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CRITICAL VALUE SETS OF GENERIC MAPPINGS

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Let Y be a real analytic set. The subset of Y consisting of all points where the local dimension of Y is maximal is called the main part of Y. A subset Y' of a real analytic manifold N is called a main semi-analytic set if Y' is the main part of some analytic set in a neighborhood of each point of N. In this paper it is shown that any proper C^∞ mapping between analytic manifolds can be approximated by an analytic mapping in the Whitney topology so that the critical value set is a main semi-analytic set. An analogue holds true for the algebraic case too.

1. Analytic results. The topology of our spaces of C^∞ or analytic mappings is the Whitney C^∞ topology [see M. Hirsch [**11**]] except for the last section, and M, N always mean real analytic manifolds of dimension n, p respectively. We denote by Σf the critical point set $\{x \in M \mid \text{rank } df_x < \min(n, p)\}$. Our main result is the following.

THEOREM 1. *Let f be a proper C^∞ mapping from M to N. Then every neighborhood of f in $C^\infty(M, N)$ contains a proper analytic mapping g such that $g(\Sigma g)$ is a main semi-analytic set of dimension $l = \min(n-1, p-1)$.*

It is natural to ask if $g(\Sigma g)$ can be analytic in the above. The answer is negative.

EXAMPLE. Let f: $\mathbf{R}^3 \to \mathbf{R}^3$ be the polynomial mapping defined by $f(x_1, x_2, x_3) = (x_1, x_2, x_3^4 + x_2x_3^2 + x_1x_3)$. Then $f(\Sigma f)$ is "Swallow's Tail" [see T. H. Bröcker [**5**] or M. Golubitsky and V. Guillemin [**8**]]. It is known that $f(\Sigma f)$ is not analytic. Moreover there exists a neighborhood U of f in $C^\infty(\mathbf{R}^3, \mathbf{R}^3)$ such that for any g of U, $g(\Sigma g)$ is not an analytic set. (See Figure 1.)

We see easily that a main semi-analytic set is semi-analytic [see S. Łojasiewicz [**12**]]. Any nowhere dense semi-analytic set is the critical value set of some analytic mapping, to say more precisely.

REMARK. Let K be a semi-analytic subset of N of codimension > 0. Then there exist an analytic manifold M and an analytic mapping f: $M \to N$ such that

$$\dim M = \dim N > \dim \Sigma f \quad \text{and} \quad f(\Sigma f) = K.$$

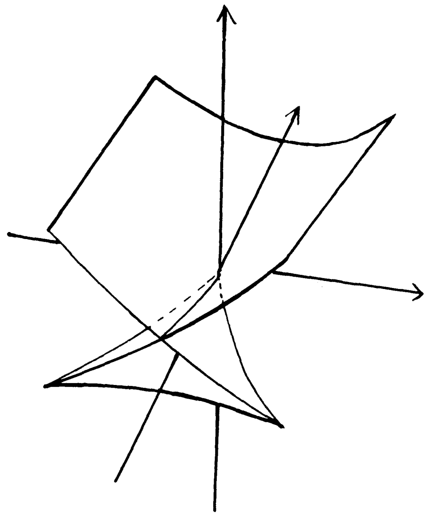

FIGURE 1

Let Y be a main semi-analytic set of dimension l. Then we define the *fundamental class* of Y in the homology group $H_l(Y; \mathbf{Z}_2)$ as follows, where we use infinite chains if Y is not compact. By **[12]** we have a triangulation of Y. Consider the homology groups of the simplicial complex with coefficient $\mathbf{Z}_2$. Then the sum of all l-simplexes defines a cycle, because any analytic set has the fundamental class [see A. Borel and A. Heafliger **[4]**]. The cycle is called the fundamental class of Y. If Y is a subset of N, we denote by $[Y]$ the image of the fundamental class of Y in $H_l(N; \mathbf{Z}_2)$.

Proper C^∞ mappings f_1, f_2: $M \to N$ are called *proper homotopic* if we have a proper C^∞ mapping F: $M \times [0, 1] \to N$ such that $F|_{M\times 0} = f_1$ and $F|_{M\times 1} = f_2$.

THEOREM 2. *There exists an open dense subset G of the set of all proper analytic mappings from M to N such*

(i) *for any f of G, $f(\Sigma f)$ is a main semi-analytic set of dimension* $l = \min(n - 1, p - 1)$,

(ii) *the fundamental class of Σf is mapped to it of $f(\Sigma f)$ by f_* for* $f \in G$,

(iii) *if f and g of G are proper homotopic, we have* $[f(\Sigma f)] = [g(\Sigma g)]$ *in* $H_l(N; \mathbf{Z}_2)$.

2. Preliminaries. First, we prepare some notations of singularities [see J. Boardman **[3]** and J. Mather **[14]**]. For any integer $r \geq 1$, $J^r(n, p)$ is the linear space of r-jets of C^∞ map germs $(\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$, and $J^r(M, N)$ is the set of all r-jets of germs $(M, x) \to (N, y)$ for any $x \in M$, $y \in N$.

Then $J^r(M, N)$ is fiber bundles over $M \times N$ and M with fibers $J^r(n, p)$, $J^r(n, p) \times N$ respectively. We put for any integer $r > 0$

$$M^{(k)} = \{(x_1,\ldots,x_k) \in M^k : x_i \neq x_j \text{ for } i \neq j\},$$
$$_kJ^r(M, N) = q^{-1}(M^{(k)})$$

where q: $J^r(M, N)^k \to M^k$ is the projection. For any C^∞ mapping f: $M \to N$, we denote by j^rf the cross section of the fiber bundle $J^r(M, N) \to M$ naturally defined by f, and by $_kj^rf$ the restriction of $(j^rf)^k$ to $M^{(k)}$. If a subset B of $J^r(n, p)$ is invariant under the coordinate transformations of $(\mathbf{R}^n, 0)$ and $(\mathbf{R}^p, 0)$, we denote by $B(M, N)$ the total space of the subbundle whose fiber is B, and by $B(f)$ the inverse image $j^rf^{-1}(B(M, N))$ for a C^∞ mapping f: $M \to N$.

Let Σ^i or $\Sigma^{i,j}$ denote the Thom-Boardman symbol (see J. Boardman [3]). We put

$$B_0(f) = \begin{cases} \Sigma^{n-p+1,0}(f) & (n \geq p), \\ \Sigma^{1,0}(f) & (n < p). \end{cases}$$

We further put

$$\Delta = \{(j^2f_1(x_1), j^2f_2(x_2)) \in {}_2J^2(M, N) : f_1(x_1) = f_2(x_2)\}.$$

Here we introduce the transversal condition concerning the Thom-Boardman singularities. We say that a C^∞ mapping f: $M \to N$ *satisfies the condition* (T-B), if f has the following properties:

(I) In the case $n \geq p$,

(1) j^1f is transversal to each $\Sigma^i(M, N)$,

(2) j^2f is transversal to each $\Sigma^{n-p+1,j}(M, N)$,

(3) $_2j^2f$ is transversal to $(\Sigma^{n-p+1,0}(M, N) \times \Sigma^{n-p+1,0}(M, N)) \cap \Delta$.

(II) In the case $n < p$,

(1) j^1f is transversal to each $\Sigma^i(M, N)$,

(2) j^2f is transversal to each $\Sigma^{1,j}(M, N)$,

(3) $_2j^2f$ is transversal to $(\Sigma^{1,0}(M, N) \times \Sigma^{1,0}(M, N)) \cap \Delta$.

3. Proof of Theorem 1. In this section we give the proof of Theorem 1.

LEMMA 3.1. [Multi Transversality Theorem [**8**], [**11**], [**14**].] *Let $\mathcal{S}$ be a Whitney stratification of a closed subset of $_kJ^r(M, N)$. Then the set of C^∞ mappings f: $M \to N$ such that $_kj^rf$ is transversal to each stratum of $\mathcal{S}$ is dense in $C^\infty(M, N)$.*

A germ of C^∞ mapping $f\colon (M, x) \to (N, f(x))$ is called *$\mathcal{K}$-finite*, if the quotient $\mathcal{E}_{M,x}/(J_{f,x} + f_x^*(\mathfrak{M}_{N,f(x)})\mathcal{E}_{M,x})$ is finite dimensional over $\mathbf{R}$, where $J_{f,x}$ is the ideal in $\mathcal{E}_{M,x}$ generated by $(p \times p)$-minors of the Jacobian matrix of f at x and $\mathfrak{M}_{N,f(x)}$ is the maximal ideal of $\mathcal{E}_{N,f(x)}$. Especially, a germ is called *$\mathcal{K}$-ν-finite*, if the quotient is at most ν dimensional. The following lemma is an easy consequence from III, Theorem 7.2 in G. Gibson et al. [7] and Lemma 3.1.

LEMMA 3.2. *For sufficiently large ν, the set of C^∞ mappings $f\colon M \to N$ such that the germ f_x is $\mathcal{K}$-ν-finite for all $x \in M$ is an open dense subset of $C^\infty(M, N)$.*

LEMMA 3.3 [H. Whitney [**19**].] *The set $C^\omega(M, N)$ of analytic mappings is dense in $C^\infty(M, N)$.*

LEMMA 3.4. *Let $g\colon M \to N$ be a proper analytic mapping which satisfies the condition* (T-B). *Then g has the following properties*:

(i) *$B_0(g)$ is dense in Σg.*

(ii) *There exists a semi-analytic subset $L \supset \Sigma g - B_0(g)$ such that* $\dim L < \dim \Sigma g$, *and $g|_{\Sigma g - L}\colon \Sigma g - L \to g(\Sigma g - L)$ is an analytic isomorphism.*

Proof. It follows from (1) and (2) of (T-B) that $B_0(g)$ is dense in Σg, has dimension $p - 1$, and the restriction of g to it is an immersion. Put

$$L_1 = \{y \in N\colon \text{there exist points } x_1, x_2 \in B_0(g) \text{ such that } x_1 \neq x_2 \text{ and } g(x_1) = g(x_2) = y\}.$$

Since g is a proper analytic mapping, $g(B_0(g))$ and L_1 are semi-analytic. By (3) of (T-B), we have $\dim L_1 < \dim g(B_0(g))$. Putting $L_2 = \Sigma g - B_0(g)$, L_2 is semi-analytic and $\dim L_2 < \dim \Sigma g$. Here we put $L = g^{-1}(L_1) \cup L_2$. Then (ii) follows.

LEMMA 3.5. *Let $g\colon M \to N$ be a proper analytic mapping such that for any point x of M, the germ of g at x is $\mathcal{K}$-finite. Suppose that $B_0(g)$ is dense in Σg and for an analytic subset L of Σg with* $\dim L < \dim \Sigma g$, *$g|_{\Sigma g - L}\colon \Sigma g - L \to g(\Sigma g - L)$ is an analytic isomorphism. Then $g(\Sigma g)$ is main semi-analytic.*

Note. From the assumption that $B_0(g)$ is dense in Σg, we see that the local dimension of $g(\Sigma g)$ is constant.

Proof. Since g is proper and $g|_{\Sigma g}$ is locally finite-to-one, $g(\Sigma g)$ is closed and $g|_{\Sigma g}$ is finite-to-one. Hence, $g(\Sigma g)$ turns out to be main semi-analytic if we show that for any point x of Σg the image by $g|_{\Sigma g}$ of a neighborhood of x in Σg is the main part of some analytic set in a neighborhood of $g(x)$.

Since the germ of g at x is analytic and $\mathcal{K}$-finite, there exists a representative $g_{\mathbf{C}}$: $U \to V$ of the complexification of g such that $g_{\mathbf{C}}|_{\Sigma g_{\mathbf{C}} \cap U}$ is proper and finite-to-one, where U [resp. V] is an open neighborhood of x [resp. $g(x)$] in a complexification $M_{\mathbf{C}}$ [resp. $N_{\mathbf{C}}$] of M [resp. N] [see C. T. C. Wall [**20**] and H. Hironaka [**10**]]. Then, using the same argument as the proof of Lemma 1.1 in R. Benedetti and A. Tognoli [**2**], we can prove that $g(\Sigma g \cap U)$ is the main part of some analytic set in $V \in N$ if we take U, V smaller, as follows:

We take a desingularization π: $X \to \Sigma g_{\mathbf{C}} \cap U$ and an irreducible component Y of X with $\dim_{\mathbf{R}} \pi(Y) \cap \Sigma g = \dim_{\mathbf{R}} \Sigma g$. Put $\sigma = g_{\mathbf{C}} \circ \pi$: $Y \to \sigma(Y)$ $(\subset V \subset N_{\mathbf{C}})$. First we prove

$$(*) \qquad \dim[(\sigma(Y) \cap N) - g(\pi(Y) \cap (\Sigma g - L))] < \dim g(\Sigma g).$$

From a reason of dimension, a regular value of $\pi|_Y$ is contained in $\Sigma g - L$. Thus, at a point of Y, σ is isomorphic. Hence there exists a complex analytic subset S' of Y with codimension > 0 such that $S' \supset (\pi|_Y)^{-1}(L)$ and $\sigma|_{Y-S'}$ is a local isomorphism. Then $S = \sigma^{-1}(\sigma(S'))$ is a complex analytic subset of codimension > 0 in Y. As Y is connected, $\sigma(Y - S)$ is connected. Furthermore, σ is proper. Thus $\sigma|_{Y-S}$: $Y \to S \to \sigma(Y - S)$ is a covering of finite degree. We claim that this degree is odd. In fact, $(\pi(Y) \cap \Sigma g) - \pi(S) \neq \varnothing$ and for a point $y \in g((\pi(Y) \cap \Sigma g) - \pi(S)) = \sigma(Y - S) \cap g(\Sigma g - L)$, $\sigma^{-1}(y)$ consists of a unique real point and several pairs of non-real conjugate points. This implies that $\#(\sigma^{-1}(y))$ is odd. Now assume inequality $(*)$ does not hold. Then the difference

$$(\sigma(Y - S) \cap N) - (g(\pi(Y)) \cap \Sigma g)$$

has an element y'. But we see that $\sigma^{-1}(y')$ consists of only several pairs of *non-real conjugate points, and* $\#(\sigma^{-1}(y'))$ *is even for the element* $y' \in \sigma(Y - S)$, which is a contradiction. We consider the analytic closure $\widetilde{g(\pi(Y) \cap \Sigma g)}$ of the germ $g(\pi(Y) \cap \Sigma g)$ at $g(x)$. Since σ is proper, $\sigma(Y)$ is a complex analytic subset of V and $\sigma(Y) \cap N \supset \widetilde{g(\pi(Y) \cap \Sigma g)}$ at $g(x)$. Thus, from $(*)$, we have

$$(**) \quad \dim[\widetilde{g(\pi(Y) - \Sigma g)} - g(\pi(Y) \cap (\Sigma g - L))] < \dim g(\Sigma g)$$

at $g(x)$. Lastly, we take a decomposition $X = \bigcup_i Y_i$ into a finite number of irreducible components. We denote by A the union of $\widetilde{g(\pi(Y_i) \cap \Sigma g)}$'s.

Then A is an analytic set and contains $g(\Sigma g \cap U)$. Furthermore, from (**), $\dim(A - g(\Sigma g - L)) < \dim g(\Sigma g)$ at $g(x)$. From *Note*, we have that $g(\Sigma g \cap U)$ is the main part of A. Thus Lemma 3.5 is proved.

Proof of Theorem 1. Let $f\colon M \to N$ be a proper C^∞ mapping. From Lemmas 3.1–3.3 and the fact that the set of proper C^∞ mappings is open in $C^\infty(M, N)$, f can be approximated by a proper analytic mapping g: $M \to N$ such that for any x of M, the germ of g at x is $\mathcal{K}$-finite, and g satisfies (T-B). Hence, by Lemmas 3.4–3.5, we see that $g(\Sigma g)$ is main semi-analytic. This completes the proof of Theorem 1.

4. Proofs of the other results.

Proof of the statement in Example. It is easy to check [see e.g. [7]] that this f is stable in Mather's sence. Hence there exists a neighborhood U of f in $C^\infty(\mathbf{R}^3, \mathbf{R}^3)$ such that for any g of U, we have C^∞ diffeomorphisms τ_1, τ_2 of $\mathbf{R}^3$ such that $f = \tau_1 \circ g \circ \tau_2$. Let $g \in U$. We want to see that $g(\Sigma g)$ is not an analytic set. Let τ be a C^∞ diffeomorphism of $\mathbf{R}^3$ such that $\tau(f(\Sigma f)) = g(\Sigma g)$. We assume $g(\Sigma g)$ to be analytic.

Now we see easily that the singular point set of $f(\Sigma f)$ contains $S_1 = \{y_1 = 0, y_2 \leq 0, 4y_3 = -y_2^2\}$ where $(y_1, y_2, y_3) = f(x_1, x_2, x_3)$ and that $S_2 \cap f(\Sigma f) = \{0\}$ where $S_2 = \{y_1 = 0, y_2 \geq 0, 4y_3 = -y_2^2\}$. We can assume $\tau(0) = 0$. Then the singular point set of $g(\Sigma g)$ contains $\tau(S_1)$. It is well-known that the singular point set of a semi-analytic set is semi-analytic [**12**]. Since $g(\Sigma g)$ is analytic, there exists a one-dimensional analytic set S in a neighborhood V of 0 such that

$$g(\Sigma g) \supset S \supset \tau(S_1) \cap V.$$

Let h be an analytic function on V such that $h^{-1}(0) = S$. As S_1 is diffeomorphic to $(-\infty, 0]$, there exists a C^∞ imbedding ϕ: $(-1, 0] \to \mathbf{R}^3$ such that $\phi(0) = 0$, $\phi((-1, 0]) = \tau(S_1) \cap V$. It follows that $h(\phi(t)) = 0$. Let $\psi = (\psi_1, \psi_2, \psi_3)$ be the Taylor expansion of ϕ at 0. Then $\psi(t)$ is a formal series solution of the equation $h(y_1, y_2, y_3) = 0$. By M. Artin Theorem [**1**], this equation has a convergent series solution $y(t) = (y_1(t), y_2(t), y_3(t))$ such that $y(t) \equiv \psi(t)$ modulo $\mathfrak{M}^c$ for any given integer c, where $\mathfrak{M}$ is the maximal ideal of the formal series ring. We see easily that the convergent solution is an analytic imbedding. This implies that S is the image of ψ in a neighborhood of 0. Hence S and $\tau(S_2)$ are not regularly situated at the origin [see [**12**] for the definition of "regularity situated"]. Therefore S_2 and $f(\Sigma f)$ are not regularly situated because of

$\tau^{-1}(S) \subset f(\Sigma f)$. This contradicts the regular situation property of closed semi-analytic sets **[12]**. Hence $g(\Sigma g)$ is not analytic.

Proof of Theorem 2. Let G' be the set of proper analytic mappings g: $M \to N$ such that for any $x \in M$, the germ of g at x is $\mathcal{K}$-ν-finite [ν: sufficiently large], and g satisfies (T-B). From the proof of Theorem 1, we see easily that G' includes an open dense subset G of the set of proper analytic mappings, and (i) holds.

For any f of G we have a closed semi-analytic subset $K \subset \Sigma f$ such that $\Sigma f - K$ is an analytic manifold, the restriction of f to which is an analytic imbedding, $\dim K < \dim \Sigma f$, $f^{-1}f(K) = K$, and $f(\Sigma f - K)$ is semi-analytic. By **[12]** there exist respective triangulations L_1, L_2 of Σf, $f(\Sigma f)$, and subcomplexes $L_1' \subset L_1$, $L_2' \subset L_2$ such that K and $f(K)$ correspond to the underlying polyhedrons of L_1' and L_2' respectively. Hence (ii) follows immediately.

From (ii), in order to prove (iii), it is sufficient to see $[\Sigma f] = [\Sigma g]$ in $H_l(M; \mathbf{Z}_2)$. This follows from the fact that $j^1 f$ and $j^1 g$ are transversal to each $\Sigma^i(M, N)$ [see Theorem 7 in **[16]** for details of the proof].

5. Algebraic results. In this section we will consider algebraic analogues in the compact open C^∞ topology. We assume this topology on any C^∞ mappings space.

Let Y be an algebraic set of $\mathbf{R}^p$. Then we see that the set $\{y \in Y: \dim Y_y = \dim Y\}$ is semi-algebraic **[12]**. We call this subset the *main part* of Y, and a semi-algebraic set of $\mathbf{R}^p$ is called *main semi-algebraic* if it is the main part of some algebraic set. We remark that for a main semi-algebraic set Y', the main part of the Zariski closure of Y' is Y'. Here, we denote by $\overline{\overline{Y'}}$ the Zariski closure of Y'.

A C^∞ *algebraic manifold* means at once an affine algebraic set and a C^∞ manifold. Restrictions on a subset of polynomial mappings or rational mappings between Euclidean spaces are called equally *polynomial* or *rational*. A rational mapping of C^∞ class is called a C^∞ *rational mapping*.

THEOREM 3. *Let M $[\subset \mathbf{R}^m]$ be a C^∞ algebraic manifold, and f: $M \to \mathbf{R}^p$ be a C^∞ mapping. Then every neighborhood of f contains a proper polynomial mapping g: $M \to \mathbf{R}^p$ such that the critical value set $g(\Sigma g)$ is main semi-algebraic.*

REMARK. If M is a closed C^∞ manifold, any C^∞ mapping f: $M \to \mathbf{R}^p$ can be approximated by one whose critical value set is main semi-algebraic. For any closed C^∞ manifold is C^∞ diffeomorphic to a C^∞ algebraic manifold [see A. Tognoli **[17]**].

Proof of Theorem 3. By Weierstrass' polynomial approximation theorem, f can be approximated by a polynomial mapping g'': $M \to \mathbf{R}^p$ [of degree s]. In this section, we take $s \geq 6$.

We denote by $P(\mathbf{R}^m, \mathbf{R}^p, l)$ the set of polynomial mappings h: $\mathbf{R}^m \to \mathbf{R}^p$ of degree at most l. Then $P(\mathbf{R}^m, \mathbf{R}^p, l)$ is identified with $\mathbf{R}^u$ naturally for some integer u. For a positive number $C > 0$, let g': $\mathbf{R}^m \to \mathbf{R}^p$ be the polynomial mapping defined by $g'(x) = (C|x|^{2s}, \ldots, C|x|^{2s})$. Put

$$Q = P(\mathbf{R}^m, \mathbf{R}^p, s) - \{h \in P(\mathbf{R}^m, \mathbf{R}^p, s) \colon h + g'|_M \text{ satisfies (T-B)}\}.$$

LEMMA 5.1. *Q is a semi-algebraic set in $\mathbf{R}^u$ of codimension* > 0.

Proof. Let F: $M \times P(\mathbf{R}^m, \mathbf{R}^p, s) \to J^1(M, \mathbf{R}^p)$ be the mapping defined by $F'(x, h) = j^1(h + g'|_M)(x)$, F': $M \times P(\mathbf{R}^m, \mathbf{R}^p, s) \to J^2(M, \mathbf{R}^p)$ the mapping defined by $F'(x, h) = j^2(h + g'|_M)(x)$, and F'': $M^{(2)} \times P(\mathbf{R}^m, \mathbf{R}^p, s) \to {}_2J^2(M, \mathbf{R}^p)$ the mapping defined by

$$F''(x, h) = {}_2j^2(h + g'|_M)(x).$$

Then F, F', and F'' are onto submersions [see T. Fukuda [**6**]]. Using arguments given in [**8**], we easily see that Q has measure zero in $\mathbf{R}^u$. By Tarski-Seidenberg Theorem, Q is semialgebraic in $\mathbf{R}^u$, and Q has codim $>$ 0.

From Lemma 5.1, g'' can be approximated by a polynomial mapping $g = h + g'|_M$ which is proper and satisfies (T-B), where $h \in P(\mathbf{R}^m, \mathbf{R}^p, s)$. As g is proper, $g(\Sigma g)$ is closed.

The next lemma follows similarly as Lemma 3.4.

LEMMA 5.2. *Let g: $M \to \mathbf{R}^p$ be a polynomial mapping which satisfies* (T-B). *Then g has the following properties*:

(i) *$B_0(g)$ is dense in Σg,*

(ii) *There exists a semi-algebraic subset $L \supset \Sigma g - B_0(g)$ such that* $\dim L < \dim \Sigma g$, *$\Sigma g - L$ is an analytic submanifold, and $g|_{\Sigma g - L}$: $\Sigma g - L \to g(\Sigma g - L)$ is an analytic isomorphism.*

LEMMA 5.3. *Let V be an algebraic subset of $\mathbf{R}^s$, and σ: $V \to \mathbf{R}^t$ a C^∞ rational mapping. Suppose L is a semi-algebraic subset of V such that* $\dim L < \dim V$ *and $\sigma|_{V-L}$: $V - L \to \sigma(V - L)$ is an analytic isomorphism. Then there exists an algebraic subset V' of $\mathbf{R}^t$ such that $V' \supset \sigma(V - L)$ and* $\dim(V' - \sigma(V - L)) < \dim V$.

For the proof see Lemma 1.1 in [**2**].

Applying Lemma 5.2 and Lemma 5.3, there exist a semi-algebraic subset $L \subset \Sigma g$ with $\dim L < \dim \Sigma g$, and an algebraic subset V' of $\mathbf{R}^p$ such that $V' \supset g(\Sigma g - L)$ and $\dim(V' - g(\Sigma g - L)) < \dim \Sigma g$. Since the closure of the set $\Sigma g - L$ is Σg, we have $\overline{g(\Sigma g)} \subset V'$. Putting $S = \overline{g(\Sigma g)} - g(\Sigma g)$, we have $\dim S < \dim g(\Sigma g)$. Set

$$C = \left\{ y \in \overline{g(\Sigma g)} \mid \dim \overline{g(\Sigma g)}_y = \dim \overline{g(\Sigma g)} \right\}.$$

For any y of C, S does not include $\overline{g(\Sigma g)}$ as germs at y. Hence the germ of $\overline{g(\Sigma g)}$ at y and it of $g(\Sigma g)$ at y intersect. Since $g(\Sigma g)$ is closed, we have $y \in g(\Sigma g)$. Thus we see that $g(\Sigma g) = C$, that is, $g(\Sigma g)$ is main semi-algebraic.

Let M and N be C^∞ algebraic manifolds, and f: $M \to N$ a C^∞ rational mapping. Then $f(\Sigma f)$ is a semi-algebraic subset of N.

Conversely, we have the following remark.

REMARK. Let N be a C^∞ algebraic manifold of dimension n, and K a semi-algebraic subset of N of codimension > 0. Then there exist a C^∞ algebraic manifold M of dimension n and a C^∞ rational mapping f: $M \to N$ with $\dim \Sigma f < n$ such that $f(\Sigma f) = K$.

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