$\mathcal{P}_p$-map* if $(K, K_0) \in \mathcal{P}_p$.

We shall say that maps f and g of a space Z into a metric space (Y, d) are *ε-close* provided $d(f(z), g(z)) < \varepsilon$ for every $z \in Z$. If Z and W are subsets of Y and the composition of f: $Z \to W$ with the inclusion of W into Y is ε-close to the inclusion of Z into Y, we call f an *ε-map*.

Two maps f, g: $Z \to Y$ of a space Z into a metric space (Y, d) are *ε-homotopic* (and we write $f \simeq_\varepsilon g$) if there is a homotopy h_t: $Z \to Y$, $0 \le t \le 1$, between f and g (called an *ε-homotopy*) such that h_0 and h_t are ε-close for all $t \in I = [0, 1]$.

For a metric space (Y, d), 2^Y denotes the hyperspace of all nonempty compacta in Y with the Hausdorff metric d_{H}, while d_c denotes Borsuk's metric of continuity defined by

$$d_c(A, B) = \inf\{\varepsilon \mid \exists\ \varepsilon\text{-maps } f: A \to B \text{ and } g: B \to A\}$$

for A, $B \in 2^Y$. We shall also need the sup-norm metric d on the collection $\mathrm{Map}(Z, Y)$ of all maps of a compact space Z into Y given by

$$d(f, g) = \sup\{d(f(z), g(z)) \mid z \in Z\}$$

for f, $g \in \mathrm{Map}(Z, Y)$.

Let A be a subset of a metric space (Y, d), let U and V, $V \subset U$, be open subsets of Y which contain A, and let $\varepsilon > 0$ and $\delta > 0$ be given. Then $\mathcal{P}^\varepsilon(U, V; A)$, $\mathcal{P}_h^\varepsilon(V, \delta; A)$, and $\mathcal{P}_p^\varepsilon(U, V; A)$ will denote the following statements.

$\boxed{\mathcal{P}^\varepsilon(U, V: A)}$ For every neighborhood W of A in Y and every $\mathcal{P}$-map f: $K \to V$ there is an ε-homotopy f_t: $K \to U$, $0 \le t \le 1$, with $f_0 = f$ and $f_1(K) \subset W$.

$\boxed{\mathcal{P}_h^\varepsilon(V, \delta; A)}$ For every neighborhood W of A in Y there is a neighborhood W_0 of A in Y, $W_0 \subset V \cap W$, such that every two δ-close $\mathcal{P}$-maps f, g: $K \to W_0$ are ε-homotopic in W.

$\boxed{\mathcal{P}_p^\varepsilon(U, V: A)}$ For every neighborhood W of A in Y there is a neighborhood W_0 of A in Y, $W_0 \subset V \cap W$, such that for every $\mathcal{P}_p$-map f: $(K, K_0) \to (V, W_0)$ there is an ε-homotopy f_t: $K \to U$, $0 \le t \le 1$, with $f_0 = f$, $f_1(K) \subset W$, and $f_1 \mid K_0 = f \mid K_0$.

A compactum A is (*strongly*) *e-movable* if for some, and hence for every, embedding of A into an ANR M the following holds. For each neighborhood U of A in M and every $\varepsilon > 0$ there is a neighborhood V of A in M, $V \subset U$, such that $(\mathcal{P}_p^\varepsilon(U, V; A))$ $\mathcal{P}^\varepsilon(U, V; A)$ is true. We proved in [**Č1**] and [**Č2**] that a compactum A is (strongly) e-movable iff it is an $\mathrm{AANR}_{\mathrm{C}}$ (an ANR).

A compactum A is *e-calm* if for some, and hence for every, embedding of A into an ANR M the following holds. For every $\varepsilon > 0$ there is a neighborhood V of A in M and a $\delta > 0$ such that $\mathcal{P}_h^\varepsilon(V, \delta; A)$ is true. If for every $\varepsilon > 0$ there is a $\delta > 0$ such that δ-close $\mathcal{P}$-maps into A are ε-homotopic in every neighborhood of A in M then A is *internally e-calm*. We proved in [**Č1**] that a compactum A is strongly e-movable iff A is e-movable and (internally) e-calm.

For a compact ANR M and an $\varepsilon > 0$, let $\Gamma(M, \varepsilon)$ ($\Gamma^*(M, \varepsilon)$) be the set of all $\delta > 0$ such that, for any δ-close maps $f, g: Y \to M$ defined on a metrizable space Y (and any δ-homotopy $j_t: A \to M$, $0 \leq t \leq 1$, defined on a closed subspace A of Y with $j_0 = f|A$ and $j_1 = g|A$), there exists an ε-homotopy $h_t: Y \to M$, $0 \leq t \leq 1$, such that $h_0 = f$, $h_1 = g$, (and $h_t|A = j_t|A$ for every $t \in I$) ([**Hu**, p. 122]).

For a map $f: A \to B$ between metric spaces, let $\Lambda(f, \varepsilon)$ be the set of all $\delta > 0$ with the property that $d(x, y) < \delta$ in A implies $d(f(x), f(y)) < \varepsilon$ in B.

Throughout the paper $\underline{X} = \{X_i, p_{i\,i+1}\}$ will denote an inverse sequence where each X_i is a compact metric space and $p_{i\,i+1}: X_{i+1} \to X_i$ is a continuous map. $X = \lim\limits_{\leftarrow} \underline{X}$ will denote the inverse limit of $\underline{X}$, while p_i: $X \to X_i$ is a projection. For $j > i$, $p_{ij} = p_{i\,i+1} \circ p_{i+1\,i+2} \circ \cdots \circ p_{j-1\,j}$ and $p_{i\,i} = \mathrm{id}_{X_i}$. If each bonding space X_i is an ANR, $\underline{X}$ will be called an *inverse* ANR-*sequence*. Inverse LC^n-sequences and inverse AANR_C-sequences are defined analogously.

3. Fort-Segal embeddings. This section describes the method due to Fort and Segal [**FS**] of embedding the bonding spaces X_i of a surjective inverse sequence $\underline{X} = \{X_i, p_{i\,i+1}\}$ into the product $P = \prod_{i>0} X_i$ in such a way that the images X_i^* converge to the inverse limit $X \subset P$. Since we shall study global properties of X in an ANR, we must slightly modify their procedure in order to get that P is nicely embedded in a Hilbert cube.

Let X_i be a compactum in a Hilbert cube Q_i for each positive integer i, let $p_{i\,i+1}$ be a mapping of X_{i+1} onto X_i, and let D_i be a metric for Q_i so that $D_i(x, y) \leq 1$ for all x and y in Q_i. Let $\bar{p}_{i\,i+1}: Q_{i+1} \to Q_i$ be an extension of $p_{i\,i+1}$. Let $(X, p_i) = \lim\limits_{\leftarrow}\{X_i, p_{i\,i+1}\}$ and let $(\bar{Q}, \bar{p}_i) = \lim\limits_{\leftarrow}\{Q_i, \bar{p}_{i\,i+1}\}$. Note that X is a subset of $\bar{Q}$ and $\bar{p}_i$ is an extension of p_i for every $i > 0$. We then define

$$d_i(x, y) = \sum_{j=1}^{i} 2^{-j} D_j(\bar{p}_{ij}(x), \bar{p}_{ij}(y))$$

for each positive integer i and all x and y in Q_i, and we define

$$d(u, v) = \sum_{j=1}^{\infty} 2^{-j} D_j(\bar{p}_j(u), \bar{p}_j(v))$$

for all u and v in $\bar{Q}$. Then d is a metric for $\bar{Q}$ and d_i is a metric for Q_i for each $i > 0$. Moreover $\lim_{i\to\infty} d_i(\bar{p}_i(u), \bar{p}_i(v)) = d(u, v)$ uniformly on $\bar{Q} \times \bar{Q}$.

Let $Q = \prod_{i>0} Q_i$. If we define a metric d^* for Q by letting

$$d^*(a, b) = \sum_{j=1}^{\infty} 2^{-j} D_j(a_j, b_j)$$

for $a = (a_1, a_2, \ldots)$ and $b = (b_1, b_2, \ldots)$, then the inclusion map is an isometry of $(\bar{Q}, d)$ (and therefore also of (X, d)) into (Q, d^*)). Choose a point $q = (q_1, q_2, \ldots)$ in Q. We now define for each positive integer i an isometry H_i of (Q_i, d_i) into (Q, d^*) by letting

$$H_i(x) = (p_{1i}(x), \ldots, p_{ii}(x), q_{i+1}, q_{i+2}, \ldots)$$

for every $x \in X_i$. We define h_i as the restriction of H_i on X_i and we put $X_i^* = h_i(X_i)$. Note that for every $j \geq i$ there is a map p_{ij}^*: $X_j^* \to X_i^*$ given by

$$\begin{aligned} p_{ij}^*\big((p_{1j}(x), \ldots, p_{ij}(x), p_{i+1j}(x), \ldots, p_{jj}(x), q_{j+1}, q_{j+2}, \ldots)\big) \\ = \big(p_{1i}(p_{ij}(x)), \ldots, p_{ii}(p_{ij}(x)), q_{i+1}, q_{i+2}, \ldots\big) \\ = \big(p_{1j}(x), \ldots, p_{ij}(x), q_{i+1}, q_{i+2}, \ldots\big) \end{aligned}$$

and that $d^*(p_{ij}^*(x_j^*), x_j^*) < 2^{-i}$ for all $x_j^* \in X_j^*$. Also, observe that there is a map p_i^*: $X \to X_i^*$ defined by

$$p_i^*((x_1, x_2, x_3, \ldots)) = (x_1, x_2, \ldots, x_i, q_{i+1}, q_{i+2}, \ldots)$$

and that

$$d^*(p_i^*(x_1, x_2, \ldots), (x_1, x_2, \ldots)) < 2^{-i}$$

for all $(x_1, x_2, \ldots) \in X$.

In §§4–6 we shall always consider the spaces X and X_i with metrics d and d_i, respectively. For example, in Definition (4.1) below, when we say that $p_{ij} \circ \psi$ is ε-close to φ we mean that $d_i(p_{ij} \circ \psi, \varphi) < \varepsilon$.

4. Approximate absolute neighborhood retracts. Here we shall characterize surjective inverse AANR_C-sequences whose limits are AANR_C's and surjective inverse ANR-sequences whose limits are AANR_N's.

(4.1) DEFINITION. An inverse sequence $\underline{X} = \{X_i, p_{i\,i+1}\}$ is called (*internally*) *e-movable* provided for every $\varepsilon > 0$ there is an index i_0 such that for every $i \geq i_0$, every $j \geq i$ and for every $\mathcal{P}$-map φ: $K \to X_i$ there is a map (ψ: $K \to X = \lim\limits_{\leftarrow} \underline{X}$) ψ: $K \to X_j$ with ($p_i \circ \psi$ ε-close to φ) $p_{ij} \circ \psi$ ε-close to φ.

(4.2) THEOREM. *For a surjective inverse $AANR_C$-sequence $\underline{X} = \{X_i, p_{i\,i+1}\}$ the following are equivalent.*

(i) $X = \lim\limits_{\leftarrow} \underline{X}$ *is e-movable or, equivalently, an $AANR_C$.*

(ii) $\underline{X}$ *is e-movable.*

(iii) $\underline{X}$ *is internally e-movable.*

(iv) *The sequence $\{X_i^*\}$ converges e-movably to X* [**Č6**].

(v) *The sequence $\{X_i^*\}$ converges to X in the metric of continuity d_c^* induced on 2^Q by the metric d^*.*

Proof. We shall prove (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), and (iii) $\Rightarrow$ (v). We already proved in [**Č6**] that (iv) and (v) are equivalent and that (iv) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii). Let an $\varepsilon > 0$ be given. Since X is an e-movable compactum in the Hilbert cube Q, by Proposition (4.2) in [**Č2**], there is a neighborhood V of X in Q such that for every neighborhood W of X in Q and every $\mathcal{P}$-map φ: $K \to V$ there is a map ψ: $K \to W$ which is $(\varepsilon/3)$-close to φ. Select an index i_0 so that $X_i^* \subset V$ and so that p_{ij}^* is an $(\varepsilon/3)$-map and p_j^* is an $(\varepsilon/6)$-map for all $j \geq i \geq i_0$. Consider arbitrary indices $j \geq i \geq i_0$ and a $\mathcal{P}$-map φ: $K \to X_i$. Since X_j is an e-movable compactum, there is a neighborhood W of X in Q and an extension $\bar{p}_j^*$: $W \to X_j^*$ of p_j^* such that $\bar{p}_j^*$ is an $(\varepsilon/3)$-map. By the choice of V, the map $h_i \circ \varphi$: $K \to X_i^*$ is $(\varepsilon/3)$-close to a map ψ_1: $K \to W$. But, then $\psi_1^* = \bar{p}_j^* \circ \psi_1$ is $(2\varepsilon/3)$-close to $h_i \circ \varphi$. Hence, $p_{ij}^* \circ \psi_1^* = p_{ij}^* \circ \bar{p}_j^* \circ \psi_1$ is ε-close to $h_i \circ \varphi$. It follows that φ is ε-close to $p_{ij} \circ \psi$, where $\psi = h_j^{-1} \circ \bar{p}_j^* \circ \psi_1$, because h_i is an isometry and the diagram

$$\begin{array}{ccc} X_i^* & \overset{p_{ij}^*}{\leftarrow} & X_j^* \\ h_i \uparrow & & \uparrow h_j \\ X_i & \underset{p_{ij}}{\leftarrow} & X_j \end{array}$$

commutes.

(ii) $\Rightarrow$ (iii). It clearly suffices to prove that for every $\varepsilon > 0$ there is an index i_0 such that for every $i \geq i_0$ there is an ε-map f: $X_i^* \to X$. For a given $\varepsilon > 0$, pick an index i_0 so that for all $j \geq i \geq i_0$ there is a map f_{ij}: $X_i \to X_j$ with $p_{ij} \circ f_{ij}$ $(\varepsilon/2)$-close to id_{X_i}. Let $i \geq i_0$ and select a sequence $i = i_1 < i_2 < \cdots$ such that for every $j > 0$ there is a map f_j: $X_{i_j} \to X_{i_{j+1}}$ with $p_{i_j i_{j+1}} \circ f_j$ $(\varepsilon/2^j)$-close to $\mathrm{id}_{X_{i_j}}$ (with the distance measured with respect to the metric d_{i_j}). Then the map $f_j^*\colon X_{i_j}^* \to X_{i_{j+1}}^*$ defined by $f_j^* = h_{i_{j+1}} \circ f_j \circ h_{i_j}^{-1}$ is an $(\varepsilon/2^j)$-map (measured in the metric d^*) for every $j > 0$. Hence $f = \lim_{n\to\infty} f_n^* \circ f_{n-1}^* \circ \cdots \circ f_1^*$ is an ε-map of X_i^* into X.

(iii) $\Rightarrow$ (v). By assumption, for every $\varepsilon > 0$ there is an index i_0 such that for every $i \geq i_0$ there is a map $f_i\colon X_i^* \to X$ with $p_i^* \circ f_i$ $(\varepsilon/2)$-close to $\mathrm{id}_{X_i^*}$. But, if i_0 is so large that each p_i^* is an $(\varepsilon/2)$-map, f_i will be an ε-map so that $d_c^*(X_i^*, X) < \varepsilon$ for all $i \geq i_0$.

(4.3) COROLLARY. *If every bonding map $p_{i\,i+1}$ in an inverse $AANR_C$-sequence $\underline{X} = \{X_i, p_{i\,i+1}\}$ is an approximately right invertible map* **[Ge1]**, **[Č1]**, *then $X = \varprojlim \underline{X}$ is an e-movable compactum.*

Proof. The assumption about bonding maps clearly implies that the inverse sequence $\underline{X}$ is e-movable so that we can apply (4.2)(ii) $\Rightarrow$ (i).

Since a compactum is an $\mathrm{AANR_N}$ iff it is an $\mathrm{AANR_C}$ and an FANR, **[Bo]**, (1.2) and (4.2) imply the following.

(4.4) COROLLARY. *Let $\underline{X} = \{X_i, p_{i\,i+1}\}$ be a surjective ANR-sequence. The following are equivalent.*

(i) *$X = \varprojlim \underline{X}$ is an $AANR_N$.*

(ii) *$\underline{X}$ is e-movable and strongly movable* **[M]**.

(iii) *The sequence $X_1^*, X_2^*, \ldots$ converges both strongly movably regularly and e-movably to X.*

5. Internally *e*-calm and *e*-calm compacta. In **[Č1]** the author defined internally e-calm and e-calm compacta in order to get a new characterization of ANR's analogous to the characterization of FANR's as compacta which are both movable and calm **[ČŠ**, Theorem (4.5)**]**. It is still unknown whether there exists an (internally) e-calm compactum which is not an ANR. This section shows how to recognize (internally) e-calm compacta as inverse limits of surjective inverse ANR-sequences.

(5.1) DEFINITION. An inverse sequence $\underline{X} = \{X_i, p_{i\,i+1}\}$ is *internally e-calm* if for every $\varepsilon > 0$ there is an index i and a $\delta > 0$ such that for every index $j \geq i$, $p_j \circ \varphi$ and $p_j \circ \psi$ are ε-homotopic in X_j whenever φ and ψ are δ-close $\mathcal{P}$-maps into X.

(5.2) DEFINITION. A sequence $\{A_i\}_{i=1}^{\infty}$ of compacta in a metric space Y converges *internally e*-calmly* to a compactum A_0, $A_0 \subset Y$, provided in some, and hence in every, ANR M which contains Y, for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every neighborhood W of A_0 in M and every $\gamma > 0$ there is an index i_0 so that $i \geq i_0$ implies $A_i \subset W$ and δ-close $\mathcal{P}$-maps φ, ψ: $K \to X$ are γ-close to maps φ', ψ': $K \to A_i$, respectively, which are ε-homotopic in A_i.

(5.3) THEOREM. *For a surjective inverse ANR-sequence* $\underline{X} = \{X_i, p_{i\,i+1}\}$ *the following are equivalent.*

(i) $X = \varprojlim \underline{X}$ *is internally e-calm.*

(ii) $\underline{X}$ *is internally e-calm.*

(iii) *The sequence* $X_1^*, X_2^*, \ldots$ *converges internally* e^**-calmly to* X.

Proof. (i) ⇒ (ii). Let an $\varepsilon > 0$ be given. Select an i so big that p_j^* is an $(\varepsilon/3)$-map for all $j \geq i$ and a $\delta > 0$ such that δ-close $\mathcal{P}$-maps into X are $(\varepsilon/3)$-homotopic in every neighborhood of X in Q. For $j \geq i$, choose a neighborhood W_j of X in Q such that p_j^*: $X \to X_j^*$ extends to an $(\varepsilon/3)$-map $\bar{p}_j^*$: $W_j \to X_j^*$. Now, δ-close maps φ, ψ: $K \to X$ are $(\varepsilon/3)$-homotopic in W_j. Hence, $\bar{p}_j^* \circ \varphi = p_j^* \circ \varphi$ and $\bar{p}_j^* \circ \psi = p_j^* \circ \psi$ are ε-homotopic in X_j^* so that $p_j \circ \varphi = h_j^{-1} \circ p_j^* \circ \varphi$ is ε-homotopic to $p_j \circ \psi = h_j^{-1} \circ p_j^* \circ \psi$ (because h_j is an isometry).

(ii) ⇒ (iii). For a given $\varepsilon > 0$, pick a $\delta > 0$ and an index i_1 such that $p_i \circ \varphi$ and $p_i \circ \psi$ are ε-homotopic in X_i $(i \geq i_1)$ whenever φ and ψ are δ-close $\mathcal{P}$-maps into X. Consider a neighborhood W of X in Q and a $\gamma > 0$. Choose an index $i_0 \geq i_1$ so that p_i^* is a γ-map and $X_i^* \subset W$ for all $i \geq i_0$.

(iii) ⇒ (i). Let an $\varepsilon > 0$ be given. Select a $\delta > 0$ with respect to $\varepsilon/3$ using the assumption. Let W be a compact ANR neighborhood of X in Q and let $\gamma \in \Gamma(W, \varepsilon/3)$. Choose an index i_0 so that $X_{i_0}^* \subset W$ and so that δ-close $\mathcal{P}$-maps φ, ψ: $K \to X$ are γ-close to maps φ', ψ': $K \to X_{i_0}^*$, respectively, with φ' and ψ' $(\varepsilon/3)$-homotopic in W. Since φ and φ' are $(\varepsilon/3)$-homotopic in W and ψ and ψ' are $(\varepsilon/3)$-homotopic in W, it follows that φ and ψ are ε-homotopic in W and therefore that X is internally e-calm.

(5.4) DEFINITION. An inverse sequence $\underline{X} = \{X_i, p_{i\,i+1}\}$ is *e-calm* if for every $\varepsilon > 0$ there is an index i and a $\delta > 0$ such that for every $j \geq i$ there is a $k \geq j$ with $p_{jk} \circ \varphi$ ε-homotopic to $p_{jk} \circ \psi$ in X_j whenever φ and ψ are δ-close $\mathcal{P}$-maps into X_k.

(5.5) DEFINITION. A sequence $\{A_i\}_{i=1}^{\infty}$ of compacta in a metric space Y converges e^**-calmly* to a compactum A_0, $A_0 \subset Y$, provided in some, and hence in every, ANR M which contains Y, for every $\varepsilon > 0$ there is a

$\delta > 0$ such that for every neighborhood W of A_0 in M there is an index i_W with the property that for every $i \geq i_W$ there is a neighborhood W_0^i of A_i in M, $W_0^i \subset W$, so that δ-close $\mathcal{P}$-maps into W_0^i are ε-homotopic in W.

(5.6) THEOREM. *Let $\underline{X} = \{X_i, p_{i\, i+1}\}$ be a surjective inverse ANR-sequence. The following are equivalent.*

(i) *$X = \varprojlim \underline{X}$ is e-calm.*

(ii) *$\underline{X}$ is e-calm.*

(iii) *The sequence $X_1^*, X_2^*, \ldots$ converges e^*-calmly to X.*

Proof. (i) $\Rightarrow$ (ii). For a given $\varepsilon > 0$, select a neighborhood V of X in Q and a $\delta > 0$ such that $\mathcal{P}_h^{\varepsilon/9}(V, \delta; X)$ holds. Then choose an index i so that $X_j^* \subset V$ and p_j^* is an $(\varepsilon/9)$-map of X onto X_j^* for all $j \geq i$. Let $j \geq i$. Extend p_j^* to an $(\varepsilon/9)$-map $\bar{p}_j^*$: $W \to X_j^*$ of a closed neighborhood W of X in Q and let $\eta \in \Lambda(\bar{p}_j^*, \gamma)$, where $\gamma \in \Gamma(X_j^*, \varepsilon/3)$. Inside $W \cap V$ pick a neighborhood W_0 of X in Q using $\mathcal{P}_h^{\varepsilon/9}(V, \delta; X)$ and take a $k \geq j$ so that $X_k^* \subset W_0$ and so that p_k^* is an η-map.

Consider δ-close $\mathcal{P}$-maps φ, ψ: $K \to X_k$ into X_k. The compositions $h_k \circ \varphi$ and $h_k \circ \psi$ are δ-close maps of K into W_0. Hence, they are $(\varepsilon/9)$-homotopic in W. Since $\bar{p}_j^*$ is an $(\varepsilon/9)$-map, $\bar{p}_j^* \circ h_k \circ \varphi$ and $\bar{p}_j^* \circ h_k \circ \psi$ are $(\varepsilon/3)$-homotopic in X_j^*. But, for every point $y \in X_k^*$, there is $x \in X$ such that $p_k^*(x) = y$ so that $\bar{p}_j^*(x) = p_j^*(x) = p_{jk}^* \circ p_k^*(x) = p_{jk}^*(y)$ and $\bar{p}_j^*(y)$ are γ-close. It follows that $p_{jk}^* \circ h_k \circ \varphi$ is $(\varepsilon/3)$-homotopic to $\bar{p}_j^* \circ h_k \circ \varphi$ in X_j^*. Thus, $p_{jk}^* \circ h_k \circ \varphi$ and $p_{jk}^* \circ h_k \circ \psi$ are ε-homotopic in X_j^*. This implies that $p_{jk} \circ \varphi$ and $p_{jk} \circ \psi$ are ε-homotopic in X_j.

(ii) $\Rightarrow$ (iii). Let $\varepsilon > 0$. Choose an i and a $\delta > 0$ such that i and 3δ satisfy (5.4) with respect to $\varepsilon/3$. Consider a compact ANR neighborhood W of X in Q and let $\eta \in \Gamma(W, \varepsilon/9)$. Pick an index $j = i_W \geq i$ such that $X_k^* \subset W$ and p_{jk}^* is an η-map for all $k \geq j$. Finally, for every $k \geq j$, let W_0^k denote a neighborhood of X_k^* in W such that there is a $\min\{\delta, \eta\}$-map r_k: $W_0^k \to X_k^*$.

(iii) $\Rightarrow$ (i). For a given $\varepsilon > 0$, select a $\delta > 0$ with respect to $\varepsilon/3$ using (iii). Let W be a compact ANR neighborhood of X in Q and let $\eta \in \Gamma(W, \varepsilon/3)$. Then choose an $i \geq i_W$ and a neighborhood W_0 of X in Q, $W_0 \subset W$, such that p_i^* extends to a $\min\{\delta/3, \eta\}$-map $\bar{p}_i^*$ of W_0 into W_0^i. It can be easily checked that $(\delta/3)$-close $\mathcal{P}$-maps into W_0 are ε-homotopic in W. Hence, X is e-calm by Proposition (4.2) in [**Č1**].

6. ANR's and LC^n compacta. In this section we shall put together results from §§4 and 5 and get conditions which characterize surjective inverse ANR-sequences whose inverse limits are ANR's. Then we shall indicate changes that one must make in our theorem to obtain analogous results for surjective inverse LC^n-sequences whose inverse limits are LC^n-compacta. In particular, the Fort-Segal characterization mentioned in the introduction appears as the 0-dimensional case of ours. Some applications of our method are also presented.

(6.1) DEFINITION. An inverse sequence $\underline{X} = \{X_i, p_{i\,i+1}\}$ is *strongly e-movable* provided for every $\varepsilon > 0$ there is an index i_0 such that for every $i \geq i_0$, for every $j \geq i$, and every $\delta > 0$ there is a $k \geq j$ with the property that for every $\mathcal{P}_p$-pair (K, K_0) and maps $\varphi\colon K \to X_i$ and $\psi_0\colon K_0 \to X_k$ with $\varphi | K_0 = p_{ik} \circ \psi_0$, there is a map $\psi\colon K \to X_j$ such that $p_{ij} \circ \psi$ is ε-close to φ and $\psi | K_0$ is δ-close to $p_{jk} \circ \psi_0$.

(6.2) DEFINITION. A sequence $\{A_i\}_{i=1}^{\infty}$ of compacta in a metric space Y converges *strongly e*-movably* to a compactum A_0, $A_0 \subset Y$, provided in some, and hence in every, ANR M which contains Y, for every neighborhood U of A_0 in M and every $\varepsilon > 0$ there is a neighborhood V of A_0 in M, $V \subset U$, such that for every neighborhood W of A_0 in M there is an index i_W with the property that for every $i \geq i_W$ there is a neighborhood W_0^i of A_i in M, $W_0^i \subset V \cap W$, so that for every $\mathcal{P}_p$-map $f\colon (K, K_0) \to (V, W_0^i)$ there is an ε-homotopy $f_t\colon K \to U$, $0 \leq t \leq 1$, with $f_0 = f$, $f_1(K) \subset W$, and $f_1 | K_0 = f | K_0$.

(6.3) THEOREM. *For a surjective inverse ANR-sequence* $\underline{X} = \{X_i, p_{i\,i+1}\}$ *the following are equivalent.*

(i) $X = \varprojlim \underline{X}$ *is an ANR.*

(ii) $\underline{X}$ *is strongly e-movable.*

(iii) $\underline{X}$ *is both e-movable and internally e-calm.*

(iv) $\underline{X}$ *is both e-movable and e-calm.*

(v) *The sequence* $X_1^*, X_2^*, \ldots$ *converges strongly e*-movably to* X.

(vi) *The sequence* $X_1^*, X_2^*, \ldots$ *converges both e-movably and internally e*-calmly to* X.

(vii) *The sequence* $X_1^*, X_2^*, \ldots$ *converges both e-movably and e*-calmly to* X.

Proof. We shall prove that (i) ⇒ (ii) and (ii) ⇒ (iv). The other implications are consequences of results in §§4 and 5 and Theorem (4.9)(a) and Lemma (4.10) in **[Č1]**.

(i) $\Rightarrow$ (ii). For a given $\varepsilon > 0$, select a compact ANR neighborhood V of X in Q such that for every neighborhood W of X in Q there is a neighborhood W_0 of X in Q, $W_0 \subset V \cap W$, with the property that every $\mathcal{P}_p$-map $\varphi_1^*: (K, K_0) \to (V, W_0)$ is $(\varepsilon/4)$-close to a map $\varphi_2^*: K \to W$ which agrees with φ_1^* on K_0 (Proposition (3.2) in **[Č1]**). Choose an index i_0 so that $X_i^* \subset V$ and p_{ij}^* and p_j^* are $(\eta/2)$-maps for all i and j, $j \geq i \geq i_0$, where $\eta \in \Gamma(V, \varepsilon/4)$, $0 < \eta < \varepsilon/4$.

Consider indices j and $i, j \geq i \geq i_0$, and a $\delta > 0$. Choose an $(\eta/2)$-map $r: W \to X$ which retracts a neighborhood W of X in Q onto X. Inside the intersection $V \cap W$ pick a neighborhood W_0 of X in Q as above and take an index $k \geq j$ such that $X_k^* \subset W_0$ and such that $r \mid X_k^*$ and p_k^* are $(\gamma/2)$-maps where $2\gamma \in \Lambda(p_j^*, \delta)$ and $0 < \gamma < \varepsilon/4$. Observe that $\gamma \in \Lambda(p_{jk}^*, \delta)$ because p_k^* is onto and $p_j^* = p_{jk}^* \circ p_k^*$.

Let (K, K_0) be a pair in $\mathcal{P}_p$ and let $\varphi: K \to X_i$ and $\psi_0: K_0 \to X_k$ be maps with $\varphi \mid K_0 = p_{ik} \circ \psi_0$. The compositions $\varphi^* = h_i \circ \varphi$ and $\psi_0^* = h_k \circ \psi_0$ satisfy $\varphi^* \mid K_0 = p_{ik}^* \circ \psi_0^*$. Since p_{ik}^* is an $(\eta/2)$-map, there is an $(\varepsilon/4)$-homotopy $g_t: K_0 \to V$, $0 \leq t \leq 1$, such that $g_0 = \varphi^* \mid K_0 = p_{ik}^* \circ \psi_0^*$ and $g_1 = \psi_0^*$. By the homotopy extension theorem, g_1 can be extended to a map $\varphi_1^*: K \to V$ which is $(\varepsilon/4)$-close to φ^*. The choice of V and W_0 implies that $(\varepsilon/4)$-close to φ_1^* there is a map $\varphi_2^*: K \to W$ which agrees with φ_1^* on K_0. Let $\psi^* = p_j^* \circ r \circ \varphi_2^*$ and let $\psi = h_j^{-1} \circ p_j^* \circ r \circ \varphi_2^*$. Since r and p_j^* are $(\varepsilon/8)$-maps and p_{ij}^* is an $(\varepsilon/4)$-map, ψ^* is $(\varepsilon/4)$-close to φ_2^* and $p_{ij}^* \circ \psi^*$ is ε-close to φ^* and hence $p_{ij} \circ \psi$ is ε-close to φ. On the other hand, $p_k^* \circ r \circ \psi_0^*$ is γ-close to ψ_0^* so that $p_{jk}^* \circ p_k^* \circ r \circ \psi_0^* = p_j^* \circ r \circ \psi_0^* = \psi^* \mid K_0$ is δ-close to $p_{jk}^* \circ \psi_0^*$ and thus $p_{jk} \circ \psi_0$ is δ-close to $\psi \mid K_0$.

(ii) $\Rightarrow$ (iv). Since every strongly e-movable ANR-sequence $\underline{X}$ is clearly e-movable, we must show that $\underline{X}$ is e-calm. For an $\varepsilon > 0$, pick an index i_0 as in Definition (6.1) but with respect to $\varepsilon/9$. Then choose an $i \geq i_0$ so that p_{ij}^* is an $(\varepsilon/9)$-map for every $j \geq i$. Let a $\delta > 0$ have the property that $3\delta \in \Lambda(p_i, \eta)$ where $\eta \in \Gamma(X_i, \varepsilon/9)$.

Consider an index $j \geq i$ and select a $k \geq j$ as in Definition (6.1) with respect to j and a $\gamma \in \Gamma(X_j, \varepsilon/3)$ and so that p_k^* is a δ-map. Let φ, ψ: $K \to X_k$ be δ-close maps of a compactum K into X_k. Then $\varphi^* = h_k \circ \varphi$: $K \to X_k^*$ and $\psi^* = h_k \circ \psi$: $K \to X_k$ are δ-close and for every $x \in K$ there are $y, z \in X$ such that $\varphi^*(x) = p_k^*(y)$ and $\psi^*(x) = p_k^*(z)$. Since y and z are 3δ-close, $p_i^*(y)$ and $p_i^*(z)$ are η-close. But $p_i^*(y) = p_{ik}^* \circ p_k^*(y) = p_{ik}^*(\varphi^*(x))$ and $p_i^*(z) = p_{ik}^* \circ p_k^*(z) = p_{ik}^*(\psi^*(x))$ so that $p_{ik}^* \circ \varphi^*$ and $p_{ik}^* \circ \psi^*$ are η-close maps into X_i^*. Hence, there is an $(\varepsilon/9)$-homotopy H: $K \times I \to X_i^*$ with $H_0 = p_{ik}^* \circ \varphi^*$ and $H_1 = p_{ik}^* \circ \psi^*$. The choice of k

implies that there is a map $G: K \times I \to X_j^*$ such that $p_{ij}^* \circ G$ is $(\varepsilon/9)$-close to H and $p_{jk}^* \circ \varphi^*$ is γ-close to G_0 and $p_{jk}^* \circ \psi^*$ is γ-close to G_1. Since H is an $(\varepsilon/9)$-homotopy and p_{ij}^* is an $(\varepsilon/9)$-map, G is an $(\varepsilon/3)$-homotopy. On the other hand, the selection of γ insures that $p_{jk}^* \circ \varphi^*$ is $(\varepsilon/3)$-homotopic to G_0 and that $p_{jk}^* \circ \psi^*$ is $(\varepsilon/3)$-homotopic to G_1. Hence, $p_{jk}^* \circ \varphi^*$ and $p_{jk}^* \circ \psi^*$ are ε-homotopic (in X_j^*) so that $p_{jk} \circ \varphi$ and $p_{jk} \circ \psi$ are ε-homotopic (in X_j).

(6.4) REMARK. In condition (v) in the above theorem the strongly e^*-movable convergence cannot be replaced by the more restrictive strongly e-movable convergence [**Č7**]. Indeed, by [**MS1**, Theorem 1 and Example 9], the simple closed curve X can be represented as an inverse limit of a surjective ANR-sequence $\underline{X} = \{X_i, p_{i\,i+1}\}$ where each X_i is the 2-dimensional torus. If the sequence $\{X_i^*\}$ were to converge strongly e-movably to X, then [**Č7**, Theorem (3.5)] implies that X is homotopy equivalent to almost all X_i, an obvious contradiction.

However, with an additional assumption that every bonding map $p_{i\,i+1}$ is ARI, the characterization involving strongly e-movable convergence holds.

(6.5) THEOREM. *Let $\underline{X} = \{X_i, p_{i\,i+1}\}$ be a surjective inverse ANR-sequence and assume that each map p_{ii+1} is ARI. Then $X = \varprojlim \underline{X}$ is an ANR iff the sequence $\{X_i^*\}$ converges strongly e-movably to X.*

Proof. Suppose that X is an ANR. By (6.3) and (4.2), $\underline{X}$ is strongly e-movable and $\lim d_c^*(X_i^*, X) = 0$. Hence, it remains to show (see [**Č7**, Theorem (3.10)]) that for every $\varepsilon > 0$ there is an index i and a $\delta > 0$ such that δ-close $\mathcal{P}$-maps into X_j^* are ε-homotopic in X_j^* $(j \geq i)$.

For a given $\varepsilon > 0$, choose an index i_0 so large that p_{ij}^* is an $(\varepsilon/9)$-map for all i and j, $j \geq i \geq i_0$, and that (6.1) holds with respect to $\varepsilon/9$ and i_0. Let $\delta > 0$ satisfy $3\delta \in \Lambda(p_{i_0}^*, \eta)$, where $\eta \in \Gamma(X_{i_0}^*, \varepsilon/27)$, and let $i \geq i_0$ be such that p_j^* is a δ-map for all $j \geq i$. Observe that $\delta \in \Lambda(p_{i_0 j}^*, \eta)$ for all $j \geq i$.

Let $j \geq i$ and let $\varphi, \psi: K \to X_j^*$ be δ-close $\mathcal{P}$-maps into X_j^*. Let $\gamma > 0$ be such that $\gamma \in \Lambda(p_{i_0 J}^*, \eta)$ and $\gamma \in \Gamma(X_j^*, \varepsilon/6)$. Pick $k \geq j$ with respect to γ and j using the way in which i_0 was chosen. Since p_{jk}^* is an ARI map, there are maps $\varphi', \psi': K \to X_k^*$ with $p_{jk}^* \circ \varphi'$ γ-close to φ and $p_{jk}^* \circ \psi'$ γ-close to ψ. Hence, there is an $(\varepsilon/9)$-homotopy $H: K \times I \to X_{i_0}^*$ joining $p_{i_0 k}^* \circ \varphi'$ and $p_{i_0 k}^* \circ \psi'$. On the other hand, $p_{jk}^* \circ \varphi'$ and φ are $(\varepsilon/6)$-homotopic in X_j^* and $p_{jk}^* \circ \psi'$ and ψ are $(\varepsilon/6)$-homotopic in X_j^*. Choose G: $K \times I \to X_j^*$ such that $p_{i_0 j}^* \circ G$ is $(\varepsilon/9)$-close to H and G_0 is γ-close to

$p^*_{jk} \circ \varphi'$ and G_1 is γ-close to $p^*_{jk} \circ \psi'$. Clearly, G is an $(\varepsilon/3)$-homotopy so that φ and ψ are ε-homotopic in X^*_j.

The converse follows from [**Č7**, Lemma (3.3)].

(6.6) COROLLARY. *If the inverse limit X of an inverse ANR-sequence $\underline{X} = \{X_i, p_{i i+1}\}$ with ARI bonding maps is an ANR, then X is (simple) homotopy equivalent to almost all bonding spaces X_i.*

Proof. Combine (6.5) and [**Č7**, Theorem (6.1)].

In order to handle LC^n compacta, we define notions of e-n-movability and (internal) e-n-calmness for an inverse sequence and notions of e-n-movable and (internally) e^*-n-calm convergence for sequences of compacta in a metric space simply by restricting K in Definitions (4.1), (5.1), (5.4), (5.2), and (5.5), respectively, to compact ANR's of dimension $\leq n$. Similarly, if we require in Definitions (6.1) and (6.2) that (K, K_0) is a pair of at most n-dimensional ANR's, we get notions of strong e-n-movability (for inverse sequences) and strong e^*-n-movable convergence.

Consistent changes from arbitrary ANR's to ANR's of dimension $\leq n$ and from ANR's to LC^n compacta in our proofs provide the proof of the following.

(6.7) THEOREM. *For a surjective inverse LC^n-sequence $\underline{X} = \{X_i, p_{i i+1}\}$ the following are equivalent.*

(i) $X = \lim\limits_{\leftarrow} \underline{X}$ *is an LC^n compactum.*

(ii) $\underline{X}$ *is strongly e-$(n+1)$-movable.*

(iii) $\underline{X}$ *is both e-$(n+1)$-movable and (internally) e-$(n+1)$-calm.*

(iv) *The sequence $X^*_1, X^*_2, \ldots$ converges strongly e^*-$(n+1)$-movably to X.*

(v) *The sequence $X^*_1, X^*_2, \ldots$ converges both e-$(n+1)$-movably and (internally) e^*-$(n+1)$-calmly to X.*

There are also versions of (6.4), (6.5), and (6.6) for LC^n compacta. However, for $n = 0$ the assumptions in (6.5) is not necessary (we can always get maps φ' and ψ' because without loss of generality K can be chosen a single point and p_{ij}'s are onto). Since strongly e-1-movable convergence is clearly equivalent to 0-regular convergence [**Wh**] for locally connected compacta (see Theorem (3.10) in [**Č7**]), it follows that (6.7) includes Theorem 3 in [**FS**] as a special case.

Another interesting consequence of the method of proof of Theorem (6.3) is the following improvement of Geoghegan's Theorem (1.3) in [**Ge2**].

(6.8) THEOREM ([**Ge2**]). *Let* $\underline{X} = \{X_i, p_{i\,i+1}\}$ *be an inverse ANR-sequence with each bonding map* $p_{i\,i+1}$ *an approximate fibration* [**CD**]. *Then* $X = \varprojlim \underline{X}$ *is an ANR iff X is an FANR.*

Proof. Since every ANR is an FANR, it remains to prove that if X is an FANR (which is equivalent to Geoghegan's condition that X is shape equivalent to a CW-complex) then it must be an ANR. The approximate homotopy lifting property of approximate fibrations [**CD**] and the proof of (6.3) imply that X is $\mathcal{P}_{h0}$-*e*-movable [**Č1**], where $\mathcal{P}_{h0}$ is a class of all pairs $(K \times [0,1], K \times \{0\})$ with $K \in \mathcal{P}$. In other words, for every neighborhood U of X in Q and every $\varepsilon > 0$ there is a neighborhood V of X in Q, $V \subset U$, such that for every neighborhood W of X in Q there is a neighborhood W_0 of X in Q, $W_0 \subset W \cap V$, with the property that for every pair $(K, K_0) \in \mathcal{P}_{h0}$ and a map $f\colon (K, K_0) \to (V, W_0)$ there is an ε-homotopy $f_t\colon K \to U$, $0 \le t \le 1$, with $f_0 = f$, $f_1(K) \subset W$, and $f_1 | K_0 = f | K_0$. Hence, the theorem follows from the following theorem which gives a new characterization of compact ANR's.

(6.9) THEOREM. *A compactum X is an ANR iff it is both an FANR and* $\mathcal{P}_{h0}$-*e-movable.*

Proof. Every ANR is clearly an FANR and $\mathcal{P}_{h0}$-*e*-movable. Conversely, suppose X is a $\mathcal{P}_{h0}$-*e*-movable FANR. We shall prove that X is strongly *e*-movable. Consider X as a subset of the Hilbert cube Q and let a compact ANR neighborhood U of X in Q and an $\varepsilon > 0$ be given. Let $\eta \in \Gamma^*(U, \varepsilon/2)$, $0 < \eta < \varepsilon/2$. Choose a neighborhood V_1 of X in Q, $V_1 \subset U$, with respect to U and η using the fact that X is $\mathcal{P}_{h0}$-*e*-movable. The pick a neighborhood V of X in Q, $V \subset V_1$, such that for every neighborhood W of X in Q, there is a neighborhood W_0 of X in Q, $W_0 \subset V \cap W$, with the property that for every $\mathcal{P}_p$-map $f\colon (K, K_0) \to (V, W_0)$ there is a homotopy $f_t\colon K \to V_1$, $0 \le t \le 1$, with $f_0 = f$, $f_1(K) \subset W$, and $f_t | K_0 = f | K_0$ for all $t \in [0,1]$ (this requires X to be a pointed FANR which follows either directly from Hastings-Heller's theorem [**HH**] or one can easily verify that X is arcwise connected and thus pointed 1-movable [**KM**]). We claim that $\mathcal{P}_p^\varepsilon(U, V; X)$ holds.

Indeed, let W be an arbitrary neighborhood of X in Q. Inside $V \cap W$ pick a neighborhood $\overline{W}_0$ of X in Q using the choice of V_1 and then a smaller neighborhood W_0 of X in Q with respect to $\overline{W}_0$ using the way in which V was chosen. If $f\colon (K, K_0) \to (V, W_0)$ is a $\mathcal{P}_p$-map, then there is a homotopy $F\colon K \times [0,1] \to V_1$ with $F_0 = f$, $F_1(K) \subset \overline{W}_0$ and $F_t | K_0 = f | K_0$

for all $t \in [0,1]$. But, we know that η-close to F there is a map G: $K \times [0,1] \to W$ with $G_1 = F_1$. Clearly, $G | K_0 \times [0,1]$ is an η-homotopy between $G_0 | K_0$ and $G_1 | K_0 = f | K_0$ while G_0 and F_0 are $(\varepsilon/2)$-homotopic in U via a homotopy which agrees with G_{1-t} on K_0. Hence, by the homotopy extension theorem, F_0 is ε-homotopic in U to a map of K into W that equals $f | K_0$ on K_0.

Theorem (6.8) has an amusing corollary which to the best of my knowledge has not appeared in the literature.

(6.10) COROLLARY. *A compact metrizable topological group G is a Lie group iff G is an FANR.*

Proof. By results in [**Sz**], a compact metrizable topological group G can be represented as the inverse limit of an inverse sequence $\underline{X} = \{X_i, p_{i\,i+1}\}$ where each bonding space X_i is a manifold and each bonding map $p_{i\,i+1}$ is a locally trivial fibre map. Theorem (6.8) implies that G will be an ANR iff G is an FANR. Hence, the theorem follows from the corollary to Theorem 4 in [**Sz**].

The next application of our methods was also observed by McAuley and Robinson [**McR**].

(6.11) COROLLARY. *Let $\underline{X} = \{X_i, p_{i\,i+1}\}$ be an inverse LC^n-sequence with each bonding map $p_{i\,i+1}$ an UV^n-map (or, equivalently, a Σ^n-trivial map* [**Č1**]). *Then $X = \varprojlim \underline{X}$ is an LC^n compactum.*

Proof. By Lemma (6.7) in [**Č1**], $\underline{X}$ is strongly e-$(n+1)$-movable, so that X is an LC^n compactum by Theorem (6.7).

7. Dimension. In this final section we shall give a new characterization of inverse AANR_C-sequences whose inverse limits have dimension $\leq n$. As in the previous sections, the idea is the same, namely, to "rigidify" the corresponding result (1.4) in shape theory. However, the technique of proof differs from the one used in §§4–6 (that relied heavily on Fort-Segal embeddings and remetrizations of bonding spaces and the inverse limit) and utilize the following improvement of Lemma 1 in [**MR**]. In view of Theorem 8 in [**M2**], this is a special case of Proposition 1 in [**M2**].

(7.1) LEMMA. *Let $\underline{X} = \{X_i, p_{i\,i+1}\}$ be an inverse sequence of compacta and let Y be an $AANR_C$. Then the following assertions hold*:

(i) *For every $\varepsilon > 0$ and for every map f: $X \to Y$ there is an i^* such that for each $i \geq i^*$ there is a map f_i: $X_i \to Y$ with $d(f_i \circ p_i, f) < \varepsilon$.*

(ii) *If* $\varepsilon > 0$ *and* $f_i, g_i\colon X_i \to Y$ *are maps such that* $d(f_i \circ p_i, g_i \circ p_i) < \varepsilon$, *then there is an* $i^* \geq i$ *such that* $d(f_i \circ p_{ij}, g_i \circ p_{ij}) < \varepsilon$ *for every* $j \geq i^*$.

Proof. Since the proof of Lemma 1(ii) in [**MR**] does not use any assumptions on Y, it remains only to prove (i).

Let X^* denote a compactum described in the proof of Lemma 1 in [**MR**]. Recall that as a set X^* in the disjoint union $X \cup (\bigcup_{i>0} X_i)$ and that the basis for the topology of X^* is given by open subsets $U_i \subseteq X_i$ and by the sets $U_i^* = p_i^{-1}(U_i) \cup \bigcup_{j\geq i} p_{ij}^{-1}(U_i)$. In this topology on X^* both X_i and X inherit their original topologies, every neighborhood of X in X^* contains almost all X_i's, and for every $\varepsilon > 0$ there is an index i such that $d(p_j(x), x) < \varepsilon$ for all $j \geq i$ and $x \in X$.

Consider Y as a subset of Q and pick a compact ANR neighborhood V of Y in Q for which there is an $(\varepsilon/2)$-map $r\colon V \to Y$ [**C1**]. Extend $f\colon X \to Y$ to a map $f^*\colon U \to V$ of a neighborhood U of X in X^* and put $\tilde{f} = r \circ f^*\colon U \to Y$. Observe that $d(\tilde{f}|X, f) < \varepsilon/2$. For sufficiently large i, $j \geq i$ implies that $X_j \subseteq U$ so that $f_j = \tilde{f}|X_j$ is defined. By uniform continuity of $\tilde{f}$ on $X_i^* = X \cup \bigcup_{j\geq i} X_i$ there is a $\delta > 0$ such that $d(x, x') < \delta$ implies that $d(\tilde{f}(x), \tilde{f}(x')) < \varepsilon/2$ $(x, x' \in X_i^*)$. Since for sufficiently large j one has $d(p_j(x), x) < \delta$ for $x \in X$, one concludes that

$$d\big(f(x), f_j \circ p_j(x)\big) \leq d\big(f(x), \tilde{f}(x)\big) + d\big(\tilde{f}(x), \tilde{f}(p_j(x))\big) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $x \in X$.

(7.2) Definition. An inverse sequence $\underline{X} = \{X_i, p_{i\,i+1}\}$ is called *e-n-tame* provided for every $i > 0$ and every $\varepsilon > 0$ there is a $j \geq i$, an at most n-dimensional compactum K, and maps $\alpha\colon X_j \to K$ and $\beta\colon K \to X_i$ with $\beta \circ \alpha$ ε-close to p_{ij}.

Recall that a compactum X *approximately dominates* a compactum Y [**Č2**] if for every $\varepsilon > 0$ there are maps $f\colon X \to Y$ and $g\colon Y \to X$ with $d(f \circ g, \mathrm{id}_Y) < \varepsilon$.

(7.3) Theorem. *Let* $\underline{X} = \{X_i, p_{i\,i+1}\}$ *and* $\underline{Y} = \{Y_i, q_{i\,i+1}\}$ *be inverse* $AANR_C$*-sequences and let* $\varprojlim \underline{X} = (X, p_i)$ *and* $\varprojlim \underline{Y} = (Y, q_i)$. *If* $\underline{X}$ *is e-n-tame and* X *approximately dominates* Y, *then* $\underline{Y}$ *is also e-n-tame.*

Proof. Let an $i > 0$ and an $\varepsilon > 0$ be given. Since X approximately dominates Y, there are maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $d(f \circ g, \mathrm{id}_Y) \in \Lambda(q_i, \varepsilon/6)$. Hence,

$$d(q_i \circ f \circ g, q_i) < \varepsilon/6. \tag{1}$$

By Lemma (7.1)(i), there is an index $i' \geq i$ and a map $f_{i'}: X_{i'} \to Y_i$ with $d(f_{i'} \circ p_{i'}, q_i \circ f) < \varepsilon/6$. Therefore,

$$(2) \qquad d(f_{i'} \circ p_{i'} \circ g, q_i \circ f \circ g) < \varepsilon/6.$$

Combining (1) and (2), we get

$$(3) \qquad d(f_{i'} \circ p_{i'} \circ g, q_i) < \varepsilon/3.$$

Now, since the inverse $\mathrm{AANR}_{\mathrm{C}}$-sequence $\underline{X}$ is e-n-tame, there is an index $j' \geq i'$, an at most n-dimensional compactum K, and maps α': $X_{j'} \to K$ and β': $K \to X_{i'}$ such that $d(\beta' \circ \alpha', p_{i'j'}) \in \Lambda(f_{i'}, \varepsilon/3)$. Hence,

$$(4) \qquad d(f_{i'} \circ \beta' \circ \alpha', f_{i'} \circ p_{i'j'}) < \varepsilon/3.$$

Next, since $p_{j'} \circ g$: $Y \to X_{j'}$ is a map of Y into an $\mathrm{AANR}_{\mathrm{C}}$, by Lemma (7.1)(i), there is a $k \geq j'$ and a map g_k: $Y_k \to X_{j'}$ with $d(p_{j'} \circ g, g_k \circ q_k) \in \Lambda(f_{i'} \circ p_{i'j'}, \varepsilon/3)$. Thus,

$$(5) \qquad d(f_{i'} \circ p_{i'} \circ g, f_{i'} \circ p_{i'j'} \circ g_k \circ q_k) < \varepsilon/3$$

because $p_{i'} = p_{i'j'} \circ p_{j'}$. From (3) and (5) we get

$$(6) \qquad d(f_{i'} \circ p_{i'j'} \circ g_k \circ q_k, q_{ik} \circ q_k) < 2\varepsilon/3,$$

and from (4) we have

$$(7) \qquad d(f_{i'} \circ \beta' \circ \alpha' \circ g_k \circ q_k, f_{i'} \circ p_{i'j'} \circ g_k \circ q_k) < \varepsilon/3.$$

If we apply Lemma (7.1)(ii) to (6) and (7), we see that there is an index $j \geq k$ so that

$$(8) \qquad d(f_{i'} \circ p_{i'j'} \circ g_k \circ q_{kj}, q_{ij}) < 2\varepsilon/3,$$

and

$$(9) \qquad d(f_{i'} \circ \beta' \circ \alpha' \circ g_k \circ q_{kj}, f_{i'} \circ p_{i'j'} \circ g_k \circ g_{kj}) < \varepsilon/3.$$

Finally, (8) and (9) give us

$$(10) \qquad d(f_{i'} \circ \beta' \circ \alpha' \circ g_k \circ q_{kj}, q_{ij}) < \varepsilon.$$

Hence, if we put $\alpha = \alpha' \circ g_k \circ q_{kj}$: $Y_j \to K$ and $\beta = f_{i'} \circ \beta'$: $K \to Y_i$, the last inequality can be rewritten as $d(\beta \circ \alpha, q_{ij}) < \varepsilon$, which proves that $\underline{Y}$ is e-n-tame.

(7.4) COROLLARY. *Let* (X, p_i) *be the inverse limit of an inverse* $AANR_{\mathrm{C}}$*-sequence* $\underline{X} = \{X_i, p_{i\,i+1}\}$. *Then* $\dim X \leq n$ *iff* $\underline{X}$ *is e-n-tame.*

Proof. Consider X as a subset of the Hilbert cube Q. The corollary follows from the above theorem and Theorem (5.2) in [Č2], which says that $\dim X \leq n$ iff the inverse AANR_C-sequence $\underline{N} = \{N_i, q_{i\, i+1}\}$, where $N_1 \supset N_2 \supset \cdots$ is a decreasing sequence of compact ANR neighborhoods N_i of X in Q with $X = \bigcap_{i>0} N_i$ and $q_{i\, i+1}\colon N_{i+1} \to N_i$ $(j \geq i)$ are inclusions, is e-n-tame.

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