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GYSIN HOMOMORPHISM AND SCHUBERT CALCULUS

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Let G be a connected reductive linear algebraic group over the field of complex numbers, and B a fixed Borel subgroup of G. The study of the homological properties of G/B can be carried out by two well-known methods. The first of these methods is due to A. Borel and involves the identification of the cohomology ring of G/B with the quotient ring of the ring of polynomials on the Lie algebra $\mathfrak{h}$ of the Cartan subgroup $H \subset G$ by the ideal generated by the W-invariant polynomials (where W is the Weyl group of G). The second method is classical, and based on the calculation of the homology with the aid of the partition of G/B into cells, the so-called Schubert cells. The correspondence between these approaches has been studied in the paper by Bernstein, Gel'fand and Gel'fand, where in the quotient ring of the polynomial ring figuring in Borel's model of the cohomology, the authors have found a symmetrical basis dual to the Schubert cells. Moreover, they have given a formula (Intersection formula) which expresses the intersection of any Schubert cell with a cell of codimension one. In the same paper, the authors have also generalized these results, except the intersection formula, to the case when B is replaced by an arbitrary parabolic subgroup $P \subset G$.

On the other hand, for any parabolic subgroup $P \subset G$ containing B, the cohomology Gysin homomorphism of π: $G/B \to G/P$ has been studied by Akyildiz and Carrell, where an explicit formula has been obtained between the rings figuring in Borel's model of the cohomologies. This formula also enables one to obtain some of the results mentioned above for G/P from the corresponding results on G/B. In this note, we consider the case where $G = \mathrm{GL}(n+1)$ and G/P is the Grassmann manifold. By using the explicit description of the Gysin homomorphism and the intersection formula given in the cohomology ring of G/B we obtain three main theorems of the symbolic formalism, known as Schubert Calculus, concerning the cohomology ring structure of the Grassmann manifold. Although there are several different approaches for proving these theorems (see Kleiman and Laksov), it seems that none of them uses the cohomology ring structure of $\mathrm{GL}(n+1)/B$, where B is the group of upper triangular matrices in $\mathrm{GL}(n+1)$. We thus hope that this alternative point of view may be used to understand the generalized Schubert Calculus.

1. Gysin homomorphism of π: $G/B \to G/P$. In this section we first give the rings figuring in Borel's model of the cohomologies of G/B, G/P, and then we write the formula for the cohomology Gysin homomorphism of π: $G/B \to G/P$ between these rings. Here G is a connected reductive linear algebraic group over the field of complex numbers, B is a fixed Borel subgroup of G, and P is a parabolic subgroup of G containing B. We adopt the following notation: B_u the unipotent radical of B, H a fixed maximal torus contained in B, $\mathfrak{g}$ the Lie algebra of G, $\mathfrak{h}$ and $\mathfrak{b}_u$ the Lie algebras of H and B_u respectively, $\Delta \subset \mathfrak{h}^*$ the root system of $\mathfrak{h}$ in $\mathfrak{g}$, Δ_+ the set of positive roots, namely the set of roots of $\mathfrak{h}$ in $\mathfrak{b}_u$, $\Sigma \subset \Delta_+$ the set of simple roots, W the Weyl group of G, Θ is the subset of Σ such that the parabolic subgroup P_Θ corresponding to Θ is equal to P, W_Θ the subgroup of W generated by the reflections σ_α, $\alpha \in \Theta$, Δ_Θ the subset of Δ_+ consisting of linear combinations of the elements of Θ, $W_\Theta^\perp$ the set of $w \in W$ such that $w\Theta \subset \Delta_+$.

Let $X(H)$ be the group of characters of H, and $R = \mathrm{Sym}(C \otimes_Z X(H))$, the symmetric algebra of $C \otimes_Z X(H)$. We denote by the same symbol an element of $X(H)$ and the corresponding element of Δ when it can be done without any ambiguity. For the basic facts about algebraic groups and the Borel model of the cohomologies of homogeneous spaces, the reader is referred to [**6**] and [**1**], [**3**], [**5**] respectively. Since W acts on $X(H)$ we get an action of W on R in the usual way. Let R^W be the ring of invariants of W, and $I = \{f \in R^W: f(0) = 0\}$. For any character α of H, let L_α be the associated homogeneous line bundle on G/B: $L_\alpha = G \times C/\sim$, $(g, z) \sim (g', z')$ if and only if $g' = gb$ for some b in B and $z' = \alpha(b^{-1})z$, where α is extended on B with $\alpha(u) = 1$ for u in B_u. It is shown in [**1**] that the characteristic homomorphism $c: R \to H^*(G/B, C)$ determined by $c(\alpha) = c_1(L_\alpha)$, the first Chern class of L_α for any $\alpha \in X(H)$, induces an isomorphism of graded algebras $\bar{c}: R/IR \xrightarrow{\sim} H^*(G/B, C)$ (see also [**5**]). Moreover, if R^{W_Θ} is the ring of invariants of W_Θ, then one also has the following commutative diagram between the graded algebras (cf. [**1**]):

$$\begin{array}{ccc} \bar{c}: R/IR & \xrightarrow{\sim} & H^*(G/B, C) \\ \uparrow & & \uparrow \pi^* \\ \tilde{c}: R^{W_\Theta}/IR^{W_\Theta} & \xrightarrow{\sim} & H^*(G/P, C), \end{array}$$

where π^* is the cohomology map of $\pi: G/B \to G/P$. This is the Borel model of the cohomologies of G/B and G/P.

For the cohomology Gysin homomorphism of $\pi: G/B \to G/P$, we first recall the operators $A_\omega: R \to R$ for $\omega \in W$ (cf. [**3**], [**4**], [**5**]). For each $\alpha \in \Delta$ the element $f - \sigma_\alpha \cdot f$ is divisible by α for any $f \in R$, where σ_α is the reflection corresponding to α. Thus $A_\alpha: R \to R$, $A_\alpha f = (f - \sigma_\alpha \cdot f)/\alpha$, is a well-defined R^W-linear operator on R. Let $\alpha_1, \ldots, \alpha_l \in \Sigma$, and let $\omega = \sigma_{\alpha_1} \cdots \sigma_{\alpha_l}$ be any element of W. Then

(i) if the length $l(\omega)$ of ω is less than l, then $A_{\alpha_1} \cdots A_{\alpha_l} = 0$,

(ii) if $l(\omega) = l$, then the operator $A_{\alpha_1} \cdots A_{\alpha_l}$ depends only on ω and not on the representation of ω in the form $\omega = \sigma_{\alpha_1} \cdots \sigma_{\alpha_l}$. In this case we put $A_\omega = A_{\alpha_1} \cdots A_{\alpha_l}$.

We note that the operator A_ω: $R \to R$ preserves the ideal IR, and thus induces an operator $\bar{A}_\omega$: $R/IR \to R/IR$ of homogeneous degree $-l(\omega)$. Moreover for $\omega_1, \omega_2 \in W$, one has $\bar{A}_{\omega_1}\bar{A}_{\omega_2} = \bar{A}_{\omega_1\omega_2}$ if $l(\omega_1\omega_2) = l(\omega_1) + l(\omega_2)$, $\bar{A}_{\omega_1}\bar{A}_{\omega_2} = 0$ otherwise (cf. [4], [5]). It is shown in [2] that the cohomology Gysin homomorphism π_*: $H^*(G/B, C) \to H^*(G/P, C)$ of the natural map π: $G/B \to G/P$ is given by the following commutative diagram:

$$
\begin{array}{lccc}
 & \bar{c}: R/IR & \tilde{\to} & H^*(G/B, C) \\
(1) & \bar{A}_{\tau_\Theta} \downarrow & & \downarrow \pi_* \\
 & \tilde{c}: R^{W_\Theta}/IR^{W_\Theta} & \tilde{\to} & H^*(G/P, C),
\end{array}
$$

where τ_Θ is the unique element of W_Θ of maximal length. Moreover, one has the following formula for $\bar{A}_{\tau_\Theta}$ (cf. [2]):

$$
(2) \qquad A_{\tau_\Theta}(f) = \sum_{\tau \in W_\Theta} \det(\tau)\tau \cdot f \Big/ \prod_{\alpha \in \Delta_\Theta} \alpha.
$$

This is the explicit description of the cohomology Gysin homomorphism of π: $G/B \to G/P$.

2. Schubert calculus. The correspondence between the Borel's model of the cohomology of G/P and the calculation of the homology with the aid of partition of G/P into Schubert cells has been studied in [3]. In the case of G/B, the authors have also given a formula (intersection formula) in R/IR which expresses the intersection of any Schubert cell with a cell of codimension one. In this section, we first give this correspondence. Then we write the intersection formula only for the case $G = \mathrm{GL}(n+1)$. Later, by using this formula and the results of §1. We obtain three main theorems, known as Schubert Calculus, concerning the cohomology ring structure of Grassmannians.

We keep the notation of §1. Let ω_0 be the unique element of W of maximal length, and $P_{\omega_0} = (1/|W|)\prod_{\alpha \in \Delta_+} \alpha \pmod{IR}$, where $|W|$ is the order of W. For each $\omega \in W$, let $P_\omega = \bar{A}_{\omega^{-1}\omega_0}(P_{\omega_0})$. It is shown in [3] that $\{P_\omega$: $\omega \in W\}$ is a free Z-basis of R/IR which is dual to the Schubert cells. Namely, for each $\omega \in W$, let X_ω denote the cycle class of the closure of $B(\omega_0\omega)x_0$ in $H_*(G/B, Z)$, $x_0 = B \in G/B$. Then under the Poincare duality map $\mathscr{P}$, $\mathscr{P}(X_\omega) = \bar{c}(P_\omega)$. Similarly, for the parabolic subgroup $P = P_\Theta$ of G containing B, one has the following: For each $\sigma \in W_\Theta^\perp$ let Y_σ be the cycle class of the closure of $B(\omega_0\sigma)\pi(x_0)$ in $H_*(G/P, Z)$, where π: $G/B \to G/P$ is the natural map. Then the Poincare dual $\mathscr{P}(Y_\sigma)$ of Y_σ, $\sigma \in W_\Theta^\perp$, is equal to $\tilde{c}$ (P_σ (mod IR^{W_Θ})). Moreover $\{P_\sigma$ (mod IR^{W_Θ}): $\sigma \in W_\Theta^\perp\}$ is a free Z-basis of $R^{W_\Theta}/IR^{W_\Theta}$. Although the proof of this fact is given in [**3**], for the sake of completeness of our approach we will give a different proof by using the Gysin homomorphism (cf. [**2**]). Let τ_Θ be the unique element of W_Θ of maximal length. Since $l(\sigma\tau) = l(\sigma) + l(\tau)$ for any $\sigma \in W_\Theta^\perp$ and $\tau \in W_\Theta$, we have

$$\bar{A}_{\tau_\Theta}(P_{\sigma\tau}) = \bar{A}_{\tau_\Theta}\bar{A}_{\tau^{-1}\sigma^{-1}\omega_0}(P_{\omega_0}) = \delta_{\tau_\Theta,\tau}A_{\sigma^{-1}\omega_0}(P_{\omega_0}) = \delta_{\tau_\Theta,\tau}P_\sigma.$$

This shows that $\{P_\sigma$ (mod IR^{W_Θ}): $\sigma \in W_\Theta^\perp\}$ is a basis of $R^{W_\Theta}/IR^{W_\Theta}$, because the Gysin homomorphism $\bar{A}_{\tau_\Theta}$ is surjective and any element ω of W can be written uniquely in the form $\omega = \sigma\tau$ for some $\sigma \in W_\Theta^\perp$ and $\tau \in W_\Theta$. On the other hand, it is clear that for $\sigma \in W_\Theta^\perp$, the cycle class $X_{\sigma\tau_\Theta}$ goes under the homology Gysin homomorphism of π: $G/B \to G/P$ to the cycle class Y_σ in $H_*(G/P, Z)$. Thus from the commutativity of the diagram (1) and $\bar{c}(P_{\sigma\tau_\Theta}) = \mathscr{P}(X_{\sigma\tau_\Theta})$, we get $\tilde{c}(P_\sigma$ (mod IR^{W_Θ})) $= \mathscr{P}(Y_\sigma)$ for any $\sigma \in W_\Theta^\perp$.

There is a formula (intersection formula) given in R/IR which expresses the product $P_{\sigma_\alpha} \cdot P_\sigma$ in terms of P_ω, where σ_α is the reflection corresponding to the simple root $\alpha \in \Sigma$ (cf. [**3**]). We will now write this formula for the group of invertible matrices GL($n + 1$). Let G be GL(n + 1), B the group of upper triangular matrices in G, and H the group of diagonal matrices in B. Then

$$R = \mathrm{Sym}(C \otimes_Z X(H)) = C[x_0,\ldots,x_n],$$

$$\Delta_+ = \{\alpha_{i,j} = x_i - x_j\colon 0 \le i < j \le n\},$$

$$\Sigma = \{\alpha_{i,i+1}\colon 0 \le i \le n-1\}.$$

Let S_{n+1} be the symmetric group in $0, 1, \ldots, n$, and for $\sigma \in S_{n+1}$ let $\sigma(I)$ be the permutation matrix obtained from $(n+1) \times (n+1)$ identity matrix I. The homomorphism $\sigma^{-1} \to \sigma(I)$ gives an isomorphism between S_{n+1} and the Weyl group W of G. Moreover it can be checked that the action of W on R is given by

$$\begin{aligned}\sigma(I)\cdot f(x_0,\ldots,x_n) &= \sigma f(x_0,\ldots,x_n)\\ &= f(x_{\sigma 0},\ldots,x_{\sigma n}), \quad \text{for any } \sigma \in S_{n+1}.\end{aligned}$$

Thus IR is the ideal generated by the elementary symmetric functions of $x_0, \ldots, x_n$.

We will now write the operators A_ω, and the elements P_ω of R/IR in terms of the elements of S_{n+1}. Let $\sigma = (a_0 \cdots a_n)$ be any element of S_{n+1}, namely $\sigma i = a_i$ for $i = 0, 1, \ldots, n$. Then the reflection $\sigma_{\alpha_{i,j}}$ corresponding to the root $\alpha_{i,j}$ is given by $\sigma_{\alpha_{i,j}} = (i, j)(I)$, where $(i, j) \in S_{n+1}$ is the transposition obtained by changing i with j. Let $A_{(i,i+1)} = A_{\alpha_{i,i+1}}$, and $P_\sigma = P_{\sigma(I)}$ for $\sigma \in S_{n+1}$. Then for any $\sigma \in S_{n+1}$, we get

$$A_{\sigma^{-1}} = A_{\sigma(I)}, \quad \text{and} \quad P_\sigma = \bar{A}_{\sigma\omega_0}(P_{\omega_0}),$$

where $\omega_0 = (n\; n-1 \cdots 1\; 0)$, and

$$P_{\omega_0} = \frac{1}{(n+1)!} \prod_{0 \le i < j \le n} (x_i - x_j) \pmod{IR}.$$

The intersection formula given in [3] can now be written as follows:

THEOREM (*Intersection formula*, cf. [3, p. 17]). *For any* $r = 0, 1, \ldots, n-1$, *and* $\sigma = (a_0 \cdots a_n) \in S_{n+1}$, *we have*

$$P_{(r,r+1)} P_\sigma = \sum P_{(i,j)\sigma},$$

where the summation is over all $0 \le i < j \le n$ *such that*

(a) $i \le r < j$,

(b) $a_i < a_j$,

(c) *if* $i < k < j$, *then either* $a_k < a_i$ *or* $a_k > a_j$.

We are now ready to discuss the cohomology ring structure of Grassmannians. For any $q = 0, 1, \ldots, n-1$, let $\Theta = \Sigma \setminus \{\alpha_{q,q+1}\}$. Then the parabolic subgroup $P = P_\Theta$ of G corresponding to Θ consists of all matrices of the form $\left(\begin{smallmatrix} A & * \\ 0 & B \end{smallmatrix}\right)$, where $A \in \mathrm{GL}(q+1)$, and $B \in \mathrm{GL}(n-q)$. Thus G/P is the Grassmann manifold $G_{q,n}$ of $(q+1)$-planes in C^{n+1}. Moreover it is easy to see that corresponding to Θ, we have

$$W_\Theta = \{(a_0 \cdots a_q\, a_{q+1} \cdots a_n) \in S_{n+1} : 0 \le a_0, \ldots, a_q \le q, \\ q+1 \le a_{q+1}, \ldots, a_n \le n\},$$

and

$$W_\Theta^\perp = \{(a_0 \cdots a_q \cdots a_n) \in S_{n+1} : a_0 < \cdots < a_q,\, a_{q+1} < \cdots < a_n\}.$$

A sequence of integers $0 \le a_0 < \cdots < a_q \le n$ corresponds to a unique element $(a_0 \cdots a_q\, a_{q+1} \cdots a_n)$ of $W_\Theta^\perp$, and we denote this element by $\sigma(a_0, \ldots, a_q)$. For any sequence of integers $0 \le a_0 < \cdots < a_q \le n$, let $\Omega(a_0, \ldots, a_q)$ denote the Poincare dual of $Y_{\sigma(a_0, \ldots, a_q)}$ in $H^*(G_{q,n}, Z) \subset H^*(G_{q,n}, C)$. $\Omega(a_0, \ldots, a_q)$ is called the Schubert cycle corresponding to

$0 \leq a_0 < \cdots < a_q \leq n$. The Schubert cycle corresponding to $0 < 1 < \cdots < q-1 < q+k \leq n$ for $k = 0, 1, \ldots, n-q$ is called the special Schubert cycle, and it is denoted by $\Omega(k)$.

THEOREM (*The basis theorem*). *The Schubert cycles* $\Omega(a_0, \ldots, a_q)$, $0 \leq a_0 < \cdots < a_q \leq n$, *form a basis of* $H^*(G_{q,n}, Z)$.

Proof. It follows from the above observations because $\Omega(a_0, \ldots, a_q) = P_{\sigma(a_0, \ldots, a_q)} \pmod{IR^{W_\Theta}}$ for any sequence of integers $0 \leq a_0 < \cdots < a_q \leq n$.

We will now compute $\Omega(a_0, \ldots, a_q) = P_{\sigma(a_0, \ldots, a_q)} \pmod{IR^{W_\Theta}}$. Let $\Theta_1 = \{\alpha_{i,i+1}: 0 \leq i \leq q-1\}$, and $P_1 = P_{\Theta_1}$ be the corresponding parabolic subgroup of G. Then we have

$$W_{\Theta_1} = \{(a_0 \cdots a_q\, q+1 \cdots n) \in S_{n+1}: 0 \leq a_0, \ldots, a_q \leq q\},$$
$$W_{\Theta_1}^{\perp} = \{(a_0 \cdots a_q \cdots a_n) \in S_{n+1}: a_0 < \cdots < a_q\},$$

and $\tau_0 = (q\; q-1 \;\cdots\; 0\; q+1 \;\cdots\; n)$ is the unique element of W_{Θ_1} of maximal length. Since $W_\Theta^\perp \subset W_{\Theta_1}^\perp$, we can also use the Gysin homomorphism $\bar{A}_{\tau_0}$ of $\pi_1: G/B \to G/P_1$ to compute P_σ for any $\sigma \in W_\Theta^\perp$. Namely, for $\sigma \in W_\Theta^\perp$ we have $P_\sigma = \bar{A}_{\tau_0}(P_{\tau_0\sigma})$. Although that it is possible to compute P_σ, $\sigma \in W_\Theta^\perp$, by using the Gysin homomorphism $\bar{A}_{\tau_\Theta}$, $P_\sigma = \bar{A}_{\tau_\Theta}(P_{\tau_\Theta\sigma})$, it is much simpler to do this by using $\bar{A}_{\tau_0}$. For that, we need to compute $P_{\tau_0\sigma}$ for any $\sigma \in W_\Theta^\perp$.

PROPOSITION. *Let* $\sigma = (a_0 \cdots a_q \cdots a_n) \in W_\Theta^\perp$. *Then we have* $P_{\tau_0\sigma} = x_0^{a_q} \cdots x_q^{a_0} \pmod{IR}$.

Proof. Let $\tau_0\sigma = (a_q \cdots a_0\, a_{q+1} \cdots a_n) = (b_0 \cdots b_n) \in S_{n+1}$. Since $b_0 > b_1 > \cdots > b_q$ and $b_{q+1} < \cdots < b_n$, it is easy to see that from the intersection formula one gets the following:

(i) $P_{(q,q+1)}P_{\tau_0\sigma} = \sum P_{(i,q+k)\tau_0\sigma}$, where the summation is all over $0 \leq i \leq q$ and $1 \leq k \leq n-q$ such that $b_{q+k-1} \leq b_i < b_{q+k}$,

(ii) $P_{(q-1,q)}P_{\tau_0\sigma} = \sum P_{(j,q+r)\tau_0\sigma}$, where the summation is all over $0 \leq j \leq q-1$ and $1 \leq r \leq n-q$ such that $b_{q+r-1} \leq b_j < b_{q+r}$. Thus

$$\left(P_{(q,q+1)} - P_{(q-1,q)}\right)P_{\tau_0\sigma} = \begin{cases} P_{(q,q+k)\tau_0\sigma}, & \text{if there exists } 1 \leq k \leq n-q \\ & \text{such that } b_{q+k-1} \leq b_q < b_{q+k}, \\ 0 \pmod{IR}, & \text{otherwise.} \end{cases}$$

We note that the fundamental dominant weight χ_r corresponding to the simple root $\alpha_{r,r+1}$ is given by $\chi_r = x_0 + \cdots + x_r$. By [3, p. 18], since $P_{(r,r+1)} = \chi_r \pmod{IR}$, we have $P_{(r,r+1)} - P_{(r-1,r)} = x_r \pmod{IR}$ for $r = 0, 1, \ldots, n-1$.

We prove the proposition by induction on q. If $q = 0$, then $\tau_0\sigma = (b_0 \cdots b_n)$, $b_1 < \cdots < b_n$. In this case we can have either $\tau_0\sigma = e$ (the identity) or $\tau_0\sigma = \omega(p+1) = (p+1\ 0\ \cdots\ p\ p+2\ \cdots\ n)$ for some $p = 0, 1, \ldots, n-1$. If $\tau_0\sigma = e$, then the claim follows from the formula (2), $P_e = \bar{A}_{\omega_0}(P_{\omega_0}) = 1 \pmod{IR}$. If $\tau_0\sigma = \omega(p+1)$ for some $p = 0, 1, \ldots, n-1$, then we claim that $P_{\omega(p+1)} = x_0^{p+1} \pmod{IR}$. We prove this by induction on p. If $p = 0$, then $\omega(1) = (0, 1)$. Since $P_{(0,1)} = \chi_0 = x_0 \pmod{IR}$, we have the claim for $p = 0$. We now assume $P_{\omega(p)} = x_0^p \pmod{IR}$. But by the intersection formula we have $P_{(0,1)}P_{\omega(p)} = P_{(0,p+1)\omega(p)}$. Since $(0, p+1)\omega(p) = \omega(p+1)$, by the induction hypothesis we have $P_{\omega(p+1)} = x_0 x_0^p = x_0^{p+1} \pmod{IR}$. This finishes the induction argument for $q = 0$. We now assume the claim for $q - 1$. Let $\tau_0\sigma = (a_q \cdots a_0\, a_{q+1} \cdots a_n) = (b_0 \cdots b_n)$. If $a_0 = b_q = 0$, then the claim follows from the induction hypothesis for $q - 1$ because $b_0 > \cdots > b_{q-1}$, $0 = b_q < \cdots < b_n$. If $b_q > 0$, then we claim that $P_{\tau_0\sigma} = x_0^{b_0} \cdots x_q^{b_q} \pmod{IR}$. This will be proved by induction on b_q. If $b_q = 1$, then $b_{q+1} = 0$. In this case $\omega = (q, q+1)\tau_0\sigma = (b_0 \cdots b_{q-1}\ 0\ 1\ b_{q+2} \cdots b_n)$, and thus (by the induction hypothesis for $q - 1$) $P_\omega = x_0^{b_0} \cdots x_{q-1}^{b_{q-1}} \pmod{IR}$. Since $P_{(q,q+1)} - P_{(q-1,q)} = x_q \pmod{IR}$, we have from above

$$\left(P_{(q,q+1)} - P_{(q-1,q)}\right)P_\omega = P_{(q,q+1)\omega} = P_{\tau_0\sigma} = x_0^{b_0} \cdots x_{q-1}^{b_{q-1}} x_q \pmod{IR}.$$

We now assume the claim for $b_q - 1 \geq 0$. If $b_q > 1$, then $b_{q+1} = 0$ and $b_q - 1 \in \{b_{q+2}, \ldots, b_n\}$. Let $b_q - 1 = b_{q+2+s}$ for some $s = 0, 1, \ldots, n-q-2$. Consider

$$\begin{aligned} \omega &= (q, q+2+s)\tau_0\sigma \\ &= \left(b_0 \cdots b_{q-1}\, b_q - 1\, b_{q+1} \cdots b_{q+1+s}\, b_q\, b_{q+3+s} \cdots b_n\right). \end{aligned}$$

Since $b_0 > \cdots > b_q - 1$, $b_{q+1} < \cdots < b_{q+1+s} < b_q < b_{q+3+s} < \cdots < b_n$, by the induction hypothesis we have $P_\omega = x_0^{b_0} \cdots x_q^{b_q - 1} \pmod{IR}$. But from above and the induction hypothesis, we get

$$\begin{aligned} \left(P_{(q,q+1)} - P_{(q-1,q)}\right)P_\omega &= P_{(q,q+2+s)\omega} = P_{\tau_0\sigma} \\ &= x_q x_0^{b_0} \cdots x_q^{b_q - 1} = x_0^{b_0} \cdots x_q^{b_q} \pmod{IR} \end{aligned}$$

which is the claim. This completes the proof of the proposition.

COROLLARY. *For any sequence of integers* $0 \leq a_0 < \cdots < a_q \leq n$, *we have*

$$P_{\sigma(a_0,\ldots,a_q)} = \begin{vmatrix} h(a_0) & \cdots\cdots\cdots & h(a_0-q) \\ \vdots & & \vdots \\ h(a_q) & \cdots\cdots\cdots & h(a_q-q) \end{vmatrix} \pmod{IR},$$

where $h(m)$ *is the mth complete homogeneous symmetric function of* $x_0, x_1,\ldots,x_q$. *In particular* $P_{\sigma(0,1,\ldots,q-1,q+k)} = h(k) \pmod{IR}$ *for any* $k = 0,1,\ldots,n-q$.

Proof. By the proposition we have $P_{\tau_0\sigma(a_0,\ldots,a_q)} = x_0^{a_q} \cdots x_q^{a_0} \pmod{IR}$. On the other hand, from formula (2) we get

$$\begin{aligned} P_{\sigma(a_0,\ldots,a_q)} &= \bar{A}_{\tau_0}\left(P_{\tau_0\sigma(a_0,\ldots,a_q)}\right) \\ &= \frac{\sum_{\tau\in W_{\Theta_1}} \det(\tau) x_{\tau 0}^{a_q} \cdots x_{\tau q}^{a_0}}{\prod_{0\leq i<j\leq q}(x_i - x_j)} \pmod{IR}. \end{aligned}$$

But this bialternant is equal to (cf. [**8**, p. 92])

$$\begin{vmatrix} h(a_q-q) & \cdots\cdots & h(a_q) \\ \vdots & & \vdots \\ h(a_0-q) & \cdots\cdots & h(a_0) \end{vmatrix} = \begin{vmatrix} h(a_0) & \cdots\cdots & h(a_0-q) \\ \vdots & & \vdots \\ h(a_q) & \cdots\cdots & h(a_q-q) \end{vmatrix},$$

which gives the claim.

THEOREM (*The determinantal formula*). *For all sequence of integers* $0 \leq a_0 < \cdots < a_q \leq n$ *the following formula holds in the cohomology ring* $H^*(G_{q,n}, Z)$:

$$\Omega(a_0,\ldots,a_q) = \begin{vmatrix} \Omega(a_0) & \cdots\cdots & \Omega(a_0-q) \\ \vdots & & \vdots \\ \Omega(a_q) & \cdots\cdots & \Omega(a_q-q) \end{vmatrix},$$

where we agree to put $\Omega(k) = 0$ *for* $k > n-q$ *or* $k < 0$.

Proof. It follows from the Corollary above the fact that; if $k > n - q$, then the kth complete homogeneous symmetric function of $x_0,\ldots,x_q$

(being equal to $\pm$ the kth elementary symmetric function of $x_{q+1},\ldots,x_n$ (mod IR)) is equal to zero (mod IR^{W_Θ}), (cf. [**8**]).

This theorem, together with the basis theorem, implies that the special Schubert cycles generate the cohomology ring $H^*(G_{q,n}, Z)$ as a Z-algebra. Moreover, it reduces the problem of determining the product of two arbitrary Schubert cycles to the case where one (or for that matter, each) is a special Schubert cycle. This case is handled by the following theorem.

THEOREM (*Pieri's formula*). *For all sequences of integers* $0 \le a_0 < \cdots < a_q \le n$ *and* $k = 0, 1, \ldots, n-q$, *the following formula holds in the cohomology ring* $H^*(G_{q,n}, Z)$:

$$\Omega(k)\Omega(a_0,\ldots,a_q) = \sum \Omega(b_0,\ldots,b_q),$$

where the sum ranges over all sequences of integers $b_0 < \cdots < b_q$ *satisfying* $0 \le a_0 \le b_0 < a_1 \le b_1 < \cdots < a_q \le b_q \le n$ *and* $\sum_{i=0}^q (b_i - a_i) = k$.

Proof. For any sequence of integers $0 \le a_0 < \cdots < a_q \le n$ and for $k = 0, 1, \ldots, n-q$, by the proposition and it's corollary we have

$$P_{\tau_0\sigma(a_0,\ldots,a_q)} = x_0^{a_q} \cdots x_q^{a_0} \quad (\text{mod } IR)$$

and $P_{\sigma(0,1,\ldots,q-1,q+k)} = h(k) = \sum x_0^{l_q} \cdots x_q^{l_0}$ (mod IR), where the sum ranges over all integers $l_i \ge 0$ such that $\sum_{i=0}^q l_i = k$. Thus we have

$$(3) \qquad P_{\sigma(0,1,\ldots,q-1,q+k)} P_{\tau_0\sigma(a_0,\ldots,a_q)} = \sum x_0^{a_q+l_q} \cdots x_q^{a_0+l_0}$$
$$= f(x_0,\ldots,x_q) \quad (\text{mod } IR)$$

where the sum ranges over all integers $l_i \ge 0$ such that $\sum_{i=0}^q l_i = k$. We will now apply the operator $\bar{A}_{\tau_0}$ to equation (3). Since $\bar{A}_{\tau_0}$ is $R^{W_{\Theta_1}}$-linear and $h(k) \in R^{W_{\Theta_1}}$ (where $R^{W_{\Theta_1}}$ is the ring of invariants of the group W_{Θ_1}), we have

$$(4) \quad \bar{A}_{\tau_0}\left(P_{\sigma(0,1,\ldots,q-1,q+k)} P_{\tau_0\sigma(a_0,\ldots,a_q)}\right) = P_{\sigma(0,1,\ldots,q-1,q+k)} \bar{A}_{\tau_0}\left(P_{\tau_0\sigma(a_0,\ldots,a_q)}\right)$$
$$= P_{\sigma(0,1,\ldots,q-1,q+k)} P_{\sigma(a_0,\ldots,a_q)}.$$

Let $J_{\Theta_1}(g) = \sum_{\tau \in W_{\Theta_1}} \det(\tau)\tau g$ for $g \in R$. Since $J_{\Theta_1}(\tau g) = \det(\tau) J_{\Theta_1}(g)$ for any $\tau \in W_{\Theta_1}$, we have in particular

(i) $J_{\Theta_1}(X_{\tau 0}^{m_0} \cdots X_{\tau q}^{m_q}) = \det(\tau) J_{\Theta_1}(X_0^{m_0} \cdots X_q^{m_q})$ for any $\tau \in W_{\Theta_1}$,

(ii) $J_{\Theta_1}(X_0^{m_0} \cdots X_q^{m_q}) = 0$ if $m_i = m_j$ for some $i \ne j$.

By using (i), (ii), and the fact that $X_0^{n+1} = 0$ (mod IR), it is to see that

$$J_{\Theta_1}(f(x_0,\ldots,x_q)) = \sum J_{\Theta_1}\left(x_0^{a_q+\lambda_q} \cdots x_q^{a_0+\lambda_0}\right),$$

where the sum ranges over all integers $\lambda_i \geq 0$ satisfying $a_0 + \lambda_0 < \cdots < a_q + \lambda_q \leq n$ and $\sum_{i=0}^{q} \lambda_i = k$. This gives by the proposition

$$\bar{J}_{\Theta_1}(\bar{f}) = \sum \bar{J}_{\Theta_1}\left(P_{\tau_0\sigma(b_0,\ldots,b_q)}\right),$$

where $\bar{f} = f(x_0,\ldots,x_q)$ (mod IR), and the sum ranges over all sequences of integers $b_0 < b_1 < \cdots < b_q$ satisfying $0 \leq a_0 \leq b_0 < a_1 \leq b_1 < \cdots < a_q \leq b_q \leq n$ and $\sum_{i=0}^{q}(b_i - a_i) = k$. Thus from formula (2) we get

$$\bar{A}_{\tau_0}(\bar{f}) = \sum \bar{A}_{\tau_0}\left(P_{\tau_0\sigma(b_0,\ldots,b_q)}\right) = \sum P_{\sigma(b_0,\ldots,b_q)}, \tag{5}$$

where the sum is as above. By comparing (3), (4) and (5) we obtain

$$P_{\sigma(0,1,\ldots,q-1,q+k)} P_{\sigma(a_0,\ldots,a_q)} = \sum P_{\sigma(b_0,\ldots,b_q)}, \tag{6}$$

where the sum ranges over all sequences of integers $b_0 < b_1 < \cdots < b_q$ satisfying $0 \leq a_0 \leq b_0 < a_1 \leq b_1 < \cdots < a_q \leq b_q \leq n$ and $\sum_{i=0}^{q}(b_i - a_i) = k$. By considering equation (6) (mod IR^{W_Θ}), we get the claim, because $\tilde{c}$ is a graded algebra isomorphism and $\tilde{c}(P_{\sigma(a_0,\ldots,a_q)}$ (mod IR^{W_Θ})) $= \Omega(a_0,\ldots,a_q)$.

References

[1] E. Akyildiz, *Vector Fields and Cohomology of G/P*, Lecture Notes in Math. **956** (1982), Springer-Verlag, pp. 1–9.

[2] E. Akyildiz and J. B. Carrell, *An algebraic formula for the Gysin homomorphism from G/B to G/P*, to appear.

[3] I. N. Bernstein, I. M. Gel'fand and S. I. Gel'fand, *Schubert cells and cohomology of the space G/P*, Russian Math. Surveys, **28** (1973), 1–26.

[4] M. Demazure, *Invariant symétriques entiers des groupes de Weyl et torsion*, Inventiones Math., **21** (1973), 287–301.

[5] ______, *Désingularisation des variétés de Schubert généralisées*, Ann. Scient. Éc. Norm. Sup., 4^e serie, t.7 (1974), 53–88.

[6] J. E. Humphreys, *Linear Algebraic Groups*, New York-Heidelberg-Berlin: Springer-Verlag, GTM 21, 1975.

[7] S. Kleiman and D. Laksov, *Schubert calculus*, Amer. Math. Monthly, **79** (1972), 1061–1082.

[8] D. E. Littlewood, *A University Algebra*, second edition, London: Heinemann 1958.

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