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ON THE REVERSE WEAK TYPE INEQUALITY FOR THE HARDY MAXIMAL FUNCTION AND THE WEIGHTED CLASSES $L(\log L)^k$

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Muckenhoupt has given a necessary and sufficient condition to be satisfied by the weight functions U and V in order that the Hardy-Littlewood maximal function Mf should satisfy a weighted weak type (1, 1) inequality. In this note, conditions on the weight functions U and V are given in order that the sense of this inequality may be reversed. This is then applied to give conditions which ensure that the integrability of Mf with respect to a weight implies that f belongs to a weighted Zygmund class $L\log L$, thus extending a result of Stein. Analogous results related to the strong maximal function and the classes $L(\log L)^k$ are also given. These extend certain results of Favo, Gatto and Gutiérrez.

If f is locally integrable on R^n, the Hardy-Littlewood maximal function Mf is defined by

$$(Mf)(x) = \sup|Q|^{-1}\int_Q |f|$$

where the supremum is taken over all cubes Q containing x. Here and henceforth, by "cube" we shall always mean "cube with sides parallel to the co-ordinate axis". As usual, $|E|$ denotes the Lebesgue measure of the measurable set E, and more generally if $U(x) \geq 0$ is defined on E we write $|E|_U = \int_E U(x)\,dx$. If Q is a given cube, RQ denotes the cube concentric with Q but with side R times as long. Q_0 will denote a fixed but arbitrary cube in R^n. Our first result is the following theorem.

THEOREM 1. *Suppose the non-negative weight functions U and V are defined on Q_0. If there is a constant C depending only on U and V such that*

$$(1)\qquad \frac{|Q|_U}{|Q|} \geq C \operatorname*{ess\,sup}_{x\in Q} V(x) \quad \textit{for all cubes } Q \subset Q_0,$$

then

$$(2)\quad |\{x \in Q_0 \colon (Mf)(x) > \lambda\}|_U \geq C2^{-n}\lambda^{-1}\int_{\{x\colon f(x)>\lambda\}} f(x)V(x)\,dx$$

holds for all non-negative f supported on Q_0 *with* $\int_{Q_0} f < \infty$ *and all* $\lambda \geq \lambda_0 = |Q_0|^{-1}\int_{Q_0} f$. *Conversely, if* (2) *holds for some constant C independent of all f which are characteristic functions of measurable sets* $E \subset Q_0$ *with* $|E| > 0$ *and for all* $\lambda \geq \lambda_0 = |Q_0|^{-1}|E|$, *then*

$$\frac{|(2Q) \cap Q_0|_U}{|Q|} \geq C4^{-n} \operatorname*{ess\,sup}_{x \in Q} V(x) \tag{3}$$

for all cubes $Q \subset Q_0$.

COROLLARY 1. *Let* $U(x) \geq 0$, $V(x) \geq 0$ *be defined on* R^n. *If there is a constant C depending only on U, V such that*

$$\frac{|Q|_U}{|Q|} \geq C \operatorname*{ess\,sup}_{x \in Q} V(x) \quad \textit{for all cubes } Q \subset R^n \tag{4}$$

then

$$|\{x \in R^n: (Mf)(x) > \lambda\}|_U \geq C2^{-n}\lambda^{-1}\int_{\{x: f(x) > \lambda\}} f(x)V(x)\,dx \tag{5}$$

holds for all non-negative f and all $\lambda > 0$. *Conversely, if* (5) *holds for all f which are characteristic functions of measurable sets* $E \subset R^n$ *with* $0 < |E| < \infty$ *and all* $\lambda > 0$, *then*

$$\frac{|2Q|_U}{|Q|} \geq C4^{-n} \operatorname*{ess\,sup}_{x \in Q} V(x) \quad \textit{for all cubes } Q \subset R^n. \tag{6}$$

If U satisfies the doubling condition

$$|2Q|_U \leq C|Q|_U \quad \text{for all cubes } Q$$

then (6) shows that (4) is both necessary and sufficient for (5). Observe that the example $U(x) = 1$ on $R^n - Q_0$, $U(x) = \infty$ otherwise, satisfies (4) with $U = V$ but does not satisfy the doubling condition. On the other hand, if U is locally integrable and satisfies (4) with $U = V$ then U necessarily satisfies the doubling condition. To see this, note that (4) is equivalent to

$$\frac{|Q|_U}{|Q|} \geq C\frac{|E|_V}{|E|} \quad \text{for all measurable } E \subset Q \text{ with } |E| > 0$$

so that with $U = V$ it follows that U satisfies the A_∞ condition of Muckenhoupt [**2**] and hence also the doubling condition.

The inequality (5) may be viewed as a reverse of the weak type (1, 1) inequality

$$|\{x \in R^n \colon (Mf)(x) > \lambda\}|_U \leq C\lambda^{-1} \int_{\{x \colon f(x) > \lambda/2\}} f(x)V(x)\, dx$$

which holds if and only if U, V satisfy the A_1 condition of Muckenhoupt [3], namely, $|Q|_U/|Q| \leq C \operatorname{ess\,inf}_{x \in Q} V(x)$ for all cubes Q. Restricting to the case $U = V$, the A_1 condition implies that U satisfies the doubling condition, and hence the weak type inequality and its reverse both hold in this case if and only if there are positive constants c_1 and c_2 such that $c_1 \leq U(x) \leq c_2$ for almost all $x \in R^n$.

If $U(x)^{-1}$ satisfies the A_1 condition, Hölder's inequality shows that (4) holds with $U = V$. The functions $U(x) = V(x) = |x|^\alpha$, $\alpha \geq 0$, provide further examples that satisfy (4).

Theorem 1 may be used to prove the following result.

THEOREM 2. *Suppose $f(x) \geq 0$ is supported on Q_0 and that there is a constant $C > 0$ such that the weight functions $U(x) \geq 0$, $V(x) \geq 0$ satisfy*

$$\frac{|Q|_U}{|Q|} \geq C \operatorname*{ess\,sup}_{x \in Q} V(x) \quad \textit{for all cubes } Q \subset Q_0 \tag{7}$$

Then

$$\int_{Q_0} (Mf)(x)U(x)\, dx < \infty$$

implies

$$\int_{Q_0} [f(x) \log^+ f(x)] V(x)\, dx < \infty.$$

Corollary 1 and Theorem 2 generalize some results of Stein [4] who considered the unweighted case, $U(x) \equiv V(x) \equiv 1$.

Let $1 \leq i \leq n$ and let M_i denote the Hardy-Littlewood maximal function in the ith variable, that is

$$(M_i f)(x) = \operatorname*{ess\,sup}_{a,\, b > 0} (a + b)^{-1} \int_{-a}^{b} |f(x_1, \ldots, x_{i-1}, x_i + t, x_{i+1}, \ldots, x_n)|\, dt$$

where $x = (x_1, \ldots, x_n)$. The following generalize certain results of Favo, Gatto and Gutiérrez [1] who considered the unweighted case, $U(x) \equiv 1$.

THEOREM 3. *Let $U(x) \geq 0$ be defined on R^n and let k be a fixed integer, $1 \leq k \leq n$. If there is a constant C depending only on U and k such that for each i, $1 \leq i \leq k$,*

$$(8)\quad (a+b)^{-1}\int_{-a}^{b} U(x_1,\ldots,x_{i-1},x_i+t,x_{i+1},\ldots,x_n)\,dt \geq C \operatorname*{ess\,sup}_{-a<t<b} U(x_1,\ldots,x_{i-1},x_i+t,x_{i+1},\ldots,x_n)$$

for all $a, b > 0$ and almost all $x \in R^n$, then

$$(9)\quad |\{x \in R^n: (M_k \cdots M_1 f)(x) > \lambda\}|_U \geq \frac{2^{-k}C^k}{(k-1)!\lambda}\int_{\{x: f(x)>\lambda\}} f(x)\left[\log^+\left(\frac{f(x)}{\lambda}\right)\right]^{k-1} U(x)\,dx$$

holds for all $\lambda > 0$ and all $f \geq 0$.

COROLLARY 2. *With the same hypothesis as Theorem 3 and $C > 0$, if $(M_k \cdots M_1 f)(x) \to 0$ as $|x| \to \infty$ and $\int_E (M_k \cdots M_1 f)(x)U(x)\,dx < \infty$ for every bounded set $E \subset R^n$, then*

$$\int_{R^n} |f(x)|\left[\log^+|f(x)|\right]^k U(x)\,dx < \infty.$$

Proof of Theorem 1. First we will prove that (1) implies (2). Since $\lambda \geq \lambda_0$ and f is supported on Q_0, $\lambda \geq |Q_0|^{-1}\int_{R^n} f$ so the Calderón-Zygmund decomposition [5, Theorem 4, p. 17] shows that there are pairwise disjoint cubes $\{Q_j\}$ satisfying

$$(10)\quad \bigcup Q_j \subset Q_0,$$

$$(11)\quad \lambda < |Q_j|^{-1}\int_{Q_j} f \leq 2^n\lambda$$

and

$$(12)\quad f(x) \leq \lambda \quad \text{if } x \notin \bigcup Q_j.$$

The definition of Mf and (11) shows that $(Mf)(x) > \lambda$ if $x \in Q_j$ and then (10) yields $\bigcup Q_j \subset \{x \in Q_0: (Mf)(x) > \lambda\}$ so that

$$(13)\quad |\{x \in Q_0: (Mf)(x) > \lambda\}|_U \geq \sum_j |Q_j|_U.$$

Now from (11) and hypothesis (1) it follows that

$$\sum_j |Q_j|_U \ge 2^{-n}\lambda^{-1}\sum_j |Q_j|_U\left(|Q_j|^{-1}\int_{Q_j} f(x)\,dx\right)$$
$$\ge 2^{-n}\lambda^{-1}C\sum_j\left(\operatorname*{ess\,sup}_{x\in Q_j} V(x)\right)\left(\int_{Q_j} f(x)\,dx\right)$$
$$\ge 2^{-n}\lambda^{-1}C\sum_j\int_{Q_j} f(x)V(x)\,dx = 2^{-n}\lambda^{-1}C\int_{\cup Q_j} f(x)V(x)\,dx,$$

while from (12) $\cup Q_j \supset \{x \in R^n: f(x) > \lambda\}$ so that

$$\int_{\cup Q_j} f(x)V(x)\,dx \ge \int_{\{x:\,f(x)>\lambda\}} f(x)V(x)\,dx.$$

Thus (1) implies (2).

Conversely, let $Q \subset Q_0$ be given. Let $\varepsilon > 0$ and select $E_\varepsilon \subset Q$ with $0 < |E_\varepsilon| < 2^{-n}|Q|$ so that $V(x) > \operatorname{ess\,sup}_{t\in Q} V(t) - \varepsilon$ for $x \in E_\varepsilon$. Then with $f(x) = 1$ if $x \in E_\varepsilon$, $f(x) = 0$ otherwise, and $\lambda = 2^n|E_\varepsilon|/|Q|$, elementary geometry shows that $(Mf)(x) \le \lambda$ if $x \notin 2Q$. Thus (2) yields

$$|(2Q)\cap Q_0|_U \ge |\{x \in Q_0: (Mf)(x) > \lambda\}|_U$$
$$\ge C4^{-n}\frac{|Q|}{|E_\varepsilon|}\int_{E_\varepsilon} V(x)\,dx \ge C4^{-n}|Q|\left(\operatorname*{ess\,sup}_{t\in Q} V(t) - \varepsilon\right).$$

Since ε is arbitrary we obtain (3). This proves Theorem 1.

Proof of Corollary 1. The proof that (5) implies (6) is similar to that used to prove that (2) implies (3) and is therefore omitted. Now if (4) holds and $f(x) \ge 0$, let Q_0 be a fixed cube and let $f_t(x) = f(x)$ if $f(x) \le t$ and $x \in tQ_0$, $f_t(x) = 0$ otherwise. For $\lambda > 0$, Theorem 1 yields

$$(14)\quad |\{x \in RQ_0: (Mf_t)(x) > \lambda\}|_U \ge C2^{-n}\lambda^{-1}\int_{\{x:\,f_t(x)>\lambda\}} f_t(x)V(x)\,dx$$

provided R satisfies $|RQ_0|^{-1}\int_{R^n} f_t \le \lambda$ and $R \ge t$. Note that $f_t(x) \uparrow f(x)$ and $(Mf_t)(x) \uparrow (Mf)(x)$ as $t \to \infty$. Hence (5) follows from (14) by the monotone convergence theorem, letting $R \to \infty$ first, then $t \to \infty$. This proves the Corollary.

Proof of Theorem 2. Assume that $V(x) > 0$ on a set of positive measure in Q_0 for otherwise there is nothing to prove. Then (7) shows that $|Q_0|_U > 0$. Hence, if $f(x) \not\equiv 0$ a.e. then $(Mf)(x) \ge |Q_0|^{-1}\int_{Q_0} f > 0$ for

$x \in Q_0$ and the hypothesis $\int_{Q_0}(Mf)(x)U(x)\,dx < \infty$ implies that f and U are integrable on Q_0, and in view of (7), V is also. Thus it suffices to show that

$$\int_{\{x:\, f(x)>\lambda_0\}} f(x)\log\left(\frac{f(x)}{\lambda_0}\right)V(x)\,dx < \infty \tag{15}$$

where $\lambda_0 = |Q_0|^{-1}\int_{Q_0} f$.

Fubini's Theorem shows that the left side of (15) is equal to

$$\int_{\lambda_0}^{\infty}\frac{d\lambda}{\lambda}\int_{\{x:\, f(x)>\lambda\}} f(x)V(x)\,dx$$

and Theorem 1 shows that this is bounded above by

$$C^{-1}2^n\int_{\lambda_0}^{\infty}|\{x\in Q_0:\,(Mf)(x)>\lambda\}|_U\,d\lambda.$$

This integral is bounded by

$$\int_0^{\infty}|\{x\in Q_0:\,(Mf)(x)>\lambda\}|_U\,d\lambda = \int_{Q_0}(Mf)(x)U(x)\,dx$$

so we obtain (15) and the theorem is proved.

Proof of Theorem 3. Observe first that for any fixed i, $1 \le i \le k$, Corollary 1 and (8) show that

$$\int_{\{x_i\in R^1:\,(M_if)(x)>\lambda\}} U(x)\,dx_i \ge \frac{2^{-1}C}{\lambda}\int_{\{x_i:\, f(x)>\lambda\}} f(x)U(x)\,dx_i.$$

Integrating this inequality over the remaining variables yields

$$|\{x\in R^n:\,(M_if)(x)>\lambda\}|_U \ge \frac{2^{-1}C}{\lambda}\int_{\{x:\, f(x)>\lambda\}} f(x)U(x)\,dx. \tag{16}$$

Now the proof proceeds by induction. As we have just proved, (9) holds with $k = 1$. Assume that (9) holds for some k, $k \le n-1$. Then (16) yields

$$\begin{aligned}
&|\{x\in R^n:\,(M_{k+1}\cdots M_1f)(x)>\lambda\}|_U\\
&\qquad\ge \frac{2^{-1}C}{\lambda}\int_{\{x:\,(M_k\cdots M_1f)(x)>\lambda\}}(M_k\cdots M_1f)(x)U(x)\,dx\\
&\qquad\ge 2^{-1}C\int_{\{x:\,(M_k\cdots M_1g)(x)>1\}}(M_k\cdots M_1g)(x)U(x)\,dx
\end{aligned}$$

where we have set $g(x) = f(x)/\lambda$ if $f(x) > \lambda$ and $g(x) = 0$ otherwise. Now

$$\begin{aligned}\int_{\{x:\,(M_k\cdots M_1 g)(x)>1\}} &(M_k \cdots M_1 g)(x) U(x)\,dx \\ &= \int_0^\infty |\{x\colon (M_k \cdots M_1 g)(x) > \max(1,\alpha)\}|_U\, d\alpha \\ &\geq \int_1^\infty |\{x\colon (M_k \cdots M_1 g)(x) > \alpha\}|_U\, d\alpha\end{aligned}$$

so that the inductive hypothesis shows this is bounded below by

$$\begin{aligned}&\frac{2^{-k}C^k}{(k-1)!}\int_1^\infty \frac{d\alpha}{\alpha}\int_{\{x:\,g(x)>\alpha\}} g(x)\left[\log\left(\frac{g(x)}{\alpha}\right)\right]^{k-1} U(x)\,dx \\ &= \frac{2^{-k}C^k}{k!}\int_{\{x:\,g(x)>1\}} g(x)[\log g(x)]^k U(x)\,dx \\ &= \frac{2^{-k}C^k}{k!\lambda}\int_{\{x:\,f(x)>\lambda\}} f(x)\left[\log\left(\frac{f(x)}{\lambda}\right)\right]^k U(x)\,dx.\end{aligned}$$

Thus we have (9) for $k+1$ and the proof is complete.

Proof of Corollary 2. Since $(M_k \cdots M_1 f)(x) \to 0$ as $|x| \to \infty$ the set $E = \{x\colon (M_k \cdots M_1 f)(x) \geq 1\}$ is bounded. Thus, Theorem 3 shows

$$\begin{aligned}\infty &> \int_E (M_k \cdots M_1 f)(x) U(x)\,dx \geq \int_1^\infty |\{x\colon (M_k \cdots M_1 f)(x) > \lambda\}|_U\, d\lambda \\ &\geq \frac{2^{-k}C^k}{(k-1)!}\int_1^\infty \frac{d\lambda}{\lambda}\int_{\{x:\,f(x)>\lambda\}} f(x)\left[\log\left(\frac{f(x)}{\lambda}\right)\right]^{k-1} U(x)\,dx \\ &= \frac{2^{-k}C^k}{k!}\int_{R^n} f(x)[\log^+ f(x)]^k U(x)\,dx\end{aligned}$$

where we have used Fubini's Theorem to obtain the last equality. This proves the Corollary.

References

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