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LOCAL SOLVABILITY OF NONSTATIONARY LEAKAGE PROBLEM FOR IDEAL INCOMPRESSIBLE FLUID, 2

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In this paper the existence and uniqueness of solutions of the initial boundary value problem for the Euler equations for an incompressible fluid in a bounded domain $\Omega \subset \mathbf{R}^3$ is proved. As boundary conditions the velocity vector and the pressure on boundary parts the fluid enters and leaves the domain through are assumed, respectively. The existence of solutions in Sobolev spaces for domains with dihedral angles π/n, $n = 2, 3, \ldots$, is shown.

1. Introduction. The existence and uniqueness of the initial boundary value problem solutions for the Euler equations (for an incompressible fluid) in a bounded domain with impermeable boundaries has been proved in [**1**], [**2**], [**5**], [**10**], [**14**], [**16**]. In this case, the only admissible boundary condition is the vanishing of the normal component of the velocity on the boundary.

The problems with a nonvanishing normal component of velocity on the boundary, which correspond to the flow through ducts and tunnels, have been considered in [**8**]–[**11**], [**20**], [**21**], [**23**], [**24**]. The investigations of the above-mentioned leakage problems answer the question: What physical quantities (the velocity, vorticity or fluid pressure) should be known at the inlet and outlet in order to determine the flow uniquely?

In this paper as a mathematical idealization, a leakage problem for an incompressible ideal fluid described by the Euler equations is considered. We consider the domains with the following parts of boundaries: S_1 — the part through which the fluid enters the domain: S_2 — the part through which fluid leaves; and S_0 — the part on which the normal component of a velocity vector is zero. We assume $S_1 \cap S_2 = \varnothing$.

It has already been stated [**8**]–[**11**], [**20**], [**21**], [**23**], [**24**] that when only the normal component of a velocity is given on the boundary, the leakage problem is not well posed. The uniqueness and existence of solutions of this problem when, additionally, the tangent components of the vorticity vector were assumed on S_1 have been proved in [**10**], [**11**], [**12**], [**20**], [**21**]. In [**8**] the tangent components of the velocity vector on S_1 were additionally assumed.

In this paper we prove the uniqueness and existence of the leakage problem solutions for a given velocity vector on S_1 and a pressure on S_2.

In §2 we introduce some notations about considered domains and some Banach spaces. In §3 the leakage problem for the Euler equations is determined, and a method of successive approximations, which is used to prove the uniqueness and existence of solutions, is formulated. In §4 we obtain some a priori estimates, and at last in §5 we prove the uniqueness and existence of the considered leakage problem solutions. The author is very indebted to Professor J. Heywood for fruitful discussions.

2. Notations. In this paper we shall consider the following two kinds of bounded domains $\Omega \subset \mathbf{R}^3$:

(I) nonsimply connected domains with a smooth boundary $\partial\Omega = S_1 \cup S_2$; $S_1 \cap S_2 = \varnothing$; S_i, $i = 1, 2$, are smooth surfaces of class C^k, $k \geq 1$ an integer;

(II) simply connected domains with a nonsmooth boundary $\partial\Omega = S_0 \cup S_1 \cup S_2$; S_ν, $\nu = 0, 1, 2$, are smooth surfaces of class C^k, $k \geq 1$ an integer; $S_1 \cap S_2 = \varnothing$; the dihedral angle between tangent spaces $T_x S_0$ and

$$T_x S_i, x \in L_i := S_i \cap S_0, i = 1, 2,$$

is equal to π/n, $n = 2, 3, \ldots$.

In a neighbourhood $U(q)$, $q \in S_1$, we introduce an orthonormal curvilinear system of coordinates, $(\tau_1(x), \tau_2(x), n(x))$, $x \in U(q)$. The surface $S_1 \cap U(q)$ is determined by the equation $n(x) = 0$. Hence, for $n(x) = 0$, τ_1, τ_2 are coordinates on S_1. $(\bar{\tau}_1(x), \bar{\tau}_2(x), \bar{n}(x))$ be the orthonormal basis corresponding to the coordinate system such that for $x \in S_1$, $\bar{\tau}_1(x)$, $\bar{\tau}_2(x)$ are vectors tangent to S_1, and $\bar{n}(x)$ is the outward vector normal to S_1.

We shall investigate the problem in Sobolev spaces. We denote the norm of the Sobolev spaces $W_r^l(\Omega)$ and $L_r(\Omega)$ by $\| \; \|_{l,r,\Omega}$ and $\| \; \|_{r,\Omega}$, respectively, and the norm of the space $W_r^{l-1/r}(S)$ by $\| \; \|_{l-1/r,r,S}$. Let B be a Banach space, k a nonnegative integer and T some positive constant. $L_\infty^k(0, T; B)$ is the Banach space of functions $f(t)$ on $[0, T]$ which have values in B for every fixed $t \in [0, T]$ and are k-times boundedly differentiable with respect to $t \in [0, T]$ in the topology of the space B. Let us introduce the space

$$\Pi_{k,r,\infty}^l(\Omega^T) = \bigcap_{i=k}^{l} L_\infty^{l-i}\big(0, T; W_r^i(\Omega)\big),$$

where $\Omega^T = \Omega \times [0, T]$ (similarly, $\Pi^{l'}_{k,r,\infty}(S^T_\nu)$, $l' = l - 1/r$, where $S^T_\nu = S_\nu \times [0, T]$, $\nu = 0, 1, 2$), with the norm

$$|u|_{l,k,r,\Omega^T} = \sup_{t\in[0,T]} \sum_{i=k}^{l} \|D_t^{l-i} u\|_{i,r,\Omega}, \tag{2.1}$$

and the space $\Gamma^l_{k,r}(\Omega)$ with the norm

$$|u|_{l,k,r,\Omega} = \sum_{i=k}^{l} \|D_t^{l-i} u\|_{i,r,\Omega}. \tag{2.2}$$

The index Ω will be omitted in notation of all norms.

3. Statement of the problem. Let us consider the Euler equations in Ω^T:

$$v_t + v \cdot \nabla v + \nabla p = f, \tag{3.1}$$

$$\operatorname{div} v = 0, \tag{3.2}$$

with initial condition

$$v|_{t=0} = a(x), \qquad \operatorname{div} a = 0, \tag{3.3}$$

and boundary conditions

$$\begin{aligned} &v|_{S_1} = \eta \quad \text{such that } \eta \cdot \bar{n} = -d,\ d \geq d_0 = \text{const} > 0, \\ &v \cdot \bar{n}|_{S_0} = 0, \end{aligned} \tag{3.4}$$

$$p|_{S_2} = \pi(x', t), \qquad x' \in S_2. \tag{3.5}$$

From (3.3) and (3.4) we have the following compatibility conditions:

$$\eta|_{t=0} = a|_{S_1}, \qquad a \cdot \bar{n}|_{S_0} = 0. \tag{3.6}$$

Our aim is to prove the existence and uniqueness of solutions of the problem (3.1)–(3.5). As we do not know any method to solve our problem directly from (3.1)–(3.5), we replace this problem by the equivalent system of problems. At the beginning let us prove the following lemma:

LEMMA 3.1. *Let v be a given function of class $C^{1,\alpha}(\Omega^t)$, $p \in C^{2,\alpha}(\Omega^t)$, $\eta \in C^{1,\alpha}(S^t_1)$, vectors $\bar{\tau}_1$, $\bar{\tau}_2$, $\bar{n}$ belong to C^1 in a neighbourhood of S_1 and equation* (3.2) *is satisfied on S_1.*

Then the initial and boundary conditions (3.3), (3.5) *and equations* (3.1), (3.2) *determine the following well-posed elliptic problem for p:*

(A)

$$\Delta p = \operatorname{div} f - v^k_{x^l} v^l_{x^k}, \tag{3.7}$$

$$P|_{S_2} = \pi(x', t), \qquad x' \in S_2, \tag{3.8}$$

$$(3.9)\quad \left.\frac{\partial p}{\partial n}\right|_{S_1} = -\eta_{n,t} + f_n|_{S_1} + \sum_{\mu=1}^{2}\left(\eta_n\eta_{\mu,\tau_\mu} + \eta_n\eta_\mu \operatorname{div}\bar{\tau}_\mu - \eta_\mu\eta_{n,\tau_\mu}\right) + \sum_{\mu,\nu=1}^{2}\eta_\mu\eta_\nu\bar{\tau}_\nu\cdot\bar{n}_{,\tau_\mu} + \eta_n\left(\sum_{\nu=1}^{2}\eta_\nu\bar{\tau}_\nu\cdot\bar{n}_{,n} + \eta_n \operatorname{div}\bar{n}\right) \equiv g(\eta,\bar{\tau},\bar{n}),$$

$$(3.10)\quad \left.\frac{\partial p}{\partial n}\right|_{S_0} = f_n|_{S_0} - v^k v\cdot\bar{n}_{x^k},$$

where $f_n = f\cdot\bar{n}$, $\eta_n = \eta\cdot\bar{n}$, $\eta_\nu = \eta\cdot\bar{\tau}_\nu$, $\nu = 1, 2$. In the case of nonsimply connected domains, condition (3.10) does not appear.

Proof. Using (3.1) and $v\cdot\bar{n}|_{S_0} = 0$, we obtain (3.7) and (3.10), respectively **[16]**. It remains to obtain (3.9) only. Multiplying (3.1) by $\bar{n}$, projecting the result on S_1 and using curvilinear coordinates we obtain

$$(3.11)\quad \left.\frac{\partial p}{\partial n}\right|_{S_1} = \left(f\cdot\bar{n} - v_t\cdot\bar{n} - v^k v_{x^k}\cdot\bar{n}\right)|_{S_1} = f_n|_{S_1} - \eta_{n,t} - \sum_{\mu=2}^{2}\eta_\mu\eta_{n,\tau_\mu} - \eta_n v_{n,n}|_{S_1} + \eta^k\eta\cdot\bar{n}_{,x^k},$$

where $\eta = \sum_{\mu=1}^{2}\eta_\mu\bar{\tau}_\mu + \eta_n\bar{n}$. In (3.11) the unknown quantity $v_{n,n}|_{S_1}$ appears. To calculate it we apply the operator div to (3.1), and using (3.7) we obtain

$$(3.12)\quad \left(\frac{\partial}{\partial t} + v^k\frac{\partial}{\partial x^k}\right)\operatorname{div} v = 0.$$

Now we introduce curves determined by the equations

$$(3.13)\quad \frac{dy}{ds} = v(y(x,t;s),s), \qquad y(x,t;t) = x,$$

where s is a parameter, $0 \le s \le t$. We classify these curves into two disjoined sets (a), (b):

(a) $y(x,t;s)\in\Omega$ for every $s\in[0,t)$,

(b) there exists a moment $t_*(x,t)\in[0,t)$ such that $y(x,t;t_*(x,t)) \in S_1$.

Equation (3.12) implies only that $\operatorname{div} v = \text{const}$ on curves (3.13). However, according to (3.2), $\operatorname{div} v = 0$ in Ω. (3.2) is satisfied on curves of family (a) because the initial values are such that $\operatorname{div} v|_{t=0} = \operatorname{div} a = 0$.

Initial values for (3.12) for curves of family (b) are given by boundary values on which (3.2) imposes the following restriction:

$$\operatorname{div} v|_{t=t_*(x,t)} = \operatorname{div} v|_{S_1} = 0. \tag{3.14}$$

Using the curvilinear coordinate system, (3.14) yields

$$v_{n,n}|_{S_1} = -\sum_{\mu=1}^{2} \left(\eta_{\mu,\tau_\mu} + \eta_\mu \operatorname{div} \bar{\tau}_\mu\right) - \eta_n \operatorname{div} \bar{n}. \tag{3.15}$$

Using (3.15) in (3.11) we get

$$\left.\frac{\partial p}{\partial n}\right|_{S_1} = f_n|_{S_1} - \eta_{n,t} + \sum_{\mu=1}^{2} \left(\eta_n \eta_{\mu,\tau_\mu} + \eta_\mu \eta_n \operatorname{div} \bar{\tau}_\mu - \eta_\mu \eta_{n,\tau_\mu}\right) + \eta_n^2 \operatorname{div} \bar{n} + \eta^k \eta \cdot \bar{n}_{,x^k}, \tag{3.16}$$

which implies (3.9). This completes the proof.

Now let us consider the system of problems (A, B), where (B) is defined as:

$$\text{(B)}\qquad \begin{aligned} &v_t + v \cdot \nabla v = -\nabla p + f, \\ &v|_{t=0} = a, \\ &v|_{S_1} = \eta, \quad v_n|_{S_0} = 0. \end{aligned}$$

Lemma 3.1 implies that problems (A, B) and (3.1)–(3.5) are equivalent.

To prove the existence and uniqueness of solutions of the problem (3.1)–(3.5), we use the following method of successive approximations for the equivalent problem (A, B):

$$(\mathrm{A}^s_m)\qquad \begin{aligned} &\Delta D_t^s \overset{m}{p} = D_t^s\left(\operatorname{div} f - \overset{m}{v}{}^i_{,x^k} \overset{m}{v}{}^k_{,x^i}\right), \\ &\left. D_t^s \overset{m}{p}\right|_{S_2} = D_t^s \pi(x', t), \qquad x' \in S_2, \\ &\left.\frac{\partial}{\partial n} D_t^s \overset{m}{p}\right|_{S_1} = D_t^s g(\eta, \bar{\tau}, \bar{n}), \\ &\left.\frac{\partial}{\partial n} D_t^s \overset{m}{p}\right|_{S_0} = D_t^s\left(f_n + \overset{m}{v}{}^k \overset{m}{v} \cdot \bar{n}_{,x^k}\right)\Big|_{S_0}, \end{aligned}$$

where for nonsimply connected domains (I) the boundary condition on S_0 does not appear, $g(\eta, \bar{\tau}, \bar{n})$ is determined by (3.9), and

$(\mathrm{B}^{\mathrm{s}}_{\mathrm{m}})$ (3.17)

$$\left(D_t^s \overset{m+1}{v}\right)_{,t} + \overset{m}{v} \cdot \nabla D_t^s \overset{m+1}{v} = -\sum_{j=0}^{s-1} D_t^{s-j} \overset{m}{v} \cdot \nabla D_t^j \overset{m+1}{v} - D_t^s\left(\nabla \overset{m}{p} - f\right),$$

(3.18)
$$D_t^s \overset{m+1}{v}\Big|_{t=0} = \begin{cases} a & \text{for } s = 0, \\ -D_t^{s-1}\left(\overset{m}{v} \cdot \nabla \overset{m+1}{v} + \nabla \overset{m}{p} - f\right)\Big|_{t=0} & \text{for } s \geq 1, \end{cases}$$

(3.19)
$$D_t^s \overset{m+1}{v}\Big|_{S_1} = D_t^s \eta,$$

where $m = 0, 1, \ldots,$ $\overset{0}{v} = a$ and $s = 0, \ldots, l-1$. For a given $\overset{m}{v}$ the problem $(\mathrm{A}^{\mathrm{s}}_{\mathrm{m}})$ constitutes an elliptic problem on $\overset{m}{p}$, and for a given $\overset{m}{p}$ the problem $(\mathrm{B}^{\mathrm{s}}_{\mathrm{m}})$ constitutes an evolution problem on $\overset{m+1}{v}$. Taking $\overset{0}{v}$, from $(\mathrm{A}^{\mathrm{s}}_{\mathrm{m}})$ we can calculate $\overset{0}{p}$, then from $(\mathrm{B}^{\mathrm{s}}_{1})$ we can calculate $\overset{1}{v}$, and so on.

On each step a solution of the problem $(\mathrm{B}^{0}_{\mathrm{m}})$ is such that

$(\mathrm{C}_{\mathrm{m}})$ (3.20)
$$\left(\partial_t + \overset{m}{v} \cdot \nabla\right) \operatorname{div} \overset{m+1}{v} = \overset{m}{v}{}^l_{,x^k}\left(\overset{m}{v}{}^k_{,x^l} - \overset{m}{v}{}^k_{,x^l}\right),$$

(3.21)
$$\operatorname{div} \overset{m+1}{v}\Big|_{t=0} = 0,$$

(3.22)
$$\operatorname{div} \overset{m+1}{v}\Big|_{S_1} = 0.$$

Problem $(\mathrm{C}_{\mathrm{m}})$ implies $\operatorname{div} \overset{m}{v} \neq 0$ for each $m > 0$.

Now let us consider problems $(\mathrm{A}^{\mathrm{s}}_{\mathrm{m}})$, $(\mathrm{B}^{\mathrm{s}}_{\mathrm{m}})$ separately. We shall prove existence and uniqueness of solutions of these problems.

LEMMA 3.2. *Let us assume $r > 3/l$, $l \geq 1$, $f \in \Gamma_{l,r}^{l+1}(\Omega)$, $\overset{m}{v} \in \Gamma_{1,r}^{l+1}(\Omega)$, $\pi \in \Gamma_{2,r}^{l+2-1/r}(S_2)$, $\eta \in \Gamma_{1,r}^{l+2-1/r}(S_1)$ and the smooth parts of the boundary* *are of class* C^{l+3}. *Then for* $t \in [0, T]$ *there exists a unique solution* $\overset{m}{p} \in \Gamma_{2,r}^{l+2}(\Omega)$ *of problem* (A_m^s), $s = 0, \dots, l$, *such that*

$$\left|\overset{m}{p}\right|_{l+2,2,r,\Omega} \le C_1 + C_2 \left|\overset{m}{v}\right|^2_{l+1,1,r,\Omega}, \tag{3.23}$$

where C_1 *depends on* l, r, Ω, $|f|_{l+1,1,r,\Omega}$, $|\eta|_{l+2-1/r,1,r,S_1}$, $|\pi|_{l+2-1/r,2,r,S_2}$ *and on bounds of* $(l+3)$*th derivative of the boundary, and* C_2 *depends on* r, Ω.

Proof. The existence of solutions of problem (A_m) for a domain with edges was shown in **[19]**. For nonsimply connected domains the existence of solutions of this problem is well known. Therefore, the following estimate for these problems is valid:

$$\begin{aligned}\left|\overset{m}{p}\right|_{l+2,2,r,\Omega} \le C\Big(&\left|\operatorname{div} f + \overset{m}{v}{}^i_{x^k} \overset{m}{v}{}^k_{x^i}\right|_{l,0,r,\Omega} \\ &+ |\pi|_{l+2-1/r,2,r,S_2} + \left|\overset{m}{v}{}^k \overset{m}{v} \cdot \bar{n}_{x^k} + f \cdot \bar{n}\right|_{l+1-1/r,1,r,S_0} \\ &+ \Big|\eta_{n,t} + \eta_\mu \eta_{n,\tau_\mu} - \eta_n\big(\eta_{\nu,\tau_\nu} + \eta_\nu \operatorname{div} \bar{\tau}_\nu + \eta_n \operatorname{div} \bar{n}\big) \\ &- \eta_k \eta \cdot \bar{n}_{x^k} - f \cdot \bar{n}\Big|_{l+1-1/r,1,r,S_1}\Big),\end{aligned} \tag{3.24}$$

where C depends on l, r, Ω. (3.23) follows (3.24). This concludes the proof.

LEMMA 3.3. *Let the initial data functions satisfy the restriction*

$$a \cdot \bar{n}|_{S_2} \ge a_0 = \text{const} > 0, \tag{3.25}$$

and

$$\overset{m}{p} \in C^{2,\alpha}(\bar{\Omega}^T), \quad f \in C^{1,\alpha}(\bar{\Omega}^T), \quad a \in C^{1,\alpha}(\bar{\Omega}),$$

$$\eta \in C^{2,\alpha}(S_1^T), \quad \overset{m}{v} \in C^{1,\alpha}(\bar{\Omega}^T),$$

where $\bar{\Omega}^T = \Omega \times [0, T]$. *If* $|\overset{m}{v}_t| \le C = \text{const}$, $m > 0$, *there exists a unique solution of problem* (B_m^0) *for* $t \in [0, T_1]$, *where*

$$T_1 = \frac{a_0}{C}, \tag{3.26}$$

such that $\overset{m+1}{v} \in C^{1,\alpha}(\bar{\Omega}^{T_1})$. *Moreover, if* $D_t^\sigma \overset{m}{p} \in C^{2,\alpha}(\bar{\Omega}^T)$, $D_t^\sigma f \in C^{1,\alpha}(\bar{\Omega}^T)$, $D_t^\sigma \eta \in C^{2,\alpha}(S_1^T)$, $D_t^\sigma \overset{m}{v} \in C^{1,\alpha}(\bar{\Omega}^T)$, $D_t^\sigma \overset{m+1}{v} \in C^{1,\alpha}(\bar{\Omega}^T)$, $\sigma = 0,\dots,s-1$, *and* (3.26) *is satisfied, there exists a solution of problem* $(\mathrm{B}_\mathrm{m}^\mathrm{s})$ *for* $t \in [0, T_1]$ *such that* $D_t^s \overset{m+1}{v} \in C^{1,\alpha}(\bar{\Omega}^{T_1})$.

Proof. At first we have to show that the fluid leaves the domain Ω through S_2. From the assumptions of the lemma $\overset{m}{v}_n|_{S_2} \geq a_0 - ct$, so for $t \leq T_1$, $\overset{m}{v}_n|_{S_2} \geq 0$. To show that problem $(\mathrm{B}_\mathrm{m}^0)$ is well posed, we introduce the characteristic curves of (3.17) determined by the equations

$$\frac{dy}{ds} = \overset{m}{v}(y(x,t;s),s), \qquad y(x,t;t) = x, \tag{3.27}$$

where s is a parameter, $0 \leq s \leq t$. We classify these curves into two disjoined sets (a), (b) (see the proof of Lemma 3.1). Then (3.17) can be written as

$$\frac{d}{ds}\overset{m+1}{v}(y(x,t;s),s) = -\nabla_y \overset{m}{p}(y(x,t;s),s) + f(y(x,t;s),s). \tag{3.28}$$

For each characteristic curve (3.27), equation (3.28) represents a corresponding ordinary differential equation. The initial values for (3.28) on characteristic curves belonging to (a) or (b) are determined by (3.18) or (3.19), respectively. This shows that $(\mathrm{B}_\mathrm{m}^0)$ is well posed. The existence and differential properties of the solution of $(\mathrm{B}_\mathrm{m}^0)$ result from the expression obtained by integrating (3.28) with respect to s using initial values determined by (3.18) and (3.19). Similar considerations are valid for problems $(\mathrm{B}_\mathrm{m}^\mathrm{s})$. This concludes the proof.

4. A priori estimates. In this section we obtain some a priori estimates of solutions

$$D_t^s \overset{m+1}{v} \in W_r^{l-s}(\Omega), \quad D_t^s \overset{m}{p} \in W_r^{l+1-s}(\Omega),$$

$$r > \frac{3}{l-1}, s = 0,\dots,l-1, m = 0,\dots, l \geq 2,$$

of problems (A^s_m), (B^s_m), respectively. We distinguish the case $l = 2$, because then the solutions belong to the largest admissible Sobolev space. Moreover, in this case calculations can be done explicitly.

LEMMA 4.1. *Let us assume*:
(a) $T \le T_1$, $r > 3$, $d \ge d_0 > 0$;
(b) $\eta \in \Gamma^2_{0,r}(S_1)$, $f \in \Gamma^2_{1,r}(\Omega)$;
(c) S_1 *is of class* C^3;
(d) $\overset{m-1}{p}, \overset{m}{p} \in \Gamma^3_{2,r}(\Omega)$, $\overset{m}{v} \in W^2_r(\Omega)$, $m \ge 1$;
(e) (3.25) *is satisfied.*
Let $\overset{m+1}{v}$ *be a solution of* (B^0_m) *then for* $t \in [0, T_1]$ *the following inequality is valid*:

$$\frac{d}{dt}\left\|\overset{m+1}{v}\right\|^r_{2,r} \le C_1 + C_2\left\|\overset{m}{p}\right\|^r_{3,r} + C_3\left\|\overset{m}{p}_t\right\|^r_{2,r} + C_4\left\|\overset{m}{p}\right\|^r_{3,r}\left\|\overset{m-1}{p}\right\|^r_{3,r} + C_5\left\|\overset{m}{v}\right\|_{2,r}\left\|\overset{m+1}{v}\right\|^r_{2,r} + \left(\left\|\overset{m}{p}\right\|_{3,r} + C_6\right)\left\|\overset{m+1}{v}\right\|^{r-1}_{2,r}, \tag{4.1}$$

where

$$C_1 = C_1(d_0, S_1, |\eta|_{2,0,r,S_1}, |f|_{2,1,r,\Omega}), \quad C_2 = C_2(d_0, S_1, |\eta|_{2,1,r,S_1}),$$
$$C_3 = C_3(d_0, S_1, \|\eta\|_{2,r,S_1}), \quad C_4 = C_4(S_1), \quad C_5 = C_5(r, \Omega), \quad C_6 = \|f\|_{2,r}.$$

Proof. From (3.17) we have

$$\sum_{s=1}^{3}\sum_{|j|\le 2}\int_\Omega D^j_x\left(\overset{m+1}{v}{}^s_{\,t} + \overset{m}{v}{}^k\,\overset{m+1}{v}{}^s_{\,x^k} + \nabla_s\overset{m}{p} - f^s\right) \times D^j_x\overset{m+1}{v}{}^s\left|D^j_x\overset{m+1}{v}{}^s\right|^{r-2} = 0, \tag{4.2}$$

where

$$D^j_x = \frac{\partial^{j_1}}{\partial x_1^{j_1}}\frac{\partial^{j_2}}{\partial x_2^{j_2}}\frac{\partial^{j_3}}{\partial x_3^{j_3}}, \qquad j_1 + j_2 + j_3 = |j|.$$

From (4.2) we have

$$(4.3)\quad \frac{1}{r}\,\frac{d}{dt}\left\|\overset{m+1}{v}\right\|_{2,r}^{r} \le -\frac{1}{r}\int_{\partial\Omega}\overset{m}{v}\cdot\bar{n}\left(\left|\overset{m+1}{v}\right|^{r}+\left|\overset{m+1}{v}_{x}\right|^{r}+\left|\overset{m+1}{v}_{xx}\right|^{2}\right)ds$$

$$+\max_{\Omega}\left|\overset{m}{v}_{x}\right|\left\|\overset{m+1}{v}_{x}\right\|_{r}^{r}+\frac{1}{r}\max_{\Omega}\left|\operatorname{div}\overset{m}{v}\right|\left\|\overset{m+1}{v}\right\|_{2,r}^{r}$$

$$+\max_{\Omega}\left|\overset{m+1}{v}_{x}\right|\left\|\overset{m}{v}_{xx}\right\|_{r}\left\|\overset{m+1}{v}_{xx}\right\|_{r}^{r-1}$$

$$+2\max_{\Omega}\left|\overset{m}{v}_{x}\right|\left\|\overset{m+1}{v}_{xx}\right\|_{r}^{r}$$

$$+\left(\left\|\overset{m}{p}\right\|_{3,r}+\|f\|_{2,r}\right)\left\|\overset{m+1}{v}\right\|_{2,r}^{r-1}.$$

Using curvilinear coordinates introduced in §2 and boundary condition (3.4), the surface integral in (4.3) is estimated by

$$-\int_{\partial\Omega}\overset{m}{v}\cdot\bar{n}\left(\left|\overset{m+1}{v}\right|^{r}+\left|\overset{m+1}{v}_{x}\right|^{r}+\left|\overset{m+1}{v}_{xx}\right|^{r}\right)ds$$

$$\le\int_{S_1}d\left(\left|\overset{m+1}{v}\right|^{r}+\left|\overset{m+1}{v}_{x}\right|^{r}+\left|\overset{m+1}{v}_{xx}\right|^{r}\right)ds$$

$$\le C(S_1)\int_{S_1}d\Big(|\eta|^{r}+|\eta_{,\tau}|^{r}+|\eta_{,\tau\tau}|^{r}$$

$$+\left|\overset{m+1}{v}_{,\tau n}\right|^{r}+\left|\overset{m+1}{v}_{,n\tau}\right|^{r}+\left|\overset{m+1}{v}_{,nn}\right|^{r}+\left|\overset{m+1}{v}_{,n}\right|^{r}\Big)ds$$

$$\equiv I.$$

From (3.17) restricted to S_1 for $s=0$ we calculate

$$\overset{m+1}{v}_{,n}\Big|_{S_1}=d^{-1}\left[\eta_t+\eta_\mu\eta_{,\tau_\mu}+\nabla\overset{m}{p}-f\right]\equiv d^{-1}A,$$

so

$$(4.4)\qquad \overset{m+1}{v}_{,n\tau_\nu}\Big|_{S_1}=-d^{-2}\,d_{,\tau_\nu}A+d^{-1}A_{,\tau_\nu},\quad \overset{m+1}{v}_{,nt}\Big|_{S_1}$$

$$=-d^{-2}d_tA+d^{-1}A_t,$$

$$\overset{m+1}{v}_{,nn}=\overset{m}{v}_{n}^{-2}\overset{m}{v}_{n,n}\left[\overset{m+1}{v}_{t}+\overset{m}{v}_{\mu}\overset{m+1}{v}_{,\tau_\mu}+\nabla\overset{m}{p}-f\right]$$

$$-\overset{m}{v}_{n}^{-1}\Big[\overset{m+1}{v}_{,nt}+\overset{m}{v}_{\mu,n}\overset{m+1}{v}_{,\tau_\mu}+\overset{m}{v}_{\mu}\overset{m+1}{v}_{,n\tau_\mu}+\nabla\overset{m}{p}_{,n}$$

$$-f_{,n}+\overset{m}{v}_{\mu}\left(\overset{m+1}{v}_{,\tau_\nu}a_{\nu\mu}+\overset{m+1}{v}_{,n}b_\mu\right)\Big].$$

Knowing that

$$\overset{m}{v}_{,n}\Big|_{S_1} = d^{-1}\Big(\eta_t + \eta_\mu \eta_{,\tau_\mu} + \nabla \overset{m-1}{p} - f\Big),$$

we obtain

$$\overset{m+1}{v}_{,nn}\Big|_{S_1} = g_1(\eta, \eta_t, \eta_\tau, \eta_{,t\tau}, \eta_{,\tau\tau}, f_{,t}, f_{,\tau}, f)$$
$$+ g_2(\eta, \eta_{,t}, \eta_{,\tau}, f)\Big(\overset{m}{p}_x \overset{m-1}{p}_x + \overset{m}{p}_x + \overset{m}{p}_{xx} + \overset{m}{p}_{xt}\Big),$$

where

$$d = -\eta_n = -\eta \cdot \bar{n}, \quad \eta_\mu = \eta \cdot \bar{\tau}_\mu,$$
$$a_{\nu\mu}(x) = [\bar{n}, \bar{\tau}_\nu] \cdot \bar{\tau}_\mu, \quad b_\nu(x) = [\bar{n}, \bar{\tau}_\nu] \cdot \bar{n}$$

and $[\bar{\tau}_\nu, \bar{\tau}_\mu]$ is the commutator of vector fields $\bar{\tau}_\nu$, $\bar{\tau}_\mu$. Substituting (4.4) into I we obtain the estimate

$$I \le C_1 + C_2\Big\|\overset{m}{p}\Big\|^r_{3,r} + C_3\Big\|\overset{m}{p}_t\Big\|^r_{2,r} + C_4\Big\|\overset{m}{p}\Big\|^r_{3,r}\Big\|\overset{m-1}{p}\Big\|^r_{3,r}, \tag{4.5}$$

where C_i, $i = 1,\dots,4$, are the same constants as in (4.1). Using the Sobolev imbeddings in (4.3) and using (4.5), we obtain(4.1). This ends the proof.

In estimate (4.1) the norm $\|\overset{m}{p}_t\|_{2,r}$ appears, so we have to consider problem (A^1_m) and, consequently, problem (B^1_m), also. Therefore, we formulate

LEMMA 4.2. *Let the assumptions of Lemma* 4.1 *be satisfied. Then for* $t \in [0, T_1]$ *the following differential inequality for solutions* $\overset{m}{v}, \overset{m+1}{v} \in W^2_r(\Omega)$ *of* (B^1_m) *is valid*:

$$\frac{d}{dt}\Big\|\overset{m+1}{v}_t\Big\|^r_{1,r} \le C_7 + C_8\Big\|\overset{m}{p}\Big\|^r_{2,r} + C_9\Big\|\overset{m}{p}_t\Big\|^r_{2,r}$$
$$+ C_{10}\Big(\Big\|\overset{m}{v}\Big\|_{2,r}\Big\|\overset{m+1}{v}_t\Big\|^r_{1,r} + \Big\|\overset{m}{v}_t\Big\|_{1,r}\Big\|\overset{m+1}{v}\Big\|_{2,r}\Big\|\overset{m+1}{v}_t\Big\|^{r-1}_{1,r}\Big)$$
$$+ \Big(\Big\|\overset{m}{p}_t\Big\|_{2,r} + C_{11}\Big)\Big\|\overset{m+1}{v}_t\Big\|^{r-1}_{1,r}, \tag{4.6}$$

where

$$C_7 = C_7(d_0, r, \Omega, S_1, |\eta|_{2,0,r,S_1}, |f|_{2,1,r,\Omega}),$$
$$C_8 = C_8(d_0, r, \Omega, S_1, |\eta|_{2,1,r,S_1}),$$
$$C_9 = C_9(d_0, r, S_1, \|\eta\|_{2,r,S_1}), \quad C_{10} = C_{10}(r, \Omega), \quad C_{11} = \|f_t\|_{1,r}.$$

Proof. From (3.17) of (B^1_m) we have

$$(4.7) \qquad \sum_{s=1}^{3} \sum_{|j|\le 1} \int_\Omega D_x^j\Big(\overset{m+1}{v}{}^s_t + \overset{m}{v}{}^k \overset{m+1}{v}{}^s_{x^k} + \nabla_s \overset{m}{p} - f^s\Big)_{,t} \times D_x^j \overset{m+1}{v}{}^s_t \Big|\overset{m+1}{v}{}_{xt}\Big|^{r-2} dx = 0,$$

which, after applying Hölder's inequality, implies

$$(4.8) \quad \frac{1}{r}\frac{d}{dt}\Big\|\overset{m+1}{v}{}_t\Big\|_{1,r}^r$$
$$\le -\frac{1}{r}\int_{\partial\Omega} \overset{m}{v}\cdot \bar n\Big(\Big|\overset{m+1}{v}{}_t\Big|^r + \Big|\overset{m+1}{v}{}_{xt}\Big|^r\Big)\,ds$$
$$+\frac{1}{r}\max_\Omega\Big|\operatorname{div}\overset{m}{v}\Big|\Big\|\overset{m+1}{v}{}_t\Big\|_{1,r}^r + \max_\Omega\Big|\overset{m+1}{v}{}_x\Big|\Big\|\overset{m}{v}{}_t\Big\|_r\Big\|\overset{m+1}{v}{}_t\Big\|_r^{r-1}$$
$$+\max_\Omega\Big|\overset{m+1}{v}{}_x\Big|\Big\|\overset{m}{v}{}_{xt}\Big\|_r\Big\|\overset{m+1}{v}{}_{xt}\Big\|_r^{r-1} + \max_\Omega\Big|\overset{m}{v}{}_x\Big|\Big\|\overset{m+1}{v}{}_{xt}\Big\|_r^r$$
$$+\max_\Omega\Big|\overset{m}{v}{}_t\Big|\Big\|\overset{m+1}{v}{}_{xx}\Big\|_r\Big\|\overset{m+1}{v}{}_{xt}\Big\|_r^{r-1} + \Big(\Big\|\overset{m}{p}{}_t\Big\|_{2,r} + \|f_t\|_{1,r}\Big)\Big\|\overset{m+1}{v}{}_t\Big\|_{1,r}^{r-1}.$$

Using the curvilinear coordinates and (4.4), the surface integral in (4.8) is estimated by

$$(4.9) \qquad -\int_{\partial\Omega}\overset{m}{v}\cdot\bar n\Big(\Big|\overset{m+1}{v}{}_t\Big|^r + \Big|\overset{m+1}{v}{}_{xt}\Big|^r\Big)\,ds$$
$$\le C(S_1)\int_{S_1} d\Big(|\eta_t|^r + |\eta_{,\tau t}|^r + \Big|\overset{m+1}{v}{}_{,nt}\Big|^r\Big)\,ds$$
$$\le C_7 + C_8\Big\|\overset{m}{p}\Big\|_{2,r}^r + C_9\Big\|\overset{m}{p}{}_t\Big\|_{1,r}^r,$$

where constants C_7, C_8, C_9 are described in (4.6). From (4.8) and (4.9) we obtain (4.6). This concludes the proof.

Lemmas 3.2, 4.1, and 4.2 imply a priori bounds on elements of sequences $\{\overset{m}{v}\}$, $\{\overset{m}{v}_t\}$, $\{\overset{m}{p}\}$, $\{\overset{m}{p}_t\}$ of the solutions of problems (A^s_m), (B^s_m), $s = 0, 1$. These bounds are local in time and independent of m, which is a consequence of the following theorem:

THEOREM 4.1. *Let us assume* $S_i \in C^4$, $i = 0, 1, 2$, $\eta \in \Pi^{3-1/r}_{1,r,\infty}(S_1^T)$, $\pi \in \Pi^{3-1/r}_{2,r,\infty}(S_2^T)$, $f \in \Pi^2_{1,r,\infty}(\Omega^T)$ *and*

$$T \le \max_{\rho > 1} \min\left\{\frac{a_0}{(\rho y_0)^{1/T}}, t'(\rho)\right\}, \tag{4.10}$$

where

$$\begin{aligned} y_0 &= \|a\|^r_{2,r} + \|-a^k a_{x^k} + \nabla p(0) + f(0)\|^r_{1,r} \\ &\le \|a\|^r_{2,r} + \|a\|^{2r}_{2,r} + \|p(0)\|^r_{2,r} + \|f(0)\|^r_{1,r} \\ &\le C(\|a\|_{2,r}, \|f(0)\|_{2,r}, \|\pi(0)\|_{2-1/r,r,S_2}, |\eta(0)|_{2-1/r,1,r,S_1}), \end{aligned}$$

$$t'(\rho) = \frac{1}{\gamma(\rho y_0)} \ln \frac{\rho y_0 + 1}{y_0 + 1}, \qquad \rho > 1,$$

and $\gamma(y) = C(y^4 + 1)$. *Moreover, we denote by* $\rho_0 > 1$ *the value for which the function*

$$T(\rho) = \min\left\{\frac{a_0}{(\rho y_0)^{1/r}}, t'(\rho)\right\}$$

has a maximum. Then the solutions of problems (A^s_m), (B^s_m), $s = 0, 1$ *are estimated in the following manner*:

$$\left|\overset{m}{v}\right|_{2,1,r} \le (\rho_0 y_0)^{1/r}. \tag{4.11}$$

Moreover, from Lemma 3.2 *we have the estimate of* $|\overset{m}{p}|_{3,2,r}$:

$$\left|\overset{m}{p}\right|_{3,2,r} \le C_1' + C_2'(\rho_0 y_0)^{1/r}, \tag{4.12}$$

where C_i', $i = 1, 2$, *are explained in* (3.23).

Proof. From inequalities (4.1) and (4.6) we have, for $m \geq 1$,

$$
\begin{aligned}
(4.13)\qquad \frac{d}{dt}\left|\overset{m+1}{v}\right|_{2,1,r}^{r} \leq C_1^1 + C_2^1\left|\overset{m}{p}\right|_{3,2,r}^{r} + C_3^1\left|\overset{m}{p}\right|_{3,2,r}^{r}\left|\overset{m-1}{p}\right|_{3,2,r}^{r} \\
+ C_4^1\left|\overset{m}{v}\right|_{2,1,r}\left|\overset{m+1}{v}\right|_{2,1,r} \\
+ \left(C_5^1\left|\overset{m}{p}\right|_{3,2,r} + C_6^1\right)\left|\overset{m+1}{v}\right|_{2,1,r}^{r-1},
\end{aligned}
$$

where C_i^1, $i = 1,\dots,6$, are constants dependent on C_k, $k = 1,\dots,11$. Moreover, Lemma 3.2 implies

$$
(4.14)\qquad \left|\overset{m}{p}\right|_{3,2,r} \leq C_1^2 + C_2^2\left|\overset{m}{v}\right|_{2,1,r}^{2},
$$

where C_i^2, $i = 1, 2$, are described in (3.23). Assuming $\overset{m}{y}(t) = |\overset{m}{v}(t)|_{2,1,r}^{r}$, from (4.13) and (4.14) we get, for $m \geq 1$,

$$
(4.15)\qquad \frac{d}{dt}\overset{m+1}{y} \leq \gamma\left(\overset{m}{y}, \overset{m-1}{y}\right)\left(\overset{m+1}{y} + 1\right),
$$

where $\gamma(\overset{m}{y}, \overset{m-1}{y}) = C(\overset{m}{y}{}^4 + \overset{m-1}{y}{}^4 + 1)$ and C is the upper bound of all constants appearing in (4.13) and (4.14). Using the method of induction, we will show that $\overset{m}{y}(t)$ is bounded independently of m. We have to obtain a differential inequality similar to (4.15) for the function $\overset{1}{v}$, knowing that $\overset{0}{v} = a$. From (3.17) for $m = 0$ and $s = 0, 1$, similarly as in Lemmas 4.1, 4.2 we obtain

$$
\begin{aligned}
(4.16)\qquad \frac{d}{dt}\left|\overset{1}{v}\right|_{2,1,r}^{r} \leq \int_{S_1} d\left(\left|\overset{1}{v}\right|^r + \left|\overset{1}{v}_x\right|^r + \left|\overset{1}{v}_t\right|^r + \left|\overset{1}{v}_{xx}\right|^r + \left|\overset{1}{v}_{xt}\right|^r\right) ds \\
+ C_{12}'\|a\|_{2,r}\left|\overset{1}{v}\right|_{2,1,r}^{r} + \left(\left|\overset{0}{p}\right|_{3,1,r} + C_{13}'\right)\left|\overset{1}{v}\right|_{2,1,r}^{r-1},
\end{aligned}
$$

where $C_{12}' = C_{12}'(r, \Omega)$, $C_{13}' = |f|_{1,0,r}$. To estimate the surface integral I in (4.16) we consider

$$
(4.17)\qquad \overset{1}{v}_{,n} = -a_n^{-1}\left[\overset{1}{v}_t + a_\mu \overset{1}{v}_{,\tau_\mu} + \nabla \overset{0}{p} - f\right], \quad \text{so}
$$

$$
\overset{1}{v}_{,n}\Big|_{S_1} = -a_n^{-1}\left[\eta_t + a_\mu \eta_{,\tau_\mu} + \nabla \overset{0}{p} - f\right],
$$

where $a_n = -d(0)$, and then $\overset{1}{v}_{,nt}$, $\overset{1}{v}_{,n\tau_\nu}$ can be calculated. At last we can get

$$\overset{1}{v}_{,nn} = a_n^{-2} a_{n,n}\left[\overset{1}{v}_{,t} + a_\mu \overset{1}{v}_{,\tau_\mu} + \nabla \overset{0}{p} - f\right] - a_n^{-1}\left[\overset{1}{v}_{,nt} + a_{\mu,n}\overset{1}{v}_{,\tau_\mu} + a_\mu \overset{1}{v}_{,\tau_\mu n} + \nabla \overset{0}{p}_{,n} - f_{,n}\right]. \tag{4.18}$$

Using (4.17), (4.18), the surface integral in (4.16) is estimated in the following way:

$$I \leq g_1\left(d_0, \|a\|_{1,r,S_1}, |\eta|_{2,0,r,S_1}, |f|_{1,0,r,S_1}\right) + g_2\left(d_0, \|a\|_{1,r,S_1}, |f|_{1,0,r,S_1}\right)\left(\left|\overset{0}{p}\right|_{3,2,r}^{2r} + \left|\overset{0}{p}\right|_{3,2,r}^{r}\right). \tag{4.19}$$

Now we calculate the functions $\overset{0}{p}$, $\overset{0}{p}_t$ from (A_0^0), (A_0^1):

$$(A_0^0) \qquad \Delta \overset{0}{p} = \operatorname{div} f - a^i_{x^k} a^k_{x^i},$$

$$\overset{0}{p}\Big|_{S_2} = \pi(x', t), \qquad \frac{\partial \overset{0}{p}}{\partial n}\Big|_{S_1} = g(\eta, \bar{n}, \bar{\tau}),$$

$$\frac{\partial \overset{0}{p}}{\partial n}\Big|_{S_0} = \left(f_n + a^k a \cdot \bar{n}_{x^k}\right)\big|_{S_0},$$

and

$$(A_0^1) \qquad \Delta \overset{0}{p}_t = \operatorname{div} f_t,$$

$$\overset{0}{p}_t\Big|_{S_2} = \pi_t(x', t), \qquad \frac{\partial \overset{0}{p}_t}{\partial n}\Big|_{S_1} = g(\eta_t, \bar{n}, \bar{\tau}), \qquad \frac{\partial \overset{0}{p}_t}{\partial n}\Big|_{S_0} = f_t \cdot \bar{n}|_{S_0}.$$

From problems (A_0^s), $s = 0, 1$, we have the following estimate:

$$\left|\overset{0}{p}\right|_{3,2,r} \leq C\left(\|a\|_{2,r}, |f|_{2,1,r}, |\pi|_{3-1/r,2,r,S_2}, |\eta|_{3-1/r,1,r,S_1}\right). \tag{4.20}$$

Therefore, from (4.16), (4.19), (4.20) we obtain

$$\frac{d}{dt}\left|\overset{1}{v}\right|_{2,1,r}^{r} \leq C_{12} + C_{13}\left|\overset{1}{v}\right|_{2,1,r}^{r}, \tag{4.21}$$

where C_{12}, C_{13} depend on $\|a\|_{2,r}$, $|f|_{2,1,r}$, $|\pi|_{3-1/r,2,r,S_2}$, $|\eta|_{3-1/r,2,r,S_1}$. Integrating (4.21) with respect to time we obtain

$$\left|\overset{1}{v}\right|^r_{2,1,r} \le (1+y_0)e^{(C_{12}+C_{13})t} - 1. \tag{4.22}$$

Demanding that $\overset{1}{y} \le \rho y_0$, $\rho > 1$, we obtain the following restriction for t:

$$t \le t_1(\rho) = \frac{1}{C_{12}+C_{13}} \ln \frac{\rho y_0 + 1}{y_0 + 1}. \tag{4.23}$$

Now we shall obtain the required estimate for $m \ge 1$. Assuming $\overset{m}{y}(t) \le \rho y_0$, $\overset{m-1}{y}(t) \le \rho y_0$ (for $m = 1$ it has been shown above), we shall show that $\overset{m+1}{y}(t) \le \rho y_0$ for a sufficiently small time interval which depends on ρ. Indeed, from (4.15) we obtain $\overset{m+1}{y}(t) \le (1+y_0)e^{\gamma(\rho y_0, \rho y_0)t} - 1$, and demanding that $\overset{m+1}{y} \le \rho y_0$, we get the following restriction for t:

$$t \le t_2(\rho) = \frac{1}{\gamma(\rho y_0, \rho y_0)} \ln \frac{\rho y_0 + 1}{y_0 + 1}. \tag{4.24}$$

The function $t_2(\rho)$ has a maximum for $\rho = \rho_*$, where ρ_* is a solution of the equation

$$1 = 4\rho^3 y_0^3 \frac{1+\rho y_0}{1+\rho^4 y_0^4} \ln \frac{\rho y_0 + 1}{y_0 + 1}.$$

Let $t'(\rho) = \min\{t_1(\rho), t_2(\rho)\}$. Then, assuming $t \le t'(\rho)$ and (3.26) is satisfied, we conclude the proof.

Now we consider an a priori estimate for the solutions of problems (B^s_m), $s = 0, \dots, 1$, where $l \ge 2$ is an arbitrary integer.

LEMMA 4.3. *Let us assume*
(a) $T \le T_1$, $r > \frac{3}{l-1}$, $d \ge d_0 > 0$, $l \ge 2$,
(b) $\eta \in \Gamma^l_{0,r}(S_1)$, $f \in \Gamma^l_{1,r}(\Omega)$,
(c) S_1 *is of class* C^{l+1},
(d) $\overset{m}{p} \in \Gamma^{l+1}_{2,r}(\Omega)$, $\overset{m}{v} \in \Gamma^l_{1,r}(\Omega)$.

Then for $t \in [0, T_1]$, *an arbitrary solution* $\overset{m+1}{v} \in \Gamma^l_{1,r}(\Omega)$ *of* (B^s_m), $s = 0, 1, \dots, l-1$, *satisfies the following differential inequality*:

$$\frac{d}{dt}\left|\overset{m+1}{v}\right|^r_{l,1,r} \le I + C(r,l,\Omega)\left|\overset{m}{v}\right|_{l,1,r}\left|\overset{m+1}{v}\right|^r_{l,1,r} + \left(\left|\overset{m}{p}\right|_{l+1,2,r} + |f|_{l,1,r}\right)\left|\overset{m+1}{v}\right|^{r-1}_{l,1,r}, \tag{4.25}$$

where I is the following surface integral:

$$I=\int_{S_1}\sum_{\substack{i+j\le l\\ j\le l-1}} d\left|D_t^jD_x^i\overset{m+1}{v}\right|^r ds, \tag{4.26}$$

where

$$D_x^i=\frac{\partial^{i_1}}{\partial x_1^{i_1}}\frac{\partial^{i_2}}{\partial x_2^{i_2}}\frac{\partial^{i_3}}{\partial x_3^{i_3}},\qquad i=i_1+i_2+i_3,\ D_t=\frac{\partial}{\partial t}.$$

Proof. To prove this lemma we assume all derivatives of all functions appearing in the problem are continuous. From (B_m^s), $s=0,1,\dots,l-1$ we have

$$\sum_{\substack{i+j\le l\\ j\le l-1}}\sum_{s=1}^{3}\int_\Omega D_t^jD_x^i\left(\overset{m+1}{v}{}^s_t+\overset{m}{v}{}^k\,\overset{m+1}{v}{}^s_{x^k}+\nabla_s\overset{m}{p}-f^s\right)$$
$$\times D_t^jD_x^i\overset{m+1}{v}{}^s\left|D_t^jD_x^i\overset{m+1}{v}\right|^{r-2}dx=0. \tag{4.27}$$

From (4.27), using Hölder's inequality and the Sobolev imbedding theorems, we conclude the proof.

Now we shall estimate the surface integral (4.26) which appears in Lemma 4.3.

LEMMA 4.4. *Let* (a), (b), (c) *of Lemma* 4.3 *be satisfied, and let* $a\in W_r^l(\Omega)$. *Then the surface integral* (4.26) *has the following estimate*:

$$I\le C(S_1)\sum_{i=0}^{l}I^i, \tag{4.28}$$

where

$$I^i\le\sum_{i_1+\cdots+i_{i'}\le i'}C_{i_1\cdots i_{i'}}\left|\overset{m}{p}\right|^{r-i_1}_{l+1,2,r,\Omega}\cdots\left|\overset{m-i}{p}\right|^{r-i_{i'}}_{l+1,2,r,\Omega}, \tag{4.29}$$

where $i'=\min(i,\min(m,l))$ *and*

$$C_{i_1\cdots i_i}=C_{i_1\cdots i_i}\big(|\eta|_{l,0,r,S_1},\ |f|_{l,1,r,\Omega},\ \|a\|_{l,r,\Omega},\ d_0\big).$$

Proof. Using curvilinear coordinates, from (4.26) we have

$$I\le C(S_1)\sum_{i=0}^{i}\sum_{\substack{|\alpha|+j\le l-i\\ j\le l-1}}\int_{S_1}d\left|D_{(\tau)}^\alpha D_t^jD_n^i\overset{m+1}{v}\right|^r ds\equiv C(S_1)\sum_{i=0}^{l}I^i, \tag{4.30}$$

where α is a multi-index and $D^{\alpha}_{(\tau)}$ denotes all possible derivatives of order α, with respect to τ_1, τ_2, $D_n = \partial/\partial n$. To estimate (4.30) we have to consider the form of $D_n^s \overset{m+1}{v}$, $s \leq 1$. From (3.17) of $(\mathrm{B}^0_{\mathrm{m}})$ we have

$$D_n \overset{m+1}{v} = -\overset{m}{v}{}_n^{-1}\left[\overset{m+1}{v}{}_{,t} + \overset{m}{v}{}_\mu \overset{m+1}{v}{}_{,\tau_\mu} + \nabla \overset{m}{p} - f\right]. \tag{4.31}$$

Differentiating (4.31) with respect to n we obtain

$$\begin{aligned} D_n^2 \overset{m+1}{v} = {} & \overset{m}{v}{}_n^{-2}\overset{m}{v}{}_{n,n}\left[\overset{m+1}{v}{}_{,t} + \overset{m}{v}{}_\mu \overset{m+1}{v}{}_{,\tau_\mu} + \nabla \overset{m}{p} - f\right] \\ & - \overset{m}{v}{}_n^{-1}\Big[\overset{m+1}{v}{}_{,nt} + \overset{m}{v}{}_{\mu,n} \overset{m+1}{v}{}_{,\tau_\mu} + \overset{m}{v}{}_\mu \overset{m+1}{v}{}_{,n\tau_\mu} \\ & + \nabla \overset{m}{p}{}_{,n} - f_{,n} + \overset{m}{v}{}_\mu\Big(\overset{m+1}{v}{}_{,\tau_\nu} a_{\nu_\mu} + \overset{m+1}{v}{}_{,n} b_\mu\Big)\Big], \end{aligned} \tag{4.32}$$

where $a_{\nu\mu}(x) = [\bar{n}, \bar{\tau}_\nu] \cdot \bar{\tau}_\mu$, $b_\mu = [\bar{n}, \bar{\tau}_\mu] \cdot \bar{n}$ and $\overset{m}{v}{}_{,n}$ is described by (3.17) of $(\mathrm{B}^0_{\mathrm{m-1}})$. Hence it has the form

$$\overset{m}{v}{}_{,n} = -\overset{m-1}{v}{}_n^{-1}\left[\overset{m}{v}{}_t + \overset{m-1}{v}{}_\mu \overset{m}{v}{}_{,\tau_\mu} + \nabla \overset{m-1}{p} - f\right]. \tag{4.33}$$

At last the $(s + 1)$st derivative with respect to n has the form

$$D_n^{s+1} \overset{m+1}{v} = -\sum_{i \leq s} D_n^i \overset{m}{v}{}_n^{-1} D_n^{s-i}\left[\overset{m+1}{v}{}_{,t} + \overset{m}{v}{}_\mu \overset{m+1}{v}{}_{,\tau_\mu} + \nabla \overset{m}{p} - f\right]. \tag{4.34}$$

To obtain the form of $D_n^{s+1} \overset{m+1}{v}$ we shall use inductive considerations. From (4.31) we see that $D_n \overset{m+1}{v}$ has the polynomial form of degree two with respect to $D_t^i D_{(\tau)}^j \overset{m}{v}$, $D_t^i D_{(\tau)}^j \overset{m+1}{v}$, $i + j \leq 1$, and that it is linear with respect to $D_t^i D_{(\tau)}^j \overset{m+1}{v}$, $i + j \leq 1$, $\overset{m}{p}{}_x$, f.

From (4.32) it follows that $D_n^2 \overset{m+1}{v}$ is the polynomial of degree four with respect to $D_t^i D_{(\tau)}^j \overset{m+1-\sigma}{v}$ t, where $\sigma = 0, 1, 2$, $i + j \leq 2$, and that it is the polynomial of degree two with respect to $D_t^i D_{(x)}^j \overset{m-\rho}{p}$, $D_t^k D_{(x)}^l f$, where $i + j \leq 2$, $i \leq 1$, $\rho = 0, 1$, $k + 1 \leq 1$. We see that $D_n^2 \overset{m+1}{v}$ depends linearly on $D_t^i D_{(\tau)}^j \overset{m+1}{v}$, where $i + j \leq 2$.

Let us introduce the inductive assumption. Let $D_n^s \overset{m+1}{v}$ be a polynomial with respect to $D_t^i D_{(\tau)}^j \overset{m+1-\sigma}{v}$, $D_t^i D_{(x)}^j \overset{m-\rho}{p}$ and $D_t^k D_{(x)}^l f$, where

$i+j\le s$, $\sigma=0,\dots,s$, $\rho=0,\dots,s-1$, $k+1\le s-1$. Then from (4.34) we see that $D_n^{s+1}\overset{m+1}{v}$, comparing with $D_n^s\overset{m+1}{v}$, contains additional terms $\overset{m-s}{v}$ $\overset{m-s}{p}$, and the order of all derivatives increases by one. Moreover, the degree of the polynomial with respect to $\overset{m+1}{v},\dots,\overset{m-s}{v}$ and $\overset{m}{p},\dots,\overset{m-s}{p}, f$ increases by two and one, respectively. At last, if $s>m$, then in $D_n^{s+1}\overset{m+1}{v}$ there appear the functions $D_{(x)}^x a$, $0\le x\le s$, in the power up to $s-m$.

From the inductive assumptions we obtain that $D_n^{s+1}\overset{m+1}{v}$ is a polynomial of degree $2(s+1)$ and $s+1$ with respect to $D_t^i D_{(\tau)}^j\overset{m+1-\sigma}{v}$, $i+j\le s+1$, $\sigma=0,\dots,s+1$, and $D_t^i D_{(x)}^j\overset{m-\rho}{p}$, $D_t^k D_{(x)}^l f$, $i+j\le s+1$, $i\le s$, $\rho=0,\dots,s$, $k+1\le s$, respectively. We must underline that $\overset{m+1}{v}$ with all its derivatives appears there linearly. Hence, using the fact that $W_r^l(\Omega)$, $r>3/l-1$, is an algebra, we obtain (4.28), (4.29), so we have proved the lemma.

Now we estimate the initial conditions $D_t^s\overset{m+1}{v}|_{t=0}$ appearing in problems $(\mathrm{B}_{\mathrm{m}}^{\mathrm{s}})$, $s=0,1,\dots,l-1$, $l\ge 2$.

LEMMA 4.5. *Assume*

$$a\in W_r^l(\Omega),\quad f(0)\in\Gamma_{l-s,r}^l(\Omega),\quad \pi(0)\in\Gamma_{l+1-s,r}^{l+1-1/r}(S_2),$$
$$g(0)\in\Gamma_{l+1-s,r}^{l+1-1/r}(S_1),\ r>\frac{3}{l-1}.$$

Then for solutions of problems $(\mathrm{A}_{\mathrm{m}}^{\mathrm{s}})$, $(\mathrm{B}_{\mathrm{m}}^{\mathrm{s}})$, $s=0,\dots,l-1$, *the following estimate is valid*:

$$\left\|D_t^s\overset{m+1}{v}\Big|_{t=0}\right\|_{l-s,r,\Omega}\le F^s\big(\|a\|_{l,r,\Omega},|f(0)|_{l,l-s,r,\Omega},|\pi(0)|_{l+1-1/r,l+1-s,r,S_2},|g(0)|_{l+1-1/r,l+1-s,r,S_1}\big),\tag{4.35}$$

where F^s *is a polynomial of degree* $s+1$ *with respect to its arguments and* $l\ge 2$.

Proof. To obtain the form of $D_t^s\overset{m+1}{v}|_{t=0}$, $s=0,\dots,l-1$, we use inductive considerations. For $s=0$ we have $\overset{m+1}{v}|_{t=0}=a$. For $s=1$ we have $D_t\overset{m+1}{v}|_{t=0}=-a\cdot\nabla a-\nabla\overset{m}{p}(0)+f(0)$, where $\overset{m}{p}(0)$ is a solution of problem $(\mathrm{A}^0_{\mathrm{m}})$ for $t=0$. Therefore, we have

$$\overset{m}{p}(0)=\mathcal{F}\big(\operatorname{div} f(0)-a^i_{x^k}a^k_{x^i},\ g(\eta(0),\bar\tau,\bar n),$$

$$\big(f(0)\cdot\bar n+a^k a\cdot \bar n_{x^k}\big)\big|_{S_0},\ \pi(x',0)\big),$$

where $\mathcal{F}$ is a linear functional which represents a solution of $(\mathrm{A}^0_{\mathrm{m}})$. Hence, the function $D_t\overset{m+1}{v}|_{t=0}$ has the polynomial form of degree two with respect to $D^i_x a$, $i\le 1$, and it depends linearly on $f(0)$, $g(0)$, $\pi(0)$. For $s=2$ we have

$$D^2_t\overset{m+1}{v}\Big|_{t=0}=-D_t\overset{m}{v}\Big|_{t=0}\cdot\nabla a-a\cdot\nabla D_t\overset{m+1}{v}\Big|_{t=0}-\nabla D_t\overset{m}{p}\Big|_{t=0}+D_t f|_{t=0},$$

where $D_t\overset{m}{v}|_{t=0}$, $D_t\overset{m+1}{v}|_{t=0}$ are calculated from the previous step (for $s=1$), and $D_t\overset{m}{p}|_{t=0}$, calculated from (A^1_m), has the form

$$D_t\overset{m}{p}\Big|_{t=0}=\mathcal{F}\Big(D_t\Big(\operatorname{div} f-\overset{m}{v}{}^i_{x^k}\overset{m}{v}{}^k_{x^i}\Big)\Big|_{t=0},\ D_t g|_{t=0},$$

$$D_t\Big(f\cdot\bar n+\overset{m}{v}{}^k\overset{m}{v}\cdot\bar n_{x^k}\Big)\Big|_{S_0\cap t=0},\ D_t\pi|_{t=0}\Big).$$

Therefore, $D^2_t\overset{m+1}{v}|_{t=0}$ is a polynomial of degree three with respect to $D^i_{(x)}a$, $0\le i\le 2$. Moreover, it depends linearly on $D^i_tD^j_{(x)}f|_{t=0}$, $i+j\le 1$, $D_t\overset{m}{p}_x|_{t=0}$, $\overset{m-1}{p}_x(0)$. At last, the first and third arguments of $D_t\overset{m}{p}|_{t=0}$ constitute a polynomial of degree three with respect to $D^i_{(x)}a$, $0\le i\le 2$, and they depend linearly on $D^i_tD^j_{(x)}f|_{t=0}$, $i+j\le 1$, $D^i_{(x)}\overset{m-1}{p}|_{t=0}$, $i=1,2$.

Now we shall consider the sth-derivative of $\overset{m+1}{v}$:

$$D^s_t\overset{m+1}{v}\Big|_{t=0}=-\sum_{j=0}^{s-1}\Big(D^j_t\overset{m}{v}\cdot\nabla D^{s-1-j}_t\overset{m+1}{v}\Big)\Big|_{t=0}-\nabla D^{s-1}_t\overset{m}{p}\Big|_{t=0}+D^{s-1}_t f|_{t=0}. \tag{4.36}$$

Knowing that $\mathcal{F}$ is a linear functional with respect to its arguments, we can treat $D^{s-1}_t\overset{m+1}{v}|_{t=0}$ as a polynomial with respect to a, $f(0)$, $\pi(0)$, $g(0)$ and their derivatives. Moreover, the derivatives of f, π, g, with respect to time for $t=0$, have to be considered also.

The expression $D^s_t\overset{m+1}{v}|_{t=0}$, comparing with $D^{s-1}_t\overset{m+1}{v}|_{t=0}$, additionally contains the derivatives $D^{s-1}_t f|_{t=0}$ and $D^{s-1}_t\overset{m}{p}|_{t=0}$, where $D^{s-1}_t\overset{m}{p}|_{t=0}$

depends on $D_t^{s-1}\pi|_{t=0}$, $D_t^{s-1}g|_{t=0}$. In the end, because of the bilinear and linear differential operators of the first order appearing in (4.36), the expression $D_t^s \overset{m+1}{v}|_{t=0}$ has a polynomial form (with respect to a, $f(0)$, $g(0)$, $\pi(0)$ and their derivatives) of degree greater than $D_t^{s-1}\overset{m+1}{v}|_{t=0}$ by one. The order of derivatives of a, f, g, π appearing in $D_t^s\overset{m+1}{v}|_{t=0}$ is greater by one also. Therefore, from induction considerations, $D_t^s\overset{m+1}{v}|_{t=0}$ is a polynomial of degree $s+1$ with respect to $a,\dots,D_{(x)}^s a$, $D_{(x)}^i D_t^j f|_{t=0}$, $D_{(x)}^i D_t^j \pi|_{t=0}$, $D_{(x)}^i D_t^j g|_{t=0}$, for $i+j \le s-1$. At last, using Lemma 3.2 and the fact that W_r^{l-1}, $r > 3/(l-1)$ is an algebra, we conclude the proof.

THEOREM 4.2. *Let us assume*

(a) $\eta \in \Pi_{1,r,\infty}^{l+1-1/r}(S_1^T)$, $f \in \Pi_{1,r,\infty}^{l}(\Omega^T)$, $\pi \in \Pi_{2,r,\infty}^{l+1-1/r}(S_2^T)$, $l \ge 2$, $r > 3/(l-1)$,

(b) $a \in W_r^l(\Omega)$,

(c) $S_i \in C^{l+2}$, $i = 0,1,2$,

and

$$T \le \max_{\rho} \min\left\{ \frac{a_0}{(\rho y_0)^{1/r}}, t''(\rho) \right\}, \tag{4.37}$$

where $\rho > 1$ and y_0 is defined in the assumptions of Theorem 4.1. Let the function $t''(\rho)$ be a solution of

$$e^{C_1(\rho Y(0))t}[C_2(\rho Y(0))t + Y^r(0)] = (\rho Y(0))^r, \tag{4.38}$$

where

$$C_1(\rho Y(0)) = C(r,l,\Omega)[\rho Y(0) + 1],$$
$$C_2(\rho Y(0)) = W(\rho Y(0),\dots,\rho Y(0)),$$

where

$$W(a_1,\dots,a_i) = \sum_{i_1+\cdots+i_i \le i} C_{i_1\cdots i_i} a_1^{2ri_1}\cdots a_i^{2ri_i}, \qquad i = \min(m,l),$$

$$C'_{i_1\cdots i_i} = C'_{i_1\cdots i_i}\big(d_0, l, r, \Omega, \|a\|_{l,r}, |\eta|_{l+1-1/r,1,r,S_1^T},$$
$$|f|_{l,1,r,\Omega^T}, |\pi|_{l+1-1/r,2,r,S_2^T}\big),$$

and

$$Y^r(0) = \sum_{s=0}^{l} \left\| D_t^s \overset{m+1}{v}\Big|_{t=0} \right\|_{l-s,1,r,\Omega}^r,$$

which is estimated in Lemma 4.5. *We denote a value for which the function*

$$T(\rho) = \min\left\{ \frac{a_0}{(\rho y_0)^{1/r}}, t''(\rho) \right\}$$

has a maximum by ρ_0^l. *Then solutions of problems* (A_m^s), (B_m^s), $s = 0, \dots, l-1$, *are estimated by*

$$\left| \overset{m}{v} \right|_{l,1,r,\Omega} \le \rho_0^l Y(0), \tag{4.39}$$

and then, from Lemma 3.2 *we have the estimate*

$$\left| \overset{m}{p} \right|_{l+1,2,r,\Omega} \le C_1' + C_2' \left(\rho_0^l Y(0) \right)^2, \tag{4.40}$$

where C_i', $i = 1, 2$, *are explained in* (3.23).

Proof. Using the Young inequality in (4.25) we obtain

$$\frac{d}{dt} \left| \overset{m+1}{v} \right|_{l,1,r}^r \le I + \frac{2^r}{r} \left(\left| \overset{m}{p} \right|_{l+1,2,r}^r + |f|_{l,1,r}^r \right) + \left(C(r, l, \Omega) \left| \overset{m}{v} \right|_{l,1,r} + \frac{r-1}{r} \right) \left| \overset{m+1}{v} \right|_{l,1,r}^r. \tag{4.41}$$

Using (3.23) and (4.28) in (4.41) we have the inequality

$$\frac{d}{dt} \left| \overset{m+1}{v} \right|_{l,1,r}^r \le W\left(\left| \overset{m}{v} \right|_{l,1,r}, \dots, \left| \overset{m-i}{v} \right|_{l,1,r} \right) + C(r, l, \Omega) \left(\left| \overset{m}{v} \right|_{l,1,r} + 1 \right) \left| \overset{m+1}{v} \right|_{l,1,r}^r, \tag{4.42}$$

where W is described above. Assuming $|\overset{i}{v}|_{l,1,r} \le \rho Y(0)$, $\rho > 1$, $i \le m$, then integrating (4.42) over time and using (4.37), (4.38), we obtain

$$\left| \overset{m+1}{v} \right|_{l,1,r}^r \le e^{C_1(\rho Y(0))t} \left[C_2(\rho Y(0))t + Y(0)^r \right] = (\rho Y(0))^r. \tag{4.43}$$

Therefore we have proved (4.39). This concludes the proof.

5. Unique solvability of the initial boundary value problem. In this part we shall prove the existence and uniqueness of solutions of the problem (3.1)–(3.5). At first we shall prove the main result:

THEOREM 5.1. *Let the assumptions of Theorem* 4.2 *be satisfied. Then, there exists a unique solution of the problem* (A, B) *such that*

$$v \in \Pi_{1,r,\infty}^l(\Omega^T), p \in \Pi_{2,r,\infty}^{l+1}(\Omega^T), \quad \text{for } l \ge 2, r > \frac{3}{l-1}, \tag{5.1}$$

where T *is described by* (4.37).

Proof. We shall use the method of successive approximations. At first, we show that the sequences $\{\overset{m}{v}\}$, $\{\overset{m}{p}\}$ are strongly convergent in spaces $L_\infty(0, T; W_r^1(\Omega))$ and $L_\infty(0, T; W_r^2(\Omega))$, respectively. Let us introduce

$$\overset{m}{\vartheta} = \overset{m}{v} - \overset{m-1}{v}, \quad \overset{m}{\mathcal{P}} = \overset{m}{p} - \overset{m-1}{p}, \quad m \geq 1, \vartheta^0 = a. \tag{5.2}$$

Then, from problems $(\mathrm{A}_{\mathrm{m}}^0)$, $(\mathrm{B}_{\mathrm{m}}^0)$ we obtain

$$\Delta \overset{m}{\mathcal{P}} = -\overset{m}{\vartheta}{}^i_{,x^k} \overset{m}{v}{}^k_{x^i} - \overset{m-1}{v}{}^i_{x^k} \overset{m}{\vartheta}_{x^i}, \quad \overset{m}{\mathcal{P}}\Big|_{S_2} = 0, \quad \frac{\partial \overset{m}{\mathcal{P}}}{\partial n}\Big|_{S_1} = 0, \tag{5.3}$$

$$\frac{\partial}{\partial n} \overset{m}{\mathcal{P}}\Big|_{S_0} = -\overset{m}{\vartheta}{}^k \overset{m}{v} \cdot \bar{n}_{x^k} - \overset{m-1}{v}{}^k \overset{m}{\vartheta} \cdot \bar{n}_{x^k},$$

and

$$\frac{d}{dt} \overset{m+1}{\vartheta} + \overset{m}{v} \cdot \nabla \overset{m+1}{\vartheta} + \overset{m}{\vartheta} \nabla \overset{m}{v} = -\nabla \overset{m}{\mathcal{P}}, \quad \overset{m+1}{\vartheta}\Big|_{t=0} = 0, \tag{5.4}$$

$$\overset{m+1}{\vartheta}\Big|_{S_1} = 0.$$

Using Lemma 3.2, from problem (5.3) we have the estimate

$$\left\| \overset{m}{\mathcal{P}} \right\|_{2,r} \leq C \left(\left\| \overset{m}{v} \right\|_{2,r} + \left\| \overset{m-1}{v} \right\|_{2,r} \right) \left\| \overset{m}{\vartheta} \right\|_{1,r}, \tag{5.5}$$

where C is a constant. From (5.4) we get

$$\begin{aligned} \frac{d}{dt} \left\| \overset{m+1}{\vartheta} \right\|_{1,r} \leq & -\int_{\partial\Omega} \overset{m}{v} \cdot \bar{n} \left(\left| \overset{m+1}{\vartheta} \right|^r + \left| \overset{m+1}{\vartheta}_x \right|^r \right) ds \\ & + \left\| \overset{m}{\mathcal{P}} \right\|_{2,r} \left\| \overset{m+1}{\vartheta} \right\|_{1,r}^{r-1} \\ & + C \left\| \overset{m}{v} \right\|_{2,r} \left(\left\| \overset{m+1}{\vartheta} \right\|_{1,r} + \left\| \overset{m}{\vartheta} \right\|_{1,r} \right) \left\| \overset{m+1}{\vartheta} \right\|_{1,r}^{r-1}, \end{aligned} \tag{5.6}$$

where C is a constant. Using the boundary condition (3.4) in (5.4), the estimate (4.39) and $D_n \overset{m+1}{\vartheta}|_{S_1} = -d_0^{-1} D_\tau \overset{m}{\mathcal{P}}|_{S_1}$, the inequality (5.6) is replaced by

$$\frac{d}{dt} \left\| \overset{m+1}{\vartheta} \right\|_{1,r}^r \leq C_1 \left\| \overset{m}{\mathcal{P}} \right\|_{2,r}^r + C_2 \left(\left\| \overset{m+1}{\vartheta} \right\|_{1,r} + \left\| \overset{m}{\vartheta} \right\|_{1,r} \right) \left\| \overset{m+1}{\vartheta} \right\|_{1,r}^{r-1}. \tag{5.7}$$

At last, using the estimate (4.39) in (5.5), from (5.5) and (5.7) we obtain

$$(5.8) \qquad \frac{d}{dt}\overset{m+1}{\sigma} \leq \alpha \overset{m+1}{\sigma} + \beta \overset{m}{\sigma},$$

where α, β are corresponding constants, $m \geq 0$ and $\overset{0}{\sigma} = e^{\alpha t} y^0$, $\overset{m}{\sigma} = \|\overset{m}{\vartheta}\|_{1,r}^r$, $y^0 = \|a\|_{1,r}^r$. Integrating (5.8), we have $\overset{m+1}{\sigma} = e^{\alpha t}\int_0^t \beta \overset{m}{\sigma}(t')e^{-\alpha t'}\,dt'$, so after the inductive considerations we obtain $\overset{m}{\sigma} \leq e^{\alpha t}(\beta t)^m/m!(y^0)$, hence the series $\Sigma_{m=0}^{\infty} \overset{m}{\sigma}$ converges. It means that the sequence $\{\overset{m}{v}\}$ converges strongly in $L_\infty(0, T;\ W_r^1(\Omega))$, and from (5.5) that the sequence $\{\overset{m}{p}\}$ converges strongly in $L_\infty(0, T;\ W_r^2(\Omega))$. Therefore, there exists the limit functions v, p in these spaces, and from estimates (4.39), (4.40) it follows that they belong to spaces $\Pi^l_{1,r,\infty}(\Omega^T)$, $\Pi^{l+1}_{2,r,\infty}(\Omega^T)$, respectively. To show that the limit functions v, p are solutions of the problem (A, B) instead of the problem (A, B), we consider the following integral identities:

$$(5.9) \qquad -\int_{\Omega^T} \overset{m+1}{v}\eta_t\,dx\,dt + \int_\Omega a(x)\eta(x,0)\,dx + \int_{\Omega^T} \overset{m}{v}{}^k \overset{m+1}{v}{}_{x^k} \cdot \eta\,dx\,dt$$
$$= \int_{\Omega^T}\left(f - \nabla \overset{m}{p}\right)\eta\,dx\,dt,$$

where $\overset{m}{p} = \mathfrak{A}(\overset{m}{v})$ is a solution of the problems $(\mathrm{A}^0_{\mathrm{m}})$ for every continuously differentiable function η such that $\eta(x, T) = 0$, $\eta|_{S_2} = 0$. Passing with m to infinity, we obtain the same identities for the limit functions. Thus the limit functions are solutions of (A, B). This concludes the proof. As problems (3.1)–(3.5) and (A, B) are equivalent, it follows that the limit function v of the sequence $\{\overset{m}{v}\}$ satisfies $\operatorname{div} v = 0$. Apart from this, considering problems (C_m), one can prove that property directly:

Lᴇᴍᴍᴀ 5.1. *Let* v, p *be a solution of* (A, B) *described by Theorem* 5.1. *Then*

$$(5.10) \qquad \operatorname{div} v = 0 \quad \text{in } \Omega.$$

Proof. Multiplying (3.20) by $\operatorname{div} \overset{m+1}{v} |\operatorname{div} \overset{m+1}{v}|^{r-2}$, integrating the result over Ω and using (3.22) we obtain

$$\frac{1}{r}\frac{d}{dt}\left\|\operatorname{div} \overset{m+1}{v}\right\|_r^r = \frac{1}{r}\int_\Omega \operatorname{div} \overset{m}{v} \left|\operatorname{div} \overset{m+1}{v}\right|^r dx$$
$$+ \int_\Omega \overset{m}{v}{}^i_{x^k}\left(\overset{m}{v}{}^k_{x^i} - \overset{m+1}{v}{}^k_{x^i}\right)\operatorname{div} \overset{m+1}{v} \left|\operatorname{div} \overset{m+1}{v}\right|^{r-2} dx.$$

Using Hölder's inequality, we get, for $r > 3$,

$$\frac{d}{dt}\left\|\operatorname{div}\overset{m+1}{v}\right\|_r^r \le \max_\Omega\left|\operatorname{div}\overset{m}{v}\right|\left\|\operatorname{div}\overset{m+1}{v}\right\|_r^r + r\max_\Omega\left|\overset{m}{v}_x\right|\left\|\overset{m+1}{v}_x - \overset{m}{v}_x\right\|_r\left\|\operatorname{div}\overset{m+1}{v}\right\|_r^{r-1}.$$

Therefore, from Young's inequality we have

$$\frac{d}{dt}\left\|\operatorname{div}\overset{m+1}{v}\right\|_r^r \le r\max_\Omega\left|\overset{m}{v}_x\right|\left\|\operatorname{div}\overset{m+1}{v}\right\|_r^r + \max_\Omega\left|\overset{m}{v}_x\right|\left\|\overset{m+1}{v}_x - \overset{m}{v}_x\right\|_r^r. \tag{5.11}$$

Integrating (5.11) with respect to time, using (3.21) and (4.11) or (4.35), we obtain

$$\left\|\operatorname{div}\overset{m+1}{v}(t)\right\|_r^r \le Ce^{Ct}\int_0^t\left\|\overset{m+1}{v}_x - \overset{m}{v}_x\right\|_r^r dt \quad \text{for } t \in [0, T]. \tag{5.12}$$

From the proof of Theorem 5.1, we know that $\overset{m}{v}$ converges strongly in $L_\infty(0, T; W_r^1(\Omega))$. Then (5.12) implies $\lim_{m\to\infty}\operatorname{div}\overset{m}{v}(t) = \operatorname{div} v(t) = 0$ a.e. in Ω. But from (5.1) it follows that $\operatorname{div} v$ is a continuous function. Therefore $\operatorname{div} v = 0$. This concludes the proof.

6. Remarks. In this paper the existence of solutions of (3.1)–(3.5), local in time, $(v(t), p(t)) \in \Gamma_{1,r}^l(\Omega) \times \Gamma_{2,r}^{l+1}(\Omega)$, $t \in [0, T]$, $r > 3/(l-1)$, $l \ge 2$, has been proved. The time T of the existence of these solutions is determined in the assumptions of Theorems 4.1, 4.2. For $l = 2$ we have solutions v, p of the smallest smoothness. However, in this case $v_t(t)$, $v_x(t)$, $p_{xx}(t)$ belong to the class $C^\alpha(\Omega)$, $t \in [0, T]$, where $\alpha = (r-3)/r$, so the Euler equations are still satisfied in the classical sense.

Condition (3.25), which assures that a fluid leaves the domain Ω through S_1 for $t \in [0, T]$ (see the proof of Lemma 3.3), is necessary to obtain Lemmas 3.3 and 4.1–4.4. Consequently, this condition is necessary to prove the existence of solutions of (3.1)–(3.5). Knowing that v is at least of class $C^{1,\alpha}(\Omega^T)$, condition (3.25) implies that the considered domain Ω must be either simply connected with edges or nonsimply connected with a smooth boundary.

Let us note that the taking of pressure on S_2, as it was done in this paper, is not the only condition necessary for the uniqueness of problem

(3.1)–(3.4). It is easily proved from (3.1)–(3.4) that the following inequality is satisfied:

$$(6.1)\qquad \frac{d}{dt}\|\vartheta\|_2^2 \le \max_{\Omega}\left|\overset{2}{v}_x\right| \|\vartheta\|_2^2 + \int_{\partial\Omega} \vartheta\cdot\bar{n}\mathcal{P}\,ds - \int_{\partial\Omega} \overset{1}{v}\cdot\bar{n}|\vartheta|^2\,ds,$$

where $\vartheta = \overset{1}{v} - \overset{2}{v}$, $\mathcal{P} = \overset{1}{p} - \overset{2}{p}$ and $\overset{i}{v}$, $\overset{i}{p}$, $i = 1,2$, are two solutions of (3.1)–(3.4). Hence, knowing

$$(6.2)\qquad \vartheta|_{S_1} = 0,\quad \vartheta\cdot\bar{n}|_{S_0} = 0,\quad \vartheta|_{t=0} = 0$$

and

$$(6.3)\qquad \overset{i}{v}\cdot\bar{n}\Big|_{S_1} < 0,\quad \overset{i}{v}\cdot\bar{n}\Big|_{S_0} = 0,\quad \overset{i}{v}\cdot\bar{n}\Big|_{S_2} > 0,\qquad i = 1,2,$$

we see that for a given normal component of the velocity on S_2, there is also a unique solution. Though the existence of solutions of this problem in Sobolev spaces cannot be proved [**24**], it was done in Hölder spaces [**8**].

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