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# CONDITIONAL EXPECTATION WITHOUT ORDER

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**In this paper we show that an arbitrary contractive projection on a $J^*$-algebra has the properties of a conditional expectation (Theorem 1). This fact is then used to solve the bicontractive projective problem (Theorem 2).**

Let $M$ be a $J^*$-algebra and let $\theta$ be an isometry (equivalently a $J^*$-automorphism [7]) of $M$ of order 2. Then $P$, defined by $Px = \frac{1}{2}(x + \theta x)$, is a bicontractive projection on $M$, i.e., $P^2 = P$, $\|P\| \leq 1$, $\|\mathrm{id}_M - P\| \leq 1$. By the bicontractive projection problem we mean the converse of this statement.

An affirmative answer to the bicontractive projection problem imposes strong symmetry properties on the Banach space $M$, so it cannot be true for a general Banach space.

In Bernau-Lacey [**2**], the problem is solved for the class of Lindenstraus spaces. In [**1**] Arazy-Friedman solved it with $M =$ the $C^*$-algebra of compact operators on a separable complex Hilbert space. In [**10**], Størmer, influenced by partial results of Robertson-Youngson [**9**], solved it with $M$ an arbitrary $C^*$-algebra and $P$ assumed positive and unital. Our Theorem 2, specialized to a $C^*$-algebra, generalizes each of these results of Arazy-Friedman and Størmer. The authors have recently solved the problem for associative Jordan triple systems [**3**].

Both Robertson-Youngson and Størmer expressed the belief that the result is true in the case of a positive unital projection with contractive complement on a $JB$-algebra. In order to prove Theorem 2, we found it necessary to first prove the conjecture of Robertson-Youngson in the case of a $JC$-algebra.

As we have pointed out [**6**], a $J^*$-algebra is the appropriate algebraic model in which to study problems not involving order. The techniques developed by us in [**4, 5**] can now be used to give a short solution of the bicontractive projection problem.

A simple analysis of this problem leads to a formulation of the conditional expectation properties proved in Theorem 1. As a corollary of Theorem 1 we obtain an analogue of the well known theorem of Tomiyama [**11**].

A *$J^*$-algebra* is a norm closed complex linear subspace of $\mathscr{L}(H, K)$, the bounded linear operators from a Hilbert space $H$ to a Hilbert space $K$, which is closed under the operation $a \to aa^*a$. By setting $\{abc\} = \frac{1}{2}(ab^*c + cb^*a)$, one can make a $J^*$-algebra into a Jordan triple system.

We now recall some notation and results from [**4**, **5**] which will be used in this paper.

Let $M$ be a $J^*$-algebra. For each $f$ in $M'$ let $v = v(f)$ be the unique partial isometry in $M''$ occurring in the enveloping polar decomposition of $f$ [**4**: Th. 1]. Then $l(f) = vv^*$ and $r(f) = v^*v$ are projections in the von Neumann algebra $A''$, where $A$ is any $C^*$-algebra containing $M$ as a $J^*$-subalgebra. More generally, for any partial isometry $v$ in $M''$, the *Peirce projections* are defined by $E(v)x = lxr$, $F(v)x = (1-l)x(1-r)$, $G(v)x = lx(1-r) + (1-l)xr$, where $l = vv^*$ and $r = v^*v$. We shall write $E(f)$ for $E(v(f))$ and similarly for $G(f)$ and $F(f)$.

The following commutativity formulas from [**4**] are fundamental: let $Q$ be a contractive projection on the dual $M'$ of a $J^*$-algebra $M$ and let $f \in Q(M')$. Then

(0.1) $$QE(f) = E(f)QE(f) \quad ([\mathbf{4}\text{: Prop. 3.3}]);$$

(0.2) $$G(f)Q = QG(f)Q \quad ([\mathbf{4}\text{: Prop. 4.3}]);$$

(0.3) $$E(f)Q = QE(f)Q \quad ([\mathbf{4}\text{; Prop. 4.3}]).$$

Let $Q$ be a contractive projection on the dual $M'$ of $M$. By an *atom* of $Q$ is meant any extreme point of the unit ball $Q(M')_1$ of $Q(M')$. The elements $v(f)$, $f$ an atom of $Q$, are called *minimal tripotents* of $Q'$. Define

$$L_0 = \sup\{l(f): f \text{ atom of } Q\}, \quad R_0 = \sup\{r(f): f \text{ atom of } Q\}.$$

Then $L_0$, $R_0$ are projections in $A''$ (where $A$ is any $C^*$-algebra containing $M$ as a $J^*$-subalgebra) and they define contractive projections $\mathscr{E}_0$ and $\mathscr{T}_0$ on $A''$ by $\mathscr{E}_0 z = L_0 z R_0$, $\mathscr{T}_0 z = (1 - L_0)z(1 - R_0)$, for $z \in A''$. We reserve the notation $L_1$, $R_1$, $\mathscr{E}_1$, $\mathscr{T}_1$ for the objects just defined in the case $Q = \mathrm{id}_{M'}$.

A fundamental result from [**5**] is the following decomposition of functionals with respect to a contractive projection $Q$ [**5**: Theorem 1]

(0.4) Let $Q$ be a contractive projection on the dual $M'$ of a $J^*$-algebra $M$. Then $Q(M') = \mathscr{A} \oplus_{l_1} \mathscr{N}$, where $\mathscr{A}$ is the norm closed linear span of the atoms of $Q$ and the unit ball of $\mathscr{N}$ has no extreme points. Moreover $\mathscr{A} = \mathscr{E}_0 Q(M')$ and $\mathscr{N} = \mathscr{T}_0 Q(M')$.

We shall use the following two consequences of this result (cf. [**5**: Cor. 4.4, Lemma 4.5, Prop. 4]).

(0.5) Let $M_{\text{fin}}$ be the set of all finite linear combinations of pairwise orthogonal minimal tripotents of $Q'$. Then $M_{\text{fin}}$ is $\sigma$-weakly dense in $\mathscr{E}_0 Q'(M'')$.

(0.6) For each $x$ in $Q'(M'')$ we have $x = \mathscr{E}_0 x + \mathscr{T}_0 x$. Then by (0.5), $\mathscr{E}_0 x, \mathscr{T}_0 x \in M''$.

The following fact is a consequence of [**4**: Remark 2.5b] and [**5**: Lemma 4.5].

(0.7) For $x \in M''$, $\mathscr{E}_0 x = 0$ implies $\mathscr{E}_0 Q'x = 0$.

Finally we shall use the following, which is a consequence of [**4**: Remark 3.2]:

(0.8) Let $P$ be a contractive projection on a $J^*$-algebra $M$, and let $f \in M'$. Then $E(f)M''$ is a $JW^*$-algebra and $E(f)P''$ restricted to $E(f)M''$ is a positive unital faithful projection.

**1. Conditional expectation without order.** In this section we prove the conditional expectation properties of an arbitrary contractive projection and prove the conjecture of Robertson-Youngson for $JC$-algebras.

THEOREM 1. *Let $P$ be a contractive projection on a $J^*$-algebra $M$. Let $a$, $x \in M$ satisfy $Pa = a$, $Px = 0$. Then*

(i) $P\{aax\} = 0$;

(ii) $P\{axa\} = 0$.

*Proof*. (i) Let $b = \sum_{i=1}^{n} \alpha_i v_i \in M_{\text{fin}}$ with $v_i$ orthogonal minimal tripotents of $P''$. We show first that $P''\{bbx\} = 0$. We have

$$\{bbx\} = \sum_{i,j} \alpha_i \bar{\alpha}_j \{v_i v_j x\} = \sum_i |\alpha_i|^2 \{v_i v_i x\},$$

and

$$P''\{v_i v_i x\} = P''\big(E(v_i) + \tfrac{1}{2}G(v_i)\big)x = P''\big(E(v_i) + \tfrac{1}{2}G(v_i)\big)P''x = 0$$

by (0.2) and (0.3). Thus $P''\{bbx\} = 0$ and by linearization we have $P''\{bcx\} = 0$ for $b, c \in M_{\text{fin}}$. By (0.6), $a = \mathscr{E}_0 a + \mathscr{T}_0 a$ so that

$$\{aax\} = \{\mathscr{E}_0 a, \mathscr{E}_0 a, x\} + \{\mathscr{T}_0 a, \mathscr{T}_0 a, x\}.$$

Set $\alpha_1 = \{\mathscr{E}_0 a, \mathscr{E}_0 a, x\}$, $\alpha_2 = \{\mathscr{T}_0 a, \mathscr{T}_0 a, x\}$. Since by Krein-Milman, $\|P\{aax\}\| = \|\mathscr{E}_0 P\{aax\}\|$, it suffices to prove $\mathscr{E}_0 P''\alpha_1 = \mathscr{E}_0 P''\alpha_2 = 0$. Since $\alpha_2 \in M''$ and $\mathscr{E}_0 \alpha_2 = 0$ we have $\mathscr{E}_0 P''\alpha_2 = 0$ by (0.7). On the other hand, with $b_n$ a net in $M_{\text{fin}}$ converging $\sigma$-weakly to $\mathscr{E}_0 a$, we have $\alpha_1 = \lim_n \lim_m \{b_n b_m x\}$ so that $P''\alpha_1 = 0$.

(ii) With $a = \mathscr{E}_0 a + \mathscr{T}_0 a$ we have $\{axa\} = \beta_1 + \beta_2 + 2\beta_3$ where $\beta_1 = \{\mathscr{E}_0 a, x, \mathscr{E}_0 a\}$, $\beta_2 = \{\mathscr{T}_0 a, x, \mathscr{T}_0 a\}$, $\beta_3 = \{\mathscr{E}_0 a, x, \mathscr{T}_0 a\}$. Since $\|P\{axa\}\| = \|\mathscr{E}_0 P\{axa\}\|$ it suffices to prove $\mathscr{E}_0 P''\beta_1 = \mathscr{E}_0 P''\beta_2 = \mathscr{E}_0 P''\beta_3 = 0$. Since $\beta_2, \beta_3 \in M''$ and $\mathscr{E}_0\beta_2 = \mathscr{E}_0\beta_3 = 0$, we have $\mathscr{E}_0 P''\beta_2 = \mathscr{E}_0 P''\beta_3 = 0$. We now prove that $P''\beta_1 = 0$. By the linearization and approximation argument in the proof of (i), it will suffice to prove $P''\{bxb\} = 0$ for $b \in M_{\text{fin}}$. Setting $b = \sum_{i=1}^n \alpha_i v_i$ with $v_i$ orthogonal minimal tripotents of $P''$ shows that it suffices to prove that $P''\{vxu\} = 0$ whenever $u$, $v$ are minimal tripotents of $P''$ which are either equal or orthogonal.

Let $w = u + v$ (or $w = v$ if $u = v$), let $A$ be the $JW^*$-algebra $E(w)M''$ with identity element $e$, and let $R$ be the unital contractive projection $E(w)P''$ on $A$. Let $z = \{vxu\}$. Since, by (0.3), $P''z = P''E(w)z = P''E(w)P''z = P''Rz$, it suffices to prove that $Rz = 0$. Let $y = E(w)x$ and note that $z = \{vxu\} = \{vyu\}$ and $y \in A$. Note also that $e$, $v$, $u \in R(A)$ and that by (0.1) $Ry = E(w)P''E(w)x = E(w)P''x = 0$. It is easy to verify that

$$\{ve\{uey\}\} = \{v\{eye\}u\} + \{v\{eue\}y\}$$

so that $z = \{vye\} = \{ve\{uey\}\} + \{vuy\}$. By (i) applied to $R$ and $A$, $R(z) = 0$. □

By considering elements $x$ of the form $z - Pz$, and linearizing we obtain:

COROLLARY 1. *Let $P$ be a contractive projection on a $J^*$-algebra $M$. For $x, y, z \in M$,*

$$P\{Px, Py, Pz\} = P\{Px, Py, z\} = P\{Px, y, Pz\}.$$

We know from [**5**: Theorem 2] that $P(M)$ is a Jordan triple system isometric to a $J^*$-algebra. If $P(M)$ happens to be a $J^*$-subalgebra of $M$ we obtain the following analogue of a well known of Tomiyama.

COROLLARY 2. *Let $N$ be a $J^*$-subalgebra of a $J^*$-algebra $M$ and let $P$ be a norm one projection of $M$ onto $N$. Then for $a, b \in N$ and $x \in M$,*

(i) $P\{abx\} = \{a, b, Px\}$,

(ii) $P\{axb\} = \{a, Px, b\}$.

We note that (ii) was proved for $JB$-algebras and unital $P$ in [**8**: Appendix].

Our final corollary solves the problem of Robertson-Youngson in the important cases of a $JC$-algebra.

COROLLARY 3. *Let $R$ be a unital bicontractive projection on a $JC$-algebra $A$. Then $R$ has the form $Rx = \frac{1}{2}(x + \theta x)$ where $\theta$ is a Jordan automorphism of $A$ of order* 2.

*Proof.* As remarked by Robertson-Youngson, such a $\theta$ exists if and only if we have the implication: $Ra = 0 \Rightarrow R(a^2) = a^2$. Since the complexification of $A$ is a $J^*$-algebra we have, with $Q = \mathrm{id} - R$, $Q(a^2) = Q\{a, 1, a\} = 0$ since $Qa = a$ and $Q1 = 0$.

## 2. Solution of the bicontractive projection problem.

In this section we prove the following, which solves the bicontractive projection problem for $J^*$-algebras.

THEOREM 2. *Let $P$ be a bicontractive projection on a $J^*$-algebra $M$. Then there is a $J^*$-automorphism $\theta$ of $M$ of order* 2 *such that*

$$Px = \tfrac{1}{2}(x + \theta x), \qquad x \in M. \tag{2.0}$$

*Proof.* Let $P$ be a bicontractive projection on a $J^*$-algebra $M$ and define $\theta$ by (2.0). We need only show that

$$\theta(xx^*x) = \theta x(\theta x)^*\theta x, \quad \text{for } x \in M. \tag{2.1}$$

Write $x = x_1 + x_2$, with $x_1 \in P(M)$ and $x_2 \in (\mathrm{id} - P)(M)$. Then $\theta x = x_1 - x_2$ and

$$xx^*x = x_1x_1^*x_1 + x_2x_2^*x_2 + 2\{x_1x_1x_2\} + 2\{x_2x_2x_1\} + x_1x_2^*x_1 + x_2x_1^*x_2, \tag{2.2}$$

$$\theta x(\theta x)^*\theta x = x_1x_1^*x_1 - x_2x_2^*x_2 - 2\{x_1x_1x_2\} + 2\{x_2x_2x_1\} - x_1x_2^*x_1 + x_2x_1^*x_2. \tag{2.3}$$

By Theorem 1 applied to $P$ and $\mathrm{id} - P$ we have

$$P\{x_1x_1x_2\} = 0, \qquad P\{x_1x_2x_1\} = 0; \tag{2.4}$$

$$P\{x_2x_2x_1\} = \{x_2x_2x_1\}, \qquad P\{x_2x_1x_2\} = \{x_2x_1x_2\}. \tag{2.5}$$

Below we shall prove

(2.6) $P(M)$ and $(\mathrm{id} - P)(M)$ are $J^*$-subalgebras of $M$.

Applying $\theta = 2P - \text{id}$ to (2.2) and using (2.4)–(2.6) we get (2.1).

Thus Theorem 2 will be proved if we can show that the range of a bicontractive projection on a $J^*$-algebra is a $J^*$-subalgebra. This will be done in Proposition 1 below, for which we prepare some lemmas.

We need two technical facts in order to prove Proposition 1. First, $P''$ fixes the atomic part of $P''$ (Lemma 4) and second, the decompositions $x = \mathscr{E}_0 x + \mathscr{T}_0 x$ of $x \in P''(M)$ and $x = \mathscr{E}_1 x + \mathscr{T}_1 x$ (defined in the introduction) coincide (Lemma 5). Lemmas 1 and 2 are preliminary to Lemma 3, which is needed to prove Lemma 5.

LEMMA 1. *Let $A$ be a $JW$-algebra and let $R$ be a normal unital bicontractive projection on $A$. Then $R(A)$ is a $JW$-subalgebra of $A$ and if $R(A)$ is purely non-atomic then so is $A$.*

*Proof*. The fact that $R(A)$ is a $JW$-subalgebra follows from [**9**]. By Corollary 3, $R = \frac{1}{2}(\text{id} + \theta)$ with $\theta$ a Jordan automorphism of $A$.

Suppose that $\varphi$ is a multiple of a normal pure state of $A$. Then $\psi \equiv R'\varphi = \frac{1}{2}(\varphi + \theta'\varphi)$ is a purely atomic normal positive functional on $A$ and can therefore be written as a linear combination of two orthogonal normal pure states of $A$. It follows that $E(\psi)A$ is a $JW$-algebra of rank $\leq 2$. Now $E(\psi)R(A)$ is a $JW$-subalgebra of $E(\psi)A$, hence also of rank $\leq 2$. Since $\psi$ is in the range of $(E(\psi)R)'$ it can be written as a linear combination of two atoms of $R'E(\psi)$, which are atoms of $R'$ by [**5**: Remark 1.3]. Since $R(A)$ is purely non-atomic we must have $\psi = 0$. But $R$ is faithful, so $\varphi = 0$. □

In the lemmas that follow, $P$ denotes a bicontractive projection on a $J^*$-algebra $M$.

LEMMA 2. *The atoms of $P'$ lie in the convex hull of the extremal points of the unit ball $M_1'$ of $M'$.*

*Proof*. Let $f$ be an atom of $P'$. Let $A$ be the $JW$-algebra which is the self-adjoint part of $E(f)M''$, and let $R = E(f)P''$ on $A$. By (0.8) and [**4**: Prop. 3.7], $R$ is a unital bicontractive projection on $A$ with $R(A) = \mathbf{R} \cdot 1_A$. According to [**9**: Prop. 2.6] there are three possible cases: $A = \mathbf{R} \cdot 1_A$, $A = \mathbf{R} \oplus \mathbf{R}$, $A =$ a spin factor. Therefore $E(f)M''$ is a Jordan algebra of rank $\leq 2$, and so $f$ is a convex combination of at most two extremal elements of $E(f)M'$, which, by [**5**: Remark 1.3] are extremal points of $M_1'$. □

We shall now use Lemmas 1 and 2 to show that the decomposition (0.4) of a functional in the image of $P'$ coincides with the decomposition corresponding to the identity projection.

LEMMA 3. *For each* $\varphi$ *in* $P'(M')$ *we have* $\mathscr{E}_0\varphi = \mathscr{E}_1\varphi$ *and* $\mathscr{T}_0\varphi = \mathscr{T}_1\varphi$. *Moreover*

$$\mathscr{T}_1 P'\mathscr{E}_0 = 0 \quad \text{and} \quad \mathscr{E}_1 P'\mathscr{T}_1 = 0. \tag{2.7}$$

*Proof*. Let $\varphi_1 = \mathscr{E}_0\varphi$, $\varphi_2 = \mathscr{T}_0\varphi$, and let $R = E(\varphi_2)P''$ restricted to $A = E(\varphi_2)M''$. By (0.8) $R(A)$ is a $JW$-subalgebra of $A$ and by the definition of $\mathscr{T}_0\varphi$, $R(A) = E(\varphi_2)P''(M'')$ is purely non-atomic. By Lemma 1, $A$ is purely non-atomic so that $\varphi_2 = \mathscr{T}_1\varphi_2$. On the other hand, by Lemma 2,

$$\mathscr{E}_0\varphi = \mathscr{E}_1\varphi_1 = \mathscr{E}_1(\varphi - \varphi_2) = \mathscr{E}_1\varphi - \mathscr{E}_1\mathscr{T}_1\varphi_2 = \mathscr{E}_1\varphi.$$

We now prove (2.7). Let $\varphi \in M'$, and write $\varphi = P'\varphi + (\mathrm{id} - P')\varphi$. Decompose $P'\varphi$ and $(\mathrm{id} - P')\varphi$ with respect to $P'$ and $\mathrm{id} - P'$ respectively:

$$P'\varphi = \varphi_1 + \varphi_2, \quad (\mathrm{id} - P')\varphi = \psi_1 + \psi_2.$$

Then

$$\begin{aligned}\mathscr{T}_1 P'\mathscr{E}_1\varphi &= \mathscr{T}_1 P'\mathscr{E}_1(\varphi_1 + \psi_1 + \varphi_2 + \psi_2)\\ &= \mathscr{T}_1 P'(\varphi_1 + \psi_1) = \mathscr{T}_1\varphi_1 = 0.\end{aligned}$$

A similar argument gives $\mathscr{E}_1 P'\mathscr{T}_1 = 0$. □

LEMMA 4. *Let* $v$ *be a minimal tripotent of* $P''$. *Then* $P''v = v$.

*Proof*. By [**4**: Prop. 1], $P''v = v + b$ where $b = \mathscr{T}P''v$ and $\mathscr{T}$ is defined in [**4**: Intro.]. Since $b = \mathscr{T}b$, $P''$ vanishes on the $J^*$-algebra $B$ generated by $b$. Since $b$ is orthogonal to $v$, the $J^*$-algebra $J = \mathbf{C}v \oplus B$ generated by $v$ and $b$ is commutative in the sense of [**3**]. By restriction $P''$ is a bicontractive projection on $J$ and so has the form $P''x = \frac{1}{2}(x + \theta x)$ for $x \in J$, where $\theta$ is a $J^*$-automorphism of $J$ of order 2 [**3**: Prop. 3.3]. Now $\theta = -\mathrm{id}$ on $B$ so $\theta(B) = B$ and therefore $\theta v$ is orthogonal to $B$. Hence $\theta v = \lambda v$ and therefore $P''v = v$. □

LEMMA 5. *Let* $x \in P''(M'')$. *Then* $\mathscr{E}_0 x = \mathscr{E}_1 x$, *and* $\mathscr{T}_0 x = \mathscr{T}_1 x$.

*Proof*. Since $x = \mathscr{E}_0 x + \mathscr{T}_0 x$, we have $x = P''x = P''\mathscr{E}_0 x + P''\mathscr{T}_0 x$. by Lemma 4 and (0.5), $P''\mathscr{E}_0 x = \mathscr{E}_0 x$, whence $\mathscr{T}_0 x = P''\mathscr{T}_0 x$. Let $y = \mathscr{T}_0 x$. If $\psi \in M'$ is arbitrary,

$$\begin{aligned}\langle y, \psi\rangle &= \langle P''y, \psi\rangle = \langle y, P'\psi\rangle = \langle \mathscr{T}_0 x, \mathscr{E}_0 P'\psi + \mathscr{T}_1 P'\psi\rangle \\ &= \langle \mathscr{T}_0 x, \mathscr{T}_1 P'(\mathscr{E}_1\psi + \mathscr{T}_1\psi)\rangle = \langle y, \mathscr{T}_1 P'\mathscr{T}_1\psi\rangle = \langle \mathscr{T}_1 P''\mathscr{T}_1 y, \psi\rangle\end{aligned}$$

(we have used (2.7)). Therefore $y = \mathscr{T}_1 y$. We now have $\mathscr{T}_0 x = y = \mathscr{T}_1 y = \mathscr{T}_1\mathscr{T}_0 x = \mathscr{T}_1 x$, and thus $\mathscr{E}_0 x = \mathscr{E}_1 x$. □

PROPOSITION 1. *Let $P$ be a bicontractive projection on a $J^*$-algebra $M$. Then $P(M)$ is a $J^*$-subalgebra of $M$.*

*Proof*. Let $x \in P(M)$. Write $x = \mathscr{E}_0 x + \mathscr{T}_0 x$. Then

$$xx^*x = \mathscr{E}_0 x(\mathscr{E}_0 x)^*\mathscr{E}_0 x + \mathscr{T}_0 x(\mathscr{T}_0 x)^*\mathscr{T}_0 x$$

and

$$P(xx^*x) = P''(\mathscr{E}_0 x(\mathscr{E}_0 x)^*\mathscr{E}_0 x) + P''(\mathscr{T}_0 x(\mathscr{T}_0 x)^*\mathscr{T}_0 x). \tag{2.8}$$

By (0.5) and Lemma 4, $P''(\mathscr{E}_0 x(\mathscr{E}_0 x)^*\mathscr{E}_0 x) = \mathscr{E}_0 x(\mathscr{E}_0 x)^*\mathscr{E}_0 x$. Also by Lemma 5 $P''(\mathscr{T}_0 x(\mathscr{T}_0 x)^*\mathscr{T}_0 x) = \mathscr{T}_1 P''(\mathscr{T}_0)x(\mathscr{T}_0 x)^*\mathscr{T}_0(x)$. Applying $\mathscr{E}_1$ to (2.8) therefore yields

$$\mathscr{E}_1 P(xx^*x) = \mathscr{E}_0 x(\mathscr{E}_0 x)^*\mathscr{E}_0 x = \mathscr{E}_1(xx^*x)$$

by Lemma 5. Since the map $y \to \mathscr{E}_1 y$ is isometric on $M$ we have proved that $P(xx^*x) = xx^*x$. □

For any partial isometry $v$ in a $J^*$-algebra, $P = E(v) + F(v)$ is a contractive projection by [4: Lemma 1.1]. Here, $\mathrm{id} - P = G(v)$ is also contractive and $\theta = 2P - \mathrm{id} = E(v) + F(v) - G(v)$ is the symmetry defined by $v$ (cf. [5: Lemma 3.1]).

Formula (ii) of Theorem 1 has been obtained recently for contractive projections on $JB^*$-triples by W. Kaup in a preprint "Contractive Projections on Jordan $C^*$-algebras and generalizations", using methods different from ours. In particular, this settles the Robertson-Youngson conjecture for $JB$-algebras.

The following question arises naturally in connection with Theorem 2: Let $P_1, P_2, P_3$ be contractive projections on a $J^*$-algebra $M$ and suppose $P_1 + P_2 + P_3 = \mathrm{id}$. Does there exist a $J^*$-automorphism $\theta$ of order 3 such that

$$(2.9)\qquad \begin{cases} P_1x = (x + \theta x + \theta^2 x)/3 \\ P_2x = (x + \omega\theta x + \omega^2\theta^2 x)/3 \\ P_3x = (x + \omega^2\theta x + \omega\theta x)/3 \end{cases}$$

where $\omega = \exp(2\pi i/3)$?

The answer is easily verified to be yes for the Peirce projections $P_1 = E(v)$, $P_2 = G(v)$, $P_3 = F(v)$ of an arbitrary partial isometry $v$. The answer can also be shown to be yes for commutative $J^*$-algebras by using [**3**]. However, the answer is no in general. To see this note that (2.9) implies

$$(2.10)\qquad \theta = P_1 + \omega P_2 + \omega^2 P_3.$$

Now let $M$ be the $J^*$-algebra of 2 by 2 complex matrices and for $x = [\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}] \in M$, let

$$P_1x = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}, \quad P_2x = \begin{bmatrix} 0 & \frac{1}{2}(b+c) \\ \frac{1}{2}(b+c) & 0 \end{bmatrix},$$

$$P_3x = \begin{bmatrix} 0 & \frac{1}{2}(b-c) \\ \frac{1}{2}(c-b) & 0 \end{bmatrix}.$$

By (2.10),

$$\theta x = \begin{bmatrix} a & \frac{1}{2}(b+c)\omega + \frac{1}{2}(b-c)\omega^2 \\ \frac{1}{2}(b+c)\omega^2 + \frac{1}{2}(c-b)\omega & d \end{bmatrix}$$

and it follows that $\theta$ is not a $J^*$-automorphism, i.e., $\theta(x)\theta(x)^*\theta(x) \neq \theta(xx^*x)$ if $x = [\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}]$ for example.

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