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ON A QUESTION OF FEIT CONCERNING CHARACTER VALUES OF FINITE SOLVABLE GROUPS

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Let χ be an irreducible character of a finite group G and let f be the smallest integer such that $\{\chi(x)|x \in G\} \subseteq Q(\sqrt[f]{1})$. The question raised by W. Feit is: Does G contain an element of order f. In this article we given an affirmative answer to the question for solvable groups.

Introduction. Let G be a finite group and χ an irreducible complex character of G. Denote by $Q(\chi)$ the field obtained by adjoining the values of χ to the rational number field Q. For every positive integer m we denote by Q_m the field $Q(\omega)$, where ω is a primitive mth root of unity. Finally, denote by $f(\chi)$ the smallest positive integer f for which $Q(\chi) \subseteq Q_f$.

The following question has been raised by Walter Feit (see e.g. [**4**] p. 178): Let χ be an irreducible complex character of a finite group G, does G contains an element of order $f(\chi)$?

In this article we show that if G is solvable the answer to the question is positive. Before stating this result we survey the known positive answers to the question.

Brauer ([**3**] Corollary 4) gave an affirmative answer in the case that $f(\chi)$ has the form $f(\chi) = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ where $\alpha_i \geq 2$ for all i and the p_i's are primes. There is no restriction on G. In [**5**] Gow gives an affirmative answer in the case that G has odd order with no restriction on $f(\chi)$. In [**1**], Brauer's and Gow's methods are generalized and an affirmative answer is given (Theorem 2.2 of [1]) in a case of which both Brauer's and Gow's cases are special cases. Also, it is fairly easy to prove ([**2**]) that if $f(\chi)$ has the form $f(\chi) = p^\alpha q^\beta$, p and q primes, the answer is also positive. The main result of this paper is:

THEOREM. *Let G be a finite solvable group and χ an irreducible complex character of G, then G contains an element of order $f(\chi)$.*

Most of our notation is standard and taken mainly from [**6**]. Some other pieces of notation will be introduced as we go along.

2. Preliminaries and proof of the theorem. The notation $o(a)$ will be used to denote the order of the element a of a group. If G is a finite group and $\chi \in \mathrm{Irr}(G)$ we let $\pi(\chi) = \{p | p \text{ a prime divisor of } f(\chi)\}$. For each $p \in \pi(\chi)$ we fix a generator, $\sigma_p(\chi)$ of the cyclic group $\mathrm{Gal}(Q_f/Q_{f/p})$, where $f = f(\chi)$. By Galois theory we have that

$$o(\sigma_p(\chi)) = \begin{cases} p & \text{if } p^2 \mid f \\ p - 1 & \text{if } p^2 \nmid f. \end{cases}$$

We note that if $p^2 \nmid f$ then $p \neq 2$. It is clear from the definitions that for all $p \in \pi(\chi)$ we have that $\chi^{\sigma_p(\chi)} \neq \chi$.

LEMMA 1. *Let H be a subgroup of the finite group G, $\chi \in \mathrm{Irr}(G)$ and $\psi \in \mathrm{Irr}(H)$.*

(a) *If $Q(\chi) \subseteq Q(\psi)$ then $f(\chi) | f(\psi)$.*

(b) *If $Q(\psi) \subseteq Q(\chi)$ and $\psi^{\sigma_p(\chi)} \neq \psi$ for all $p \in \pi(\chi)$, then $f(\chi) = f(\psi)$.*

Proof. If $Q(\chi) \subseteq Q(\psi)$ then $Q_{f(\chi)} \subseteq Q_{f(\psi)}$ and (a) follows. As $\psi^{\sigma_p(\chi)} \neq \psi$ is equivalent to $Q(\psi) \not\subseteq Q_{f(\chi)/p}$ we get that (b) holds.

PROPOSITION 2. *Let $\chi \in \mathrm{Irr}(G)$, $f = f(\chi)$ and $\pi = \pi(\chi)$. If G contains no element of order $f(\chi)$ then there exist $p \in \pi$ such that:*

(a) *$p^2 \nmid f$, and*

(b) *$\chi^{\sigma_p(\chi)} = \chi^\tau$ for some $\tau \in \mathrm{Gal}(Q_f/Q_p)$.*

Proof. Let $\sigma_q = \sigma_q(\chi)$ for each $q \in \pi$ and set $\mathscr{G} = \mathrm{Gal}(Q_f/Q)$. Denote by A the abelian subgroup of the ring of class functions of G that is generated by $\{\chi^\sigma | \sigma \in \mathscr{G}\}$. For each $\sigma \in \mathscr{G}$ and $\alpha \in A$ define $\alpha \cdot \sigma = \alpha^\sigma$. Then A becomes a $Z\mathscr{G}$-module, where Z is the ring of integers.

Let $g \in G$. Then $o(g)$ is not divisible by the full q-part of f for some $q \in \pi$. Then $\alpha(g)^{\sigma_q} = \alpha(g)$ for all $\alpha \in A$. It follows that if $\beta \in A \cdot (\sigma_q - 1)$ then $\beta(g) = 0$. Since each $g \in G$ has such a $q \in \pi$, we get that if $\beta \in A \cdot \prod_{q \in \pi}(\sigma_q - 1)$, then $\beta(g) = 0$ for all $g \in G$. This shows that $\prod_{q \in \pi}(\sigma_q - 1)$ annihilates A and in particular it annihilates χ.

Let π_0 be a subset of π minimal such that $\chi \cdot \prod_{q \in \pi_0}(\sigma_q - 1) = 0$. Let p be the largest prime in π_0 and set $\pi_1 = \pi_0 - \{p\}$. Write $\varepsilon = \prod_{q \in \pi_1}(\sigma_q - 1)$, then $\chi \cdot (\sigma_p - 1)\varepsilon = 0$ and the minimality of π_0 implies that $\chi \cdot \varepsilon \neq 0$. Hence $\chi \cdot \sigma_p \varepsilon = \chi \cdot \varepsilon \neq 0$. An irreducible constituent of $\chi \cdot \sigma_p \varepsilon = \chi^{\sigma_p} \cdot \varepsilon$ has a form $\chi \cdot \sigma_p \mu = \chi \cdot \nu$ where μ, ν are in the abelian group $B = \langle \sigma_q | q \in \pi_1 \rangle$. Thus $\chi \cdot \sigma_p = \chi \cdot \nu\mu^{-1}$. Let $\tau = \nu\mu^{-1}$, then

$\chi^{\sigma_p} = \chi^{\tau}$ and $\tau \in B$. We note that if $q \in \pi_1$, then $\sigma_q \in \mathrm{Gal}(Q_f/Q_{f/q}) \subseteq \mathrm{Gal}(Q_f/Q_p)$ as $q \neq p$. It follows that $\tau \in B \subseteq \mathrm{Gal}(Q_f/Q_p)$ as required.

Finally, we claim that $p^2 \nmid f$. For if $p^2 | f$, then $o(\sigma_p) = p$ and the equality $\chi^{\sigma_p} = \chi^{\tau}$ implies that $p | o(\tau)$. On the other hand the maximality of p implies that $o(\sigma_q) \leq q < p$ for all $\sigma_q \in B$. As τ is a product of elements of B we get that $p \nmid o(\tau)$, a contradiction.

DEFINITIONS. (1) Let G be a finite solvable group. A p-chief factor of G, K/L, is called distinguished if $p \nmid |G:K|$. There is in this case a unique conjugacy class of complements of K/L in G, a complement being a subgroup H of G such that $G = KH$ and $K \cap H = L$, $|H| < |G|$.

(2) If $N \lhd G$ and $\theta \in \mathrm{Irr}(N)$ we define $\mathrm{Irr}(G|\theta) = \{\chi \in \mathrm{Irr}(G) | [\chi_N, \theta] \neq 0\}$.

The next lemma sums up some known facts from character correspondence theory that will be needed in the proof of the Theorem.

LEMMA 3. *Let K/L be a distinguished chief factor of the solvable finite group G and let H be a complement of K/L. Suppose that $\chi \in \mathrm{Irr}(G)$ is primitive. Then χ_K and χ_L have, each, a unique irreducible constituent, θ and ϕ respectively, and there are just two possiblities*:

(i) *$\theta_L = \phi$. In this case the mapping $\mu \to \mu_H$ is a bijection from $\mathrm{Irr}(G|\theta)$ to $\mathrm{Irr}(H|\phi)$. In particular: $\chi_H = \xi \in \mathrm{Irr}(H|\phi)$ and ξ and θ together uniquely determine χ. Thus if $\sigma \in \mathrm{Gal}(Q_{f(\chi)}/Q)$ and $\xi^{\sigma} = \xi$ and $\theta^{\sigma} = \theta$, then $\chi^{\sigma} = \chi$.*

(ii) *$\theta_L = e\phi$ with $e^2 = |K:L|$. In this case there is a canonically defined bijection $\mathrm{Irr}(G|\theta) \to \mathrm{Irr}(H|\phi)$. If $\chi \to \xi$ in this bijection then each of χ and ξ uniquely determines θ and ϕ and so each determines the other. It follows by Galois theory that $Q(\chi) = Q(\xi)$.*

Proof. See **[6]**, [7] and **[8]**.

PROPOSITION 4. *Let G be a finite solvable group and $\chi \in \mathrm{Irr}(G)$. Assume that there exists no proper subgroup X of G and $\psi \in \mathrm{Irr}(X)$ such that $f(\chi)$ divides $f(\psi)$. For a $p \in \pi = \pi(\chi)$, let K/L be a distinguished p-chief factor and let H be a complement of K/L. If $\sigma_p = \sigma_p(\chi)$, then*

(a) *$\chi_H = \xi \in \mathrm{Irr}(H)$ and $\xi^{\sigma_p} = \xi$.*

(b) *$\chi_K = a\theta$, $\theta \in \mathrm{Irr}(K)$, a a positive integer and $\theta^{\sigma_p} \neq \theta$.*

(c) *$\theta_L = \phi \in \mathrm{Irr}(L)$ and $\phi^{\sigma_p} = \phi$.*

Proof. Assume that the Proposition is false and choose p as large as possible to get a counterexample. Then the conclusions of the Proposition are false for some distinguished p-chief factor, K/L, and they hold for distinguished q-chief factors for $q > p$, $q \in \pi$.

If χ is induced from some proper subgroup X of G, say $\chi = \psi^G$, $\psi \in \mathrm{Irr}(X)$, then $Q(\chi) \subseteq Q(\psi)$ and so $f(\chi)$ divides $f(\psi)$. This is a contradiction. Thus χ is primitive so we apply Lemma 3. Hence $\chi_K = a\theta$ for some $\theta \in \mathrm{Irr}(K)$ and a natural number a. Let ϕ be the unique irreducible constituent of χ_L. If $\theta_L = e\phi$ with $e^2 = |K: L|$ then there is a $\xi \in \mathrm{Irr}(H)$ with $Q(\chi) = Q(\xi)$. Therefore $f(\chi) = f(\xi)$, a contradiction. Thus $\theta_L = \phi \in \mathrm{Irr}(L)$ and $\chi_H = \xi \in \mathrm{Irr}(H)$.

Now $Q(\xi) \subseteq Q(\chi)$ but $f(\xi) \neq f(\chi)$ and thus there exist $q \in \pi$ with $\xi^{\sigma_q} = \xi$ (see Lemma 1). Note that $\xi_L = \chi_L = a\phi$ and hence $\phi^{\sigma_q} = \phi$. Since $\chi^{\sigma_q} \neq \chi$ and χ is uniquely determined by ξ and θ, we must have $\theta^{\sigma_q} \neq \theta$. If $q = p$, then K/L is not a counterexample contrary to hypothesis. Therefore $q \neq p$.

Next $(\theta^{\sigma_q})_L = \phi^{\sigma_q} = \phi$ and so θ and θ^{σ_q} are two distinct extensions of ϕ and hence θ and θ^{σ_q} are two distinct irreducible constituents of ϕ^K. By [**6**] Corollary 6.17 we get that $\theta^{\sigma_q} = \lambda\theta$ for some $\lambda \in \mathrm{Irr}(K/L)$, $\lambda \neq 1$. Since $o(\lambda) = p \neq q$ we have that $\lambda^{\sigma_q} = \lambda$. Thus for every positive integer k we have $\theta^{(\sigma_q)^k} = \theta\lambda^k$. By taking $k = o(\sigma_q)$ we obtain that $\theta = \theta\lambda^k$ and hence $\lambda^k = 1$ by (6.17) of [**6**]. It follows that $p | o(\sigma_q)$. Recall that $o(\sigma_q) = q$ or $q - 1$ and $p \neq q$. Therefore $p | q - 1$ and $q > p$.

Hence, if K_0/L_0 is any distinguished q-chief factor, then the conclusions of the Proposition hold. This means that $\chi_{K_0} = a_0\theta_0$, $(\theta_0)_{L_0} = \phi_0$, $\theta_0^{\sigma_q} \neq \theta_0$, $\phi_0^{\sigma_q} = \phi_0$ where $\theta_0 \in \mathrm{Irr}(K_0)$, $\phi_0 \in \mathrm{Irr}(L_0)$ and a_0 is a positive integer. If $q \,|\, |G: K|$, then we can choose K_0/L_0 with $K \subseteq L_0$. Then $\chi_{L_0} = a_0\phi_0$ and so $a\theta = \chi_K = a_0(\phi_0)_K$ and hence $(\phi_0)_K = (a/a_0)\theta$. Since $\phi_0^{\sigma_q} = \phi_0$ we get that $\theta^{\sigma_q} = \theta$, a contradiction. Therefore $q \nmid |G: K|$. Now we choose K_0/L_0 with $K_0 \subseteq L$ and as above we get that ϕ_{K_0} is a multiple of θ_0. But $\phi^{\sigma_q} = \phi$ and this yields $(\theta_0)^{\sigma_q} = \theta_0$, a contradiction. This completes the proof.

Proof of the Theorem. Let G be a minimal counterexample. If G contains a proper subgroup H with $\psi \in \mathrm{Irr}(H)$ such that $f(\chi) | f(\psi)$, then by induction H contains an element h with $o(h) = f(\psi)$. Then there exist $g \in \langle h \rangle$ with $o(g) = f(\chi)$, a contradiction. Hence G satisfies the assumptions of Proposition 4 and therefore its conclusions. Set $f = f(\chi)$, $\pi = \pi(\chi)$ and $\sigma_q = \sigma_q(\chi)$ for all $q \in \pi$. By Proposition 2 we can choose $p \in \pi$ with $p^2 \nmid f$ and $\tau \in \mathrm{Gal}(Q_f/Q_p)$ such that $\chi^{\sigma_p} = \chi^\tau$. Clearly $p \neq 2$. Let K/L be a distinguished p-chief factor and H a complement of K/L. Then by Proposition 4 we get: $\chi_H = \xi$, $\chi_K = a\theta$, $\theta_L = \phi$ where $\xi \in$

$\mathrm{Irr}(H)$, $\theta \in \mathrm{Irr}(K)$, $\phi \in \mathrm{Irr}(L)$ and a a positive integer. Moreover $\xi^{\sigma_p} = \xi$, $\phi^{\sigma_p} = \phi$ but $\theta^{\sigma_p} \neq \theta$.

Since $\phi^{\sigma_p} = \phi$, we have that $Q(\phi) \subseteq Q_{f/p}$ and as $p^2 \nmid f$ we conclude that ϕ is p-rational. As $p \neq 2$, Theorem (6.30) of **[6]** implies that ϕ has a unique p-rational extension $\mu \in \mathrm{Irr}(K)$. As $\theta_L = \mu_L = \phi$ we get by (6.17) of **[6]** that $\theta = \lambda\mu$ for some $\lambda \in \mathrm{Irr}(K/L)$. Note that ϕ and μ uniquely determine each other so that $Q(\mu) = Q(\phi) \subseteq Q(\theta) \subseteq Q(\chi) \subseteq Q_f$. Also μ^{σ_p} is a p-rational extension of $\phi^{\sigma_p} = \phi$ and by the uniqueness we have $\mu^{\sigma_p} = \mu$.

Note that $\lambda(g) \in Q_p$ for all $g \in K/L$ and so $\lambda^\tau = \lambda$. Also, σ_p agrees with τ on $Q(\chi)$. Since $Q(\theta) \subseteq Q(\chi)$, this yields

$$\theta^{\sigma_p} = \theta^\tau = (\lambda\mu)^\tau = \lambda^\tau\mu^\tau = \lambda\mu^{\sigma_p} = \lambda\mu = \theta.$$

This contradiction completes the proof.

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