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NON CELL LIKE DECOMPOSITIONS OF S^3

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Let $\mathcal{G}$ be an upper semi continuous decomposition of the three-sphere S^3. The purpose of this note is to describe two conditions under which we can identify the quotient space $X = S^3/\mathcal{G}$, without assuming either that $\mathcal{G}$ is cell like or that X is finite dimensional.

1. Introduction. The following two theorems are our main results.

THEOREM A. *Suppose each element of $\mathcal{G}$ has the shape of the circle S^1 and that X is an ANR. Then $X \cong S^2$ if and only if each decomposition element is linked by some other element.*

THEOREM B. *Suppose the quotient map $f\colon S^3 \to X$ is a (non-constant) approximate fibration whose fiber is a pointed FANR. Then f factors as a composition $S^3 \xrightarrow{\tilde{f}} \tilde{X} \xrightarrow{p} X$ where p is a covering map and $\tilde{f}$ is either cell like or can be approximated by maps topologically equivalent to the Hopf fibration $h\colon S^3 \to S^2$. If f is monotone, p is the identity.*

Note that Theorems A and B do not assume that X is finite dimensional. In the cases covered by Theorem B, we can draw the following conclusions about $X = S^3/\mathcal{G}$. If $\tilde{f}$ can be approximated by fibrations onto S^2, then $X = S^2$ or the projective plane. For the other case, if $\tilde{f}$ is cell-like and in addition the decomposition of S^3 into point inverses of $\tilde{f}$ is shrinkable, then $\tilde{X} = S^3$ **[B2]** and X is a 3-manifold with finite fundamental group. More generally if $\tilde{f}$ is cell-like, then $\tilde{X}$ and X are generalized 3-manifolds (that is 3-dimensional ANRs with $H_i(X, X - \{*\}) \cong H_i(R^3, R^3 - \{0\})$ for every $x \in X$) **[KW]**, **[L2]**. Also $\tilde{X}$ has the homotopy type of S^3 **[L1]** and $\pi_1 X$ is a finite group whose order is the number of components of the fiber of f.

Through this paper, $f\colon S^3 \to X$ will be the quotient map of an upper semicontinuous decomposition of S^3. The definitions of these terms are standard: see **[HY]** for example. It is known that X must be a locally connected, compact, metric space. For a subset $A \subset X$, $\tilde{A}$ will denote $f^{-1}(A)$. Likewise for $p \in X$, $F_p = f^{-1}(p)$. The n-sphere will be denoted S^n; the unit interval $[0, 1]$, I; and the integers, Z. The symbols H_i, H^i and

π_i (respectively $\check{H}_i$, $\check{H}^i$, $\check{\pi}_i$) denote the singular integral homology and cohomology groups and the homotopy groups (respectively Čech groups). We suggest **[Bo]** or **[DS]** as a reference for shape theoretic notions including FANR. When we say X is a FANR, we mean that (X, x) is a pointed FANR (ANSR in **[DS]**) for each $x \in X$. If Σ_1 and Σ_2 are circle shaped compacta (i.e. $\operatorname{Sh}\Sigma_i = \operatorname{Sh} S^1$), we say that Σ_2 *links* Σ_1 if the inclusion induced homomorphism $\check{H}_1(\Sigma_2) \to H_1(S^3 - \Sigma_1)$ is non-trivial. A map is an *approximate fibration* if it has the approximate homotopy lifting property for all spaces. See **[C]** or **[CD1]** for these definitions. A mapping is *circle-shaped* if every point invese has the shape of a circle.

2. Three lemmas. In this section, we present three lemmas which will be needed in the proofs of the main results. We begin by giving a brief summary of the theory of winding functions for the special case of circle-shaped mappings defined on S^3. For further details see **[CD3]**, **[CD4]**, and **[CD5]**.

Let $f\colon S^3 \to X$ be a map with circle-shaped point inverses. Given a point $b \in X$, there is a neighborhood V of b and a shape retract $r\colon \tilde{V} \to F_b$. For $c \in V$, $(r|F_c)_*\colon \check{\pi}_1(F_c) \to \check{\pi}_1(F_b)$ is multiplication by an integer n which is well-defined (up to sign). We set $\alpha_b(c) = |n|$ and refer to $\alpha_b(c)$ as the *winding number* of F_c about F_b. If for each neighborhood U of b, there is a point $c \in U$ such that $\alpha_b(c) = 0$, we say that b is a *degenerate point* and denote the set of degenerate points by K. If $b \in X$ is a point such that for each neighborhood U of b, there is a point $c \in U$ such that $\alpha_b(c) \neq 1$, b is said to be an *exceptional point*. If b is not an exceptional point, we say that b is a *regular point*. It should be clear that the notion of degenerate, exceptional, and regular point are well-defined. If the exceptional point $b \in X$ has a neighborhood U such that each point in $U - \{b\}$ is regular, we say that b is an *isolated exceptional point*. If b is an isolated exceptional point, it follows from **[CD3]** that $\alpha_b(x)$ is constantly equal to some integer d in a neighborhood of b, $x \neq b$. We then say that b has *degree d*. We summarize the facts about winding numbers for later use.

LEMMA 2.1. *Let $f\colon S^3 \to X$ be a circle shaped mapping. If X is an ANR, then*

(i) *K is closed and nowhere dense*;

(ii) *If $\alpha_b(c)$ is defined, there is a neighborhood V of c such that $\alpha_b(x)$ and $\alpha_c(x)$ are defined whenever $x \in V$ and $\alpha_b(x) = \alpha_b(c)\alpha_c(x)$*;

(iii) *There is a dense open set $C \subset X$ such that each point in C is a regular point*;

(iv) *for any open set* $U \subset X$, $f \mid \tilde{U}$ *is an approximate fibration if and only if each point of* U *is a regular point; and*

(v) *the set of nondegenerate exceptional points is countable.*

Proof. The five parts are proven in [**CD5**, L.3.1], [**CD5**, L.3.2], [**CD3**, L.3], [**CD3**, L.4] and [**CD3**, L.6] respectively.

LEMMA 2.2. *Let* $f\colon S^3 \to X$ *be a circle-shaped mapping which is an approximate fibration over an open set* C. *If* A *is an arc with endpoints* c *and* d *such that* $A - \{d\} \subset C$, *then*

(a) *The inclusion* $F_d \to \tilde{A}$ *is a shape equivalence, and*

(b) *the restriction* $\check{H}^1\tilde{A} \to \check{H}^1F_c$ *is multiplication by* $\alpha_d(e)$, *where* e *is any point on* A *such that* $\alpha_d(e)$ *is defined. In particular, if* d *is an isolated exceptional point,* $\alpha_d(e)$ *is the degree of* d.

Proof. The proof of Lemma 5 of [**CD3**] applies almost verbatim, with the obvious modification when d is a degenerate point.

LEMMA 2.3. *Suppose* $f\colon S^3 \to X$ *is a circle-shaped mapping which is an approximate fibration over all but a finite subset of* X. *Then* X *is homeomorphic to* S^2.

Proof. We use Bing's Kline Sphere Theorem [**B1**]. Clearly, X is locally connected, connected, and metric. No pair of points can separate X since no pair of circle-shaped sets can separate S^3. To complete the argument, we must show that each simple closed curve J in X separates X. Let C be the set over which f is an approximate fibration. If $J \subset C$, write J as the union of two arcs meeting in their endpoints. It is easy to see, using 2.2 and a Mayer-Vietoris argument, that $\check{H}^2(\tilde{J}) \cong Z$. Hence $\tilde{H}_0(S^3 - \tilde{J}) \cong Z$ by duality so $\tilde{J}$ separates S^3. Since f is monotone, J separates X.

If J is not contained in C, the spirit of the argument is the same, but the algebra is more complicated. Let $J - C = \{b_1,\ldots,b_n\}$. By (iii), the degrees p_i of the exceptional points b_i are relatively prime in pairs. In particular, if some $p_i = 0$, then $J - C$ is a singleton. In any case where $J - C$ is a singleton $\{b\}$, add a "dummy" b_2 with $p_2 = 1$. Let the b_i be indexed consecutively around J with indices reduced modulo n, and choose points c_i between b_i and b_{i+1} on J. Finally, let A_{i+1} be the arc between c_i and c_{i+1}, $F_i = f^{-1}(c_i)$, and $G_i = f^{-1}(b_i)$. Then $G_i \subset \tilde{A}_i$ is a shape equivalence and $\check{H}^1\tilde{A}_i \to \check{H}^1F_i$ and $\check{H}^1\tilde{A}_{i+1} \to \check{H}^1F_i$ are multiplication by p_i and p_{i+1} respectively.

If $i < n$ we prove inductively that

$$\check{H}^1(\tilde{A}_1 \cup \cdots \cup \tilde{A}_i) \cong Z, \qquad \check{H}^2(\tilde{A}_1 \cup \cdots \cup \tilde{A}_i) \cong 0$$

and $\check{H}^1(\tilde{A}_1 \cup \cdots \cup \tilde{A}_i) \to \check{H}^1 F_i$ is multiplication by $p_1 p_2 \cdots p_i$. This is true for $i = 1$, so we assume it for $i - 1$. Now consider the Mayer-Vietoris sequence where $B_i = \tilde{A}_1 \cup \cdots \cup \tilde{A}_i$:

$$0 \leftarrow \check{H}^2 B_i \leftarrow \check{H}^1 F_i \overset{\phi}{\leftarrow} \check{H}^1 B_{i-1} \oplus \check{H}^1 \tilde{A}_i \overset{\psi}{\leftarrow} \check{H}^1 B_i \leftarrow 0.$$

We have

$$\check{H}^1 F_{i-1} \cong \check{H}^1 B_{i-1} \cong \check{H}^1 \tilde{A}_i \cong Z \quad \text{and} \quad \phi(s, t) = (p_1 p_2 \cdots p_{i-1} s, -p_i t).$$

Since $p_1 p_2 \cdots p_{i-1}$ is relatively prime to p_i, ϕ is onto and $\check{H}^2 B_i \cong 0$. Also

$$\check{H}^1 B_i \cong \operatorname{im} \psi \cong \ker \phi = \{(s, t) | p_1 p_2 \cdots p_{i-1} s = p_i t\} \cong Z.$$

Finally $\psi(r) = (p_i r, p_1 \cdots p_{i-1} r)$, so $\check{H}^1 B_i \to \check{H}^1 \tilde{A}_i \to \check{H}^1 F_i$ is multiplication by $p_1 p_2 \cdots p_i$.

Now consider the effect of adding the last arc A_n, thus closing J. Again the Mayer-Vietoris sequence yields

$$0 \leftarrow \check{H}^2 \tilde{J} \leftarrow \check{H}^1 F_{n-1} \oplus \check{H}^1 F_n \overset{\phi}{\leftarrow} \check{H}^1 B_{n-1} \oplus \check{H}^1 A_n.$$

Here

$$\check{H}^1 F_{n-1} \cong \check{H}^1 F_n \cong \check{H}^1 B_{n-1} \cong \check{H}^1 A_n \cong Z$$

and

$$\phi(s, t) = (p_1 p_2 \cdots p_{n-1} s - p_n t, p_{n-1} \cdots p_2 p_1 s - p_n t).$$

Hence $\check{H}^2 \tilde{J} \cong Z$. As before, this means J separates X. We therefore conclude that $X \cong S^2$.

3. Proof of Theorem A. Suppose f: $S^3 \to X$ is a circle-shaped mapping, X is an ANR, and each point inverse of f is linked by some other point inverse. To see that $X \cong S^2$, it suffices, by Lemma 2.3, to show that f is an approximate fibration over the complement of a finite set.

Case I. Suppose there are no degenerate points for f. By Lemma 2.1 (iii) and (v) if we write $X = C \cup D$, where D is the set of exceptional points and C the regular points, then C is open and D is countable. Now suppose that D has a limit point d. Let U be a connected neighborhood of d on which α_d is defined. It follows from Lemma 2.1 (ii) that α_d is constant on $C \cap U$, say $\alpha_d \equiv p$. If $d' \in D \cap U$, and $x \in C$ is sufficiently close to d', then $\alpha_d(d') \cdot \alpha_{d'}(x) = \alpha_d(x) = p$ by Lemma 2.1 (ii) again. Thus α_d is bounded by p on U, and we have $p > 1$ since d is an exceptional point. Furthermore, if $\alpha_d(x) = p$ for some $x \in U$, x must be a regular point (since Lemma 2.1 (ii) implies that α_x would be constantly 1 on some neighborhood of x). Therefore, there are two points d', d'' in $D \cap U$ such that $\alpha_d(d') = \alpha_d(d'') = q < p$. Let A be an arc with endpoints d' and d'' such that $A - \{d', d''\} \subset U \cap C$. If we write $A = A_1 \cup A_2$, where A_1 and A_2 are arcs meeting in a common endpoint, if follows from Lemma 2.2 and a Mayer-Vietoris argument that $\check{H}^2(\tilde{A})$ has torsion, contradicting Alexander duality. This contradiction completes the proof in Case I. Note that the linking assumption was not needed in this case.

Case II. There are degenerate points of f. In this case, let C denote the regular points, E the non-degenerate exceptional points, and K the degenerate points. By Lemma 2.1 (v), (iii) and (i), E is countable and C is open and dense. Suppose first that K contains more than one point. Then a simple point-set argument gives points d_1 and d_2 in K, a point $c \in C$, and arcs A_1 and A_2 such that A_i has endpoints c and d_i, $A_i - \{d_i\} \subset C$, and $A_1 \cap A_2 = \{c\}$. By the linking hypothesis, there is a point c' such that $F_{c'}$ links F_c. We may assume that $c' \neq d_1$ and thus $c' \notin A_1$. According to Lemma 2.2, the restriction $\check{H}^1(\tilde{A}_1) \to \check{H}^1(F_c)$ is the zero homomorphism. Then we have the commutative diagram

$$\begin{array}{ccccc} \check{H}_1(F_{c'}) & \xrightarrow{i_*} & H_1(S^3 - \tilde{A}_1) & \xrightarrow{\phi} & H_1(S^3 - F_c) \\ & & \downarrow & & \downarrow \\ & & \check{H}^1(\tilde{A}_1) & \xrightarrow{\psi} & \check{H}^1(F_c) \end{array}$$

where the vertical arrows are duality isomorphisms and the horizontal arrows are inclusion induced. Tracing the composition $\check{H}_1(F_{c'}) \to \check{H}^1(F_c)$ gives a contradiction, since ϕi_* is non-zero and ψ is the zero map. It follows that K is the singleton set $\{d_1\}$. We may now assume that if E is non-empty, E contains an isolated exceptional point e of degree $p > 1$. We can then construct an arc A_3 with endpoints d_1 and e such that $A_3 - \{d_1, e\} \subset C$ and conclude as above that $\check{H}^2(\tilde{A}_3)$ has torsion. Thus $X - C = \{d_1\}$ and the proof is complete in Case II.

For the converse, suppose that f: $S^3 \to S^2$ is a circle-shaped map, $p \in S^2$. A simple modification of the proof of Theorem 2 of [**L3**] shows that for some $q \in S^2$, F_p links F_q. (Lacher's argument is given for the case where F_p, F_q are homeomorphic to S^1. One need only make the obvious translation using Čech cohomology.)

4. Proof of Theorem B. Let f: $S^3 \to X$ be a nonconstant approximate fibration. It follows from **[CD6]** that X is LC^n for all n, and from **[CD3]** that any two fibers of f have the same shape.

PROPOSITION 4.1. *For each $x \in X$ and $y \in F_x$, there is a long exact sequence*

$$\to \check{\pi}_i(F_x, y) \to \pi_i(S^3, y) \to \pi_1(X, x) \to \cdots .$$

Proof. The argument in **[CD1]** applies directly.

PROPOSITION 4.2. *$\check{\pi}_1(F_x)$ is abelian for each $x \in X$.*

Proof. Proposition 4.1 implies that $\check{\pi}_1(F_x) \cong \pi_2(X)$.

PROPOSITION 4.3. *$\check{H}^2(F_x) \cong 0$ for all $x \in X$.*

Proof. If $\check{H}^2(F_x) \ncong 0$, then F_x separates S^3 by duality. Since f is a proper map, x is a cut point of X. Since all fibers have the same shape, this contradicts the classical fact that every nondegenerate continuum must have at least two non-cut points.

Suppose, until further notice, that f is monotone. Fix x and let $F = F_x$.

PROPOSITION 4.4. *For each neighborhood U of F, there is a compact connected manifold N, $F \subset \operatorname{int} N \subset N \subset U$, such that*

(a) *the boundary of N is connected*;

(b) $H^2(N) = H^2(S^3 - (\operatorname{int} N)) = 0$

(c) $\pi_i(N) = 0$, $i \geq 2$.

Proof. Since $\check{H}^2(F) = 0$, F does not separate U. A standard argument joining the boundary components of a simplicial neighborhood of F in U yields a manifold neighborhood N of F in U satisfying (a). A simple Mayer-Vietoris argument shows that N satisfies (b). Suppose that $\pi_2(N) \neq 0$. According to the Sphere Theorem **[P]**, there is a tame 2-sphere $\Sigma \subset N$ which does not bound a 3-ball in N. But Σ bounds 3-balls A and B in S^3 and if neither is contained in N, the fact that $S^3 - (\operatorname{int} N)$ is connected would imply that Σ does not separate S^3. Since $\pi_2(N) \cong 0$, (c) follows by applying the Hurewicz and Whitehead Theorems to the universal cover of N.

PROPOSITION 4.5. *The fiber F of f has the shape of either a point or S^1.*

Proof. Since F is a FANR, F has the shape of a CW complex **[EG]**, so that we may appeal to standard algebraic topology. Thus $\check{\pi}_1(F) \cong \check{H}_1(F)$ by the Hurewicz Theorem and Proposition 4.2. Since the Universal Coefficient Theorem injects the torsion of $\check{H}_1(F)$ into $\check{H}^2(F) \cong 0$, $\check{\pi}_1(F)$ is free abelian. By the finiteness theorem of **[EG]**, F has the shape of a finite complex K. By (c) of Proposition 4.4, K is an Eilenberg-Mac Lane $K(Z^n, 1)$ space. Hence K is homotopy equivalent to the product of n circle and the conclusion is immediate from the fact that $H^2(K) \cong 0$.

PROPOSITION 4.6. *If the quotient map $f\colon S^3 \to X$ is a (nonconstant) monotone approximate fibration whose fiber is an FANR, then either f is a cell-like map or $X \cong S^2$ and f can be approximated by maps topologically equivalent to the Hopf fibration.*

Proof. If f is not cell-like, it follows from Proposition 4.5 and Lemma 2.3 that $X \cong S^2$. The conclusion follows from **[CD3]**.

To complete the proof of Theorem B, suppose that $f\colon S^3 \to X$ is a nonconstant approximate fibration with non-connected, FANR point inverses. Let $p\colon \tilde{X} \to X$ be the universal cover of X. Since S^3 is simply connected, f lifts to a map $\tilde{f}\colon S^3 \to \tilde{X}$. It is not difficult to show, using path lifting, that $\tilde{f}$ is surjective.

PROPOSITION 4.7. *The fibers of $\tilde{f}$ are connected FANR's.*

Proof. Suppose that for some x, $\tilde{f}^{-1}(x)$ has at least two distinct components P and Q. Let $A \subset S^3$ be an arc with one endpoint in each of P and Q with int $A \subset S^3 - \tilde{f}^{-1}(x)$. Since $\tilde{X}$ is simply connected, $f(A)$ must be homotopic rel $p(x)$ to a constant map. By the regular lifting property of approximate fibrations **[CD]**, A can be homotoped into an arbitrarily small neighborhood of $\tilde{f}^{-1}(x)$ holding the endpoints of A fixed. This gives a contradiction. Thus the fibers of $\tilde{f}$ are connected. Since the fibers of $\tilde{f}$ are components of the fibers of f, each fiber of $\tilde{f}$ is an FANR.

LEMMA 4.8. *Let $p\colon \tilde{X} \to X$ be a finite sheeted covering space and ε be an open cover of $\tilde{X}$. There exists an open cover δ of X such that if $F\colon Z \times I \to \tilde{X}$ and $G\colon Z \times I \to \tilde{X}$ are maps such that $F \mid Z \times \{0\} = G \mid Z \times \{0\}$ and pF is δ-close to pG, then F is ε-close to G.*

Proof. Let η be an open cover of X by evenly covered sets such that the components of $p^{-1}(\eta)$ refine ε. Since X is LC^1 there is an open cover δ of X such that δ-close paths in X are η-homotopic **[H]**. Now given F and G as in the hypothesis, fix $z \in Z$ and consider paths ϕ and γ defined by $\phi(t) = F(z, t)$ and $\gamma(t) = G(z, t)$. Note $\phi(0) = \gamma(0)$ and $p\phi$ is δ-close to $p\gamma$. Hence there exists a homotopy H: $I \times I \to X$ such that $H(t, 0) = p\phi(t)$, $H(t, 1) = p\gamma(t)$, $H(0, s) = p\phi(0) = p\gamma(0)$ and $H(\{t\} \times I)$ is contained in some element of η for each $t \in I$, $s \in I$. Choose a subdivision $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $H([t_i, t_{i+1}] \times I)$ is contained in some element U_i of η. Since p is a covering projection, there is a lifting $\tilde{H}$: $I \times I \to \tilde{X}$ of H such that $\tilde{H}(t, 0) = \phi(t)$ and $\tilde{H}(t, 1) = \gamma(t)$. Since $H([t_i, t_{i+1}] \times I) \subset U_i$ and $\tilde{H}([t_i, t_{i+1}] \times I)$ is connected, $\tilde{H}([t_i, t_{i+1}] \times I)$ is contained in a component of $\tilde{U}_i$. Therefore ϕ is ε-close to γ and since z was arbitrary F is ε-close to G.

PROPOSITION 4.9. *The map $\tilde{f}$ is an approximate fibration.*

Proof. Given a space Z, a map h: $Z \times \{0\} \to S^3$, a homotopy H: $Z \times I \to \tilde{X}$ and an open cover ε of $\tilde{X}$ such that $H | Z \times \{0\} = \tilde{f}h$, choose an open cover δ of X satisfying Proposition 4.8. Since f is an approximate fibration, there exists a homotopy $\tilde{H}$: $Z \times I \to S^3$ which extends h and for which $f\tilde{H}$ is δ-close to pH. Since $f = p\tilde{f}$, $p\tilde{f}\tilde{H}$ is δ-close to pH and $\tilde{f}\tilde{H} | Z \times \{0\} = \tilde{f}h = H | Z \times \{0\}$. By Lemma 4.8, $\tilde{f}\tilde{H}$ is ε-close to H which proves $\tilde{f}$ is an approximate fibration.

Theorem B now follows by applying Proposition 4.6 to the map $\tilde{f}$. If f is monotone, then X is simply connected so $\tilde{X} = X$ and p is the identity.

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