ORTHOGONAL PROJECTIONS ONTO SUBSPACES
OF THE HARMONIC BERGMAN SPACE

EMIL J. STRAUBE

Let $\Omega \subset \mathbf{R}^m$ be a bounded, smooth domain. We construct a continuous linear operator T: $W^0(\Omega) \to W^0(\Omega)$ which for all $k \in (\mathbf{N} \cup \{\infty\})$ is actually continuous from $W^k(\Omega) \to W_0^k(\Omega)$, and which moreover has the property that $ST = S$, for any orthogonal projection S of $W^0(\Omega)$ onto a subspace of the harmonic Bergman space. That is, the operator assigns to each function a function vanishing to high (infinite if $k = \infty$) order at $b\Omega$, but with the same projection. S can in particular be the harmonic Bergman projection, or, when $\Omega \subset \mathbf{C}^n$, the (analytic) Bergman projection. The question whether such an operator exists arises for example in connection with regularity properties of the Bergman projection and their intimate connection with boundary regularity of holomorphic mappings.

1. Introduction and results. Let $\Omega \subset \mathbf{R}^m$ be a bounded domain with smooth boundary. For $k \in \mathbf{N}$, we denote by $W^k(\Omega)$ the usual Sobolev spaces of order k on Ω (see [10]), and by $W_0^k(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^k(\Omega)$. $h^k(\Omega)$ denotes the closed subspace of $W^k(\Omega)$ consisting of harmonic functions; the harmonic Bergman projection Q is the orthogonal projection of $W^0(\Omega)$ $(= \mathscr{L}^2(\Omega))$ onto $h^0(\Omega)$. We are interested in projections onto subspaces of $h^0(\Omega)$. The most interesting examples will be Q itself and, in the case where Ω lies in complex euclidean space $\mathbf{C}^n \cong \mathbf{R}^{2n}$, the Bergman projection P. This is the orthogonal projection of $W^0(\Omega)$ onto $A^0(\Omega)$, the subspace of $W^0(\Omega)$ consisting of analytic functions. The purpose of the present paper is to construct a continuous linear operator T from $W^0(\Omega)$ to $W^0(\Omega)$, which to each function in $W^k(\Omega)$ assigns a function in $W_0^k(\Omega)$ (i.e. "vanishing to order $k - 1$"), but with the same projection. More precisely, we have

THEOREM 1.1. *Let Ω as above. There is a continuous linear operator T:* $W^0(\Omega) \to W^0(\Omega)$ *which satisfies*

(i) *for all $k \in (\mathbf{N} \cup \{\infty\})$, T maps $W^k(\Omega)$ continuously into $W_0^k(\Omega)$.*

(ii) *if S is the orthogonal projection of $W^0(\Omega)$ onto an arbitrary closed subspace of $h^0(\Omega)$, then*

(1) $$ST = S.$$

REMARK 1.2. The point of the theorem is really that there exists T with (i) and such that $QT = Q$. For S as in the theorem, $S = SQ$, so that then trivially $ST = SQT = SQ = S$.

REMARK 1.3. By the standard interpolation argument, T is continuous from $W^r(\Omega)$ to $W_0^r(\Omega)$, for r real, ≥ 0. For $r = $ integer $+ 1/2$, we even get continuity from $W^r(\Omega)$ to $W_{00}^r(\Omega)$; see [10] for details and definition of the last space.

REMARK 1.4. The smoothness of Tg depends only on the smoothness of g near $b\Omega$. More precisely: if $g \in W^k(\Omega \setminus K)$ for some compact subset K, then $Tg \in W_0^k(\Omega)$. This will be clear from the proof of Theorem 1.1.

The main source of motivation for constructing operators like T are questions revolving about the Bergman projection P, and its intimate connection with boundary behavior of holomorphic mappings ([4] and its references, [7]). One of the key steps was Bell's construction of (differential) operators φ^k: $W^{k+N(k)}(\Omega) \to W_0^k(\Omega)$, such that $P\varphi^k = P$. φ^k is also bounded from $A^k(\Omega) \to W_0^k(\Omega)$ ($A^k(\Omega)$ is the analytic subspace of $W^k(\Omega)$). A revised version of these operators is in [1]. Harmonic and pluriharmonic versions were given in [2] and [3], respectively. From this circle of ideas, the question also arises whether the conditions R^k: P maps $W^k(\Omega)$ into itself, and R_0^k: P maps $W_0^k(\Omega)$ into $W^k(\Omega)$, are equivalent. The φ^k do not give an answer. This question also arises from [9], where R_0^k rather than R^k appeared naturally. The question was answered affirmatively by the author in [12]. It was shown that there exist continuous operators T^k: $W^k(\Omega) \to W_0^k(\Omega)$, such that $PT^k = P$. For $k = \infty$, it was shown in [6] and [12] that for $g \in W^\infty(\Omega)$, there is always $h \in W_0^\infty(\Omega)$ with $Ph = Pg$, but it was not clear whether the function in $W_0^\infty(\Omega)$ could be chosen in a continuous, linear way (in [12] a continuous linear map into a quotient of $W_0^\infty(\Omega)$ was obtained). The operator T gives a unified approach to all the above. In addition, we write down T quite explicitly (in contrast to [12], where the author's T^k were obtained by abstract arguments). This clarifies the situation; in fact it is precisely this feature which allows to check the regularity properties. Note that the equivalence for the harmonic Bergman projection, corresponding to $R^k \Leftrightarrow R_0^k$, gives nothing interesting, because Q always maps $W^k(\Omega)$ into itself ([2]). In intermediate cases however, such as the projection onto the pluriharmonic functions, the corresponding equivalence (also obtained from T) is of interest.

We also briefly mention that the operator T may be used to obtain equivalence of certain negative Sobolev norms on harmonic functions (no geometric assumptions on Ω), compare [5], §4. We do not elaborate on this, because this equivalence also follows directly from the Sobolev estimate on an "improper" $\mathscr{L}^2$-pairing given in [12], and it is our opinion that the approach via the pairing is more natural in that context.

The construction of the operator T rests upon an observation about the projections, which we now proceed to state. It says, loosely speaking, that the condition of having the same projection as some given function may be reformulated as a certain boundary condition. Consider the Dirichlet problem for the Laplacian

$$\begin{aligned} \Delta h &= g \quad \text{on } \Omega \\ h &= 0 \quad \text{on } b\Omega, \end{aligned} \tag{2}$$

with $g \in W^k(\Omega)$, so that the solution $h \in W^{k+2}(\Omega)$. Let Ψ be any function in $W^{k+2}(\Omega)$ such that

$$\begin{aligned} \Psi &= 0 \\ \frac{\partial \Psi}{\partial \nu} &= \frac{\partial h}{\partial \nu} \quad \text{on } b\Omega \end{aligned} \tag{3}$$

Here, $\partial/\partial \nu$ denotes the normal derivative (normal to $b\Omega$, oriented inward); the boundary values are, as usual, to be understood as traces. Then we have

PROPOSITION 1.5. *Let* $g \in W^k(\Omega)$ *and let* h *be the solution of the Dirichlet problem* (2). *For* Ψ *satisfying* (3),

$$S(\Delta\Psi) = Sg, \tag{4}$$

for any projection S *as in Theorem* 1.1.

As we shall see, the main point here is that (4) is implied by a *boundary condition* on Ψ (namely (3)).

Proposition 1.5 allows essentially to reduce the problem to finding functions with prescribed normal derivatives on $b\Omega$. However, *infinitely many derivatives will be involved*, and in order to get extensions depending linearly on the data, special care has to be taken. In §3, we construct sequences of extension operators (roughly one operator for each normal derivative) whose norms are well controlled. For certain boundary data spaces (including the ones arising from our problem), they can be summed up to yield a *linear* operator which gives functions with the infinitely many prescribed normal derivatives. The construction is an infinite version of a construction in [8]; it is also somewhat motivated by the construction in [11].

2. Proofs. We first prove Proposition 1.5. We have

$$(1) \qquad\qquad \Delta\Psi - g = \Delta(\Psi - h).$$

Use (2) and (3) (§1) to conclude that $\Psi - h = \partial(\Psi - h)/\partial\nu = 0$ on $b\Omega$ so that $\Psi - h \in W_0^2(\Omega)$ ([**10**]). Therefore, $\Delta(\Psi - h)$ is orthogonal to $h^0(\Omega)$ (integrate by parts, no boundary terms appear). Thus, by (1), $\Delta\Psi - g$ is orthogonal to $h^0(\Omega)$ and thus to the image of S, whence the result.

Now we prove Theorem 1.1. Let $L: W^0(\Omega) \to W^2(\Omega)$ be the solution operator of the Dirichlet problem

$$(2) \qquad\qquad \begin{aligned} \Delta Lg &= g \quad \text{on } \Omega \\ Lg &= 0 \quad \text{on } b\Omega. \end{aligned}$$

For $g \in W^0(\Omega)$, Tg will be defined as $\Delta\Psi$, for suitable Ψ, with

$$(3) \qquad\qquad \begin{aligned} \Psi &= 0 \\ \frac{\partial\Psi}{\partial\nu} &= \frac{\partial}{\partial\nu}Lg \quad \text{on } b\Omega \end{aligned}$$

(3) will ensure, by Proposition 1.5, that $S\Delta\Psi = Sg$. The condition that $\Delta\Psi = Tg \in W_0^k(\Omega)$, may also be formulated as a boundary condition; it is equivalent to the condition

$$(4) \qquad\qquad \frac{\partial^J}{\partial\nu^J}\Delta\Psi = 0 \quad \text{on } b\Omega, \qquad 0 \le j \le k - 1,$$

see [**10**]. The next step is to observe that (3) and (4) together may equivalently be written by prescribing only normal derivatives of Ψ. It will be convenient to work in local coordinates near $b\Omega$, so we choose a partition of unity $\{\varphi_s\}$ of $b\Omega$, so that supp φ_s is contained in a coordinate neighborhood U_s of $b\Omega$, which is "small" so that its local coordinates $(\theta_1, \ldots, \theta_{m-1})$ together with ν (signed distance to $b\Omega$) can be used as coordinates of $\mathbf{R}^m$ near U_s. Then the task is reduced (by linearity) to constructing $\Psi_s \in W^{k+2}(\Omega)$ satisfying the boundary condition

$$\Psi_s = 0$$

$$(5) \qquad\qquad \frac{\partial\Psi_s}{\partial\nu} = \varphi_s\frac{\partial}{\partial\nu}Lg$$

$$\frac{\partial^J}{\partial\nu^J}\Delta\Psi_s = 0 \qquad 0 \le j \le k - 1,$$

whenever the function g we start out with is in $W^k(\Omega)$. Near U_s, the Laplacian is expressed in the local coordinates (the index s is suppressed)

as

$$\Delta = \frac{\partial^2}{\partial \nu^2} + \frac{1}{2g}\frac{\partial g}{\partial \nu}\frac{\partial}{\partial \nu} + \frac{1}{\sqrt{g}}\sum_{i,j=1}^{m-1}\frac{\partial}{\partial \theta_j}\left(g^{ij}\sqrt{g}\,\frac{\partial}{\partial \theta_i}\right) \tag{6}$$

where

$$g_{ij} = \sum_{l=1}^{m}\frac{\partial x_l}{\partial \theta_i}\frac{\partial x_l}{\partial \theta_j},$$

$$\left(g^{ij}\right) = \left(g_{ij}\right)^{-1} \quad \text{and} \quad g = \det(g_{ij}).$$

Thus

$$\frac{\partial^j}{\partial \nu^j}\Delta = \frac{\partial^{2+j}}{\partial \nu^{2+j}} + \frac{\partial^j}{\partial \nu^j}\left(\frac{1}{2g}\frac{\partial g}{\partial \nu}\frac{\partial}{\partial \nu} + \sum_{i,l=1}^{m-1}\frac{\partial}{\partial \theta_l}\left(g^{il}\sqrt{g}\,\frac{\partial}{\partial \theta_i}\right)\right). \tag{7}$$

Note that the second term on the right side of (7) involves only normal derivatives of order at most $j + 1$. So if one knows the traces of Ψ and $\partial \Psi / \partial \nu$ on $b\Omega$, and those of $\partial^j \Delta \Psi / \partial \nu^j$, one may recursively calculate the traces of $\partial^j \Psi / \partial \nu^j$. Thus we obtain a sequence $(B_j^s)_{j=2}^{\infty}$ of differential operators, such that for all $k \in \mathbf{N}$, the boundary condition

$$\Psi_s = 0$$

$$\frac{\partial \Psi_s}{\partial \nu} = \varphi_s \frac{\partial}{\partial \nu}Lg \tag{8}$$

$$\frac{\partial^j \Psi_s}{\partial \nu^j} = B_j^s\left(\varphi_s \frac{\partial}{\partial \nu}Lg\right), \quad 2 \leq j \leq k + 1$$

is equivalent to the boundary condition (5) (in U_s). For convenience, we set $B_0^s := 0$ and $B_1^s = \mathrm{id}$ (the identity operator), so that (8) can be rewritten as

$$\frac{\partial^j \Psi_s}{\partial \nu^j} = B_j^s\left(\varphi_s \frac{\partial}{\partial \nu}Lg\right), \quad 0 \leq j \leq k + 1. \tag{9}$$

The order of B_j^s does not exceed $j - 1$, for $j \geq 1$. Since $B_j^s(\varphi_s(\partial Lg/\partial \nu))$ is compactly supported in U_s, we may continue it to $b\Omega$ by setting it equal to 0 outside U_s. In this way, (9) may be viewed as a boundary condition on all of $b\Omega$. Then, any Ψ_s is $W^{k+2}(\Omega)$ which satisfies this condition, will also satisfy (5) on *all* of $b\Omega$ (in U_s by construction of the B_j^s, on $b\Omega \setminus \mathrm{supp}\,\varphi_s$ because there all normal derivatives are 0 on $b\Omega$). We point out once more that the B_j^s are easily calculated explicitly, by recursion, starting with B_2^s.

In §3, we will for every given sequence $(\alpha_j)_0^\infty$ of positive numbers construct a sequence $(R^j)_0^\infty$ of linear operators defined on $C^\infty(b\Omega)' \cong \bigcup_{k=0}^\infty W^{-k}(b\Omega)$, such that for all $k \in \mathbf{N}$:

    (i)

(10) R^j is continuous from $W^{k-j-1/2}(b\Omega)$ to $W^k(\Omega)$;

        we denote the corresponding operator norm by $\|R^j\|_k$.

    (ii) for $0 \le s \le k - 1$,

(11)
$$\left(\frac{\partial^s}{\partial\nu^s}R^ju\right)\bigg|_{b\Omega} = \begin{cases} 0 & \text{if } s \ne j \\ u & \text{if } s = j \end{cases}$$

Note that these traces are well defined, since $R^ju \in W^k(\Omega)$.
    (iii)

(12)
$$\sum_{j \ge 0} \|R^j\|_k \alpha_j < \infty$$

This property is independent of the boundary norm used (see §3).

This will put us in a position to define the operator T. Denote by C^s_{jk} the operator norm of

$$B^s_j\left(\varphi_s\frac{\partial}{\partial\nu}L\right): W^k(\Omega) \to W^{k+3/2-j}(b\Omega).$$

We choose the sequence $(\alpha_j)_0^\infty$ with $\alpha_j := \sup_s \sup_{k \le j} C^s_{jk}$ and construct the R^j corresponding to this sequence. Now we set for $g \in W^0(\Omega)$

(13)
$$Tg := \Delta\left(\sum_s \sum_{j \ge 0} R^jB^s_j\left(\varphi_s\frac{\partial}{\partial\nu}Lg\right)\right)$$

T is well defined: for $k \in \mathbf{N}$ fixed, we have

$$\left\| R^jB^s_j\left(\varphi_s\frac{\partial}{\partial\nu}Lg\right)\right\|_{W^{k+2}(\Omega)} \le \|R^j\|_{k+2}C^s_{jk}\|g\|_{W^k(\Omega)}.$$

Therefore,

(14)
$$\left\|\sum_{j \ge 0} R^jB^s_j\left(\varphi_s\frac{\partial}{\partial\nu}Lg\right)\right\|_{W^{k+2}(\Omega)} \le \left(\sum_{j \ge 0}\|R^j\|_{k+2}C^s_{jk}\right)\|g\|_{W^k(\Omega)}$$

$$\le \left(\sum_{j=0}^k \|R^j\|_{k+2}C^s_{jk} + \sum_{j > k}\|R^j\|_{k+2}\alpha_j\right)\|g\|_{W^k(\Omega)}$$

$$\le C(k)\|g\|_{W^k(\Omega)}.$$

Here, we have used (12); also note that s ranges only over finitely many integers. From (14) it follows that T is continuous from $W^k(\Omega)$ to itself. If

$$(15) \qquad \Psi_s = \sum_{j \geq 0} R^j B_j^s \left(\varphi_s \frac{\partial}{\partial \nu} Lg \right)$$

and

$$(16) \qquad \Psi := \sum_s \Psi_s,$$

we have to check (3) and (4), or, after what we've done, just (9). Let k still fixed. Let $0 \leq t \leq k + 1$. Then we calculate the traces

$$(17) \qquad \frac{\partial^t \Psi_s}{\partial \nu^t} = \sum_{j \geq 0} \frac{\partial^t}{\partial \nu^t} R^j B_j^s \left(\varphi_s \frac{\partial}{\partial \nu} Lg \right)$$

$$= \sum_{j \geq 0} \delta_{tj} B_j^s \left(\varphi_s \frac{\partial}{\partial \nu} Lg \right) = B_t^s \left(\varphi_s \frac{\partial}{\partial \nu} Lg \right).$$

This is (9). Thus T has all the properties required in Theorem 1.1, except possibly those relating to the case $k = \infty$. However, these are a consequence of the properties for *all k*. Therefore, the proof of Theorem 1.1 is complete.

REMARK 2.1. There is an abstract argument which gives an operator $T^\infty: W^\infty(\Omega) \to W_0^\infty(\Omega)$, such that $ST^\infty = S$, for S as in Theorem 1.1. We briefly indicate it, as it has some interest of its own. Again, it suffices to treat the case $S = Q$. Moreover, we only need to find a right inverse $\tilde{T}$ for $Q: W_0^\infty(\Omega) \to h^\infty(\Omega)$, then $T^\infty := \tilde{T}Q$ will do the job, since Q is continuous from $W^\infty(\Omega) \to h^\infty(\Omega)$ ([2]). To find this right inverse, consider the sequence

$$(18) \qquad 0 \to \ker Q \to W_0^\infty(\Omega) \overset{Q}{\to} h^\infty(\Omega) \to 0.$$

It is exact (for the surjectivity of Q, see [12]). Now $h^\infty(\Omega) \cong C^\infty(b\Omega)$, by the Poisson extension, which in turn is isomorphic to s, the space of rapidly decreasing sequences ([14], Theorem 2.3). It is easy to show that $\ker Q \cong W_0^\infty(\Omega)$, which is again isomorphic to s ([14], Theorem 2.3). Therefore, Vogt's splitting theorem ([13], Theorem 2.2, see also Theorem 1.3) applies and yields a continuous right inverse $\tilde{T}$ of Q. The drawback of this approach is that it does not yeild exact preservation of differentiability (measured in Sobolev norms) for $\tilde{T}$.

3. Prescribing infinitely many normal derivatives. In this section, we construct the operators R^j with properties (10), (11) and (12) of §2. With the help of a partition of unity and local flattenings of the boundary, we reduce the problem to the case where Ω is a euclidean half space, $\Omega = \{(x, t) \in \mathbf{R}^m \times \mathbf{R}/t > 0\}$ (m here does not denote the same integer as in the previous sections). This causes no problems as far as properties (i) and (ii) (i.e. (10) and (11)) are concerned, but property (iii) needs some attention, since cutoffs affect norms. Also, $\|R^j\|_k$ depends on the boundary norms used, so it is not a priori clear that property (iii) is independent of the choice of norm for the boundary Sobolev spaces. However, for both problems the relevant norms are estimated by factors C_{jk}, and considering a new sequence $\tilde{\alpha}_j := (\sup_{k \le j} C_{jk})\alpha_j$ (a similar "diagonal process' was used in §2) shows that (iii) is preserved. With these considerations done, we only state the result in the setting of a euclidean half space.

THEOREM 3.1. *Let $(\alpha_j)_0^\infty$ be an arbitrary sequence of positive numbers. Then there exists a sequence $(R^j)_0^\infty$ of operators defined on $W^{-\infty}(\mathbf{R}^m)$, valued in $W^{-\infty}(\Omega)$, such that for all $k \in \mathbf{N}$:*

(i)

(1)   *R^j is continuous from $W^{k-j-1/2}(\mathbf{R}^m)$ to $W^k(\Omega)$, $j = 0, 1, 2, \ldots$,*

(ii) *for $0 \le s \le k - 1$,*

(2) $$\left. \frac{\partial^s}{\partial t^s} R^j a \right|_{\mathbf{R}^m} = \begin{cases} 0 & \text{if } s \ne j \\ a & \text{if } s = j \end{cases} \qquad j = 0, 1, 2, \ldots.$$

(iii) *If $\|\mathbf{R}^j\|_k$ denotes the operator norm of R^j as an operator from $W^{k-j-1/2}(\mathbf{R}^m)$ to $W^k(\Omega)$, then*

(3) $$\sum_{j \ge 0} \|R^j\|_k \alpha_j < \infty.$$

REMARK 3.2. We identify $b\Omega$ with $\mathbf{R}^m$. The left side of (2) is understood in the sense of traces. These are well defined for $0 \le s \le k - 1$, since $R^j a \in W^k(\Omega)$.

Proof. First note that

(4) $W^k(\Omega) = \Big\{ u \,|\, u \in \mathscr{L}^2\big((0, \infty), W^k(\mathbf{R}^m)\big),$

$$\frac{\partial^k}{\partial t^k} u \in \mathscr{L}^2\big((0, \infty), W^0(\mathbf{R}^m)\big) \Big\}.$$

Here, $\mathscr{L}^2((0,\infty),E)$ denotes the space of square integrable E-valued functions on $(0,\infty)$, with norm

$$
(5) \qquad \|u\|_{\mathscr{L}^2((0,\infty),E)} = \left(\int_0^\infty \|u(t)\|_E^2 \, dt \right)^{1/2}.
$$

Equality in (4) means "same elements and equivalent norms", i.e.

$$
(6) \qquad \|u\|_{W^k(\Omega)} \approx \left(\int_0^\infty \left(\|u(t)\|_{W^k(\mathbf{R}^m)}^2 + \left\| \frac{\partial^k}{\partial t^k} u(t) \right\|_{W^0(\mathbf{R}^m)}^2 \right) dt \right)^{1/2}
$$

For a proof, see [10], Theorem 7.4, Chapter 1. We identify the Sobolev space $W^k(\mathbf{R}^m)$ in the usual way with its Fourier transform, consisting of all functions $a(\xi)$ which are square integrable with respect to the weight $(1 + |\xi|^2)^k \, d\xi$. Thus $u \in \mathscr{L}^2((0,\infty), W^k(\mathbf{R}^m))$ is an $\mathscr{L}^2$ function with respect to t with values in a weighted $\mathscr{L}^2$-space with respect to ξ, and the norm (5) becomes

$$
(7) \qquad \|u\|_{\mathscr{L}^2((0,\infty), W^k(\mathbf{R}^m))} = \left(\int_0^\infty \left(\int_{\mathbf{R}^m} |u(t,\xi)|^2 \left(1 + |\xi|^2\right)^k d\xi \right) dt \right)^{1/2}.
$$

After these preparations, we are ready to define the R^j. For $a \in W^{-\infty}(\mathbf{R}^m)$, set

$$
(8) \qquad R^j a(t,\xi) := \frac{t^j}{j!} \varphi\left(\beta_j t \left(1 + |\xi|^2\right)^{1/2} \right) a(\xi)
$$

with

$$
(9) \qquad \beta_j := \max\left(1, (j^2)\alpha_j\right)
$$

φ is a fixed $[0,1]$-valued function in $C_0^\infty(\mathbf{R})$, supported in $[-1,1]$, and identically 1 on $[-1/2, 1/2]$.

Clearly, $R^j a \in W^{-\infty}(\Omega)$. Fix now $k \in \mathbf{N}$. Assume that $a \in W^{k-j-1/2}(\mathbf{R}^m)$. To show (1), we shall use (4). For $t \in (0,\infty)$ fixed, the right-hand side of (8) is in $W^k(\mathbf{R}^m)$, since a is locally integrable and φ has compact support. So $R^j a$ is a $W^k(\mathbf{R}^m)$-valued function on $(0,\infty)$; it is easy to see that it is smooth. To estimate $\|R^j a\|_{W^k(\Omega)}$, we have to estimate the right-hand side of (6). The first contribution is (after squaring (6)):

$$
(10) \qquad \int_0^\infty \|R^j a(t,\xi)\|_{W^k(\mathbf{R}^m)}^2 \, dt = \int_0^\infty \left(\int_{\mathbf{R}^m} |R^j a(t,\xi)|^2 \left(1 + |\xi|^2\right)^k d\xi \right) dt
$$

$$
= \int_0^\infty \int_{\mathbf{R}^m} \frac{t^{2j}}{(j!)^2} \varphi^2\left(\beta_j t \left(1 + |\xi|^2\right)^{1/2} \right) |a(\xi)|^2 \left(1 + |\xi|^2\right)^k d\xi \, dt.
$$

Since $\operatorname{supp} \varphi \subset [-1, 1]$, we are in fact only integrating over the subset of $(0, \infty) \times \mathbf{R}^m$ where

(11)
$$\beta_j t \left(1 + |\xi|^2\right)^{1/2} \leq 1.$$

Using this and Fubini's theorem, the last integral in (10) can be estimated by

(12)
$$\int_{\mathbf{R}^m} \left(\int_0^{(\beta_j)^{-1}(1+|\xi|^2)^{-1/2}} dt \right) \frac{\left(1 + |\xi|^2\right)^k}{(j!)^2 (\beta_j)^{2j} \left(1 + |\xi|^2\right)^j} |a(\xi)|^2 \, d\xi$$

$$= \frac{1}{(\beta_j)^{2j+1}(j!)^2} \int_{\mathbf{R}^m} |a(\xi)|^2 \left(1 + |\xi|^2\right)^{k-j-1/2} d\xi$$

$$= \frac{1}{(\beta_j)^{2j+1}(j!)^2} \|a\|^2_{W^{k-j-1/2}(\mathbf{R}^m)}.$$

The calculation for the second contribution in (6) is essentially the same. Note that

(13)
$$\frac{\partial^k}{\partial t^k} R^j a = \frac{\partial^k}{\partial t^k} \left(\frac{t^j}{j!} \varphi \left(\beta_j t \left(1 + |\xi|^2\right)^{1/2} \right) \right) a(\xi)$$

$$= \left(\sum_{s=0}^k \binom{k}{s} \frac{\partial^s}{\partial t^s} \left(\frac{t^j}{j!} \right) \left(\beta_j \left(1 + |\xi|^2\right)^{1/2} \right)^{k-s} \right.$$

$$\times \left. \left(\frac{\partial^{k-s}}{\partial t^{k-s}} \varphi \right) \left(\beta_j t \left(1 + |\xi|^2\right)^{1/2} \right) \right) a(\xi).$$

Therefore, in order to estimate

$$\int_0^\infty \int_{\mathbf{R}^m} \left| \left(\frac{\partial^k}{\partial t^k} R^j a \right)(t, \xi) \right|^2 d\xi \, dt,$$

it suffices to consider the terms

(14)
$$\int_0^\infty \int_{\mathbf{R}^m} \left(\frac{\partial^s}{\partial t^s} \left(\frac{t^j}{j!} \right) \right)^2 \left(\left(\frac{\partial^{k-s}}{\partial t^{k-s}} \varphi \right) \left(\beta_j t \left(1 + |\xi|^2\right)^{1/2} \right) \right)^2$$

$$\times \beta_j^{2k-2s} \left(1 + |\xi|^2\right)^{k-s} |a(\xi)|^2 \, d\xi \, dt.$$

For $s > j$, the integrand is 0, so nothing needs to be estimated. For $s \leq j$, (14) may be treated exactly as (10), and we find that it is estimated by

(15)
$$\frac{C_k}{(\beta_j)^{2j-2k+1}} \|a\|^2_{W^{k-j-1/2}(\mathbf{R}^m)},$$

where C_k is a constant depending on k only. This shows property (i) of the theorem. From (12) and (15), we also read off that

$$(16) \qquad \| R^j \|_k \le \frac{C_k}{(\beta_j)^{j-k}},$$

with a different C_k (note that $\beta_j \ge 1$). Thus

$$(17) \qquad \sum_{j \ge 0} \| R^j \|_k \alpha_j \le C_k \sum_{j \ge 0} \frac{\alpha_j}{(\beta_j)^{j-k}}.$$

Now for $j > k$, $(\beta_j)^{j-k} \ge \beta_j \ge (j)^2 \alpha_j$, so that the right-hand side of (17) is finite. This proves (iii).

To prove (ii), we first observe that the trace of $\partial^s R^j a / \partial t^s$ on $\mathbf{R}^m$ is the limit, in $W^{k-s-1/2}(\mathbf{R}^m)$, as $t_0 \to 0^+$, of the traces on the hyperplanes $t = t_0$ ([10]). Since $0 \le s \le k - 1$, this convergence takes a fortiori place in $W^0(\mathbf{R}^m) = \mathscr{L}^2(\mathbf{R}^m)$. Thus the trace we look for is the $\mathscr{L}^2$-limit, as $t \to 0^+$, of

$$(18) \qquad \frac{\partial^s}{\partial t^s} \left(\frac{t^j}{j!} \varphi \left(\beta_j t \left(1 + |\xi|^2 \right)^{1/2} \right) \right) a(\xi).$$

Because the limit exists, we may calculate it by taking the pointwise limit a.e.; but this limit is $\delta_{sj} a(\xi)$, since for t small enough, $\varphi(\beta_j t (1 + |\xi|^2)^{1/2}) \equiv 1$ (because ξ is fixed). This proves (ii), and the proof of Theorem 3.1 is complete.

REFERENCES

[1]   D. Barrett, *Regularity of the Bergman projection on domains with transverse symmetries*, Math. Ann., **258** (1982), 441–446.

[2]   S. Bell, *A duality theorem for harmonic functions*, Michigan Math. J., **29** (1982), 123–128.

[3]   _____, *A Sobolev inequality for pluriharmonic functions*, Proc. Amer. Math. Soc., **85**, Nr. 3 (1982), 350–352.

[4]   _____, *Boundary behavior of proper holomorphic mappings between non-pseudoconvex domains*, Amer. J. Math., **106**, Nr. 3 (1984), 639–643.

[5]   S. Bell and H. Boas, *Regularity of the Bergman projection in weakly pseudoconvex domains*, Math. Ann., **257** (1981), 13–30.

[6]   S. Bell and D. Catlin, *Boundary regularity of proper holomorphic mappings*, Duke Math. J., **49** (1982), 385–396.

[7]   K. Diederich and I. Lieb, *Konvexität in der komplexen Analysis*, DMV Seminar, Band 2, Birkhäuser 1981.

[8]   L. Hörmander, *Linear Partial Differential Operators*, Grundl. d. math. Wissenschaften, Band 116, Springer 1976.

[9]   G. Komatsu, *Boundedness of the Bergman projector and Bell's duality theorem*, Tôhoku Math. J., **36** (1984), 452–467.

[10]   J. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications I*, Grundl. d. math. Wissenschaften, Band 181, Springer 1972.

[11]   R. Seeley, *Extension of C^∞-functions defined in a half space*, Proc. Amer. Math. Soc., **15**, Nr. 4 (1964), 625–626.

[12]   E. Straube, *Harmonic and analytic functions admitting a distribution boundary value*, Ann. Scuola Norm. Sup. Pisa, **11**, Nr. 4 (1984), 559–591.

[13]   D. Vogt, *Subspaces and Quotient Spaces of* (s), in *Functional Analysis; Surveys and Recent Results*; Bierstedt K. and Fuchssteiner B. editors, North-Holland Mathematics Studies **27** (1977).

[14]   _____ , *Sequence Space Representations of Spaces of Test Functions and Distributions*, in *Functional Analysis, Holomorphy and Approximation Theory*; Zapata G. editor, Lecture Notes in Pure and Appl. Math. **83**, Marcel Dekker, 1983.

INDIANA UNIVERSITY
BLOOMINGTON, IN 47405