PACIFIC JOURNAL OF MATHEMATICS
Vol. 125, No. 1, 1986

LINES HAVING HIGH CONTACT WITH A PROJECTIVE VARIETY

GEORGE JENNINGS

Let $\mathscr{U}$ open $\subset \mathbf{P}^n = \mathbf{P}^n(\mathbf{C})$, $X \subset \mathscr{U}$ an analytic subvariety,

$$\begin{array}{ccc} & J = \{(p,l) \in \mathbf{P}^n \times \mathbf{G}(1,n) \mid p \in l\} & \\ \pi \swarrow & & \searrow \lambda \\ \mathbf{P}^n & & \mathbf{G} = \mathbf{G}(1,n), \end{array}$$

the incidence correspondence with induced projections π, λ, where $\mathbf{G} = \mathbf{G}(1,n)$ is the Grassmannian of lines in $\mathbf{P}^n$.

0. Definition. The *contact cones* of X are

$$C^r = \{(p,l) \in \pi^{-1}\mathscr{U} \mid l \text{ has contact} \geq r+1 \text{ with } X \text{ at } p\}$$

$$C^\infty = \bigcap_{r=0}^{\infty} C^r.$$

The contact cones may be thought of as schemes of cones in the tangent space of $\mathbf{P}^n$ which reflect the local geometry of the embedding $X \to \mathscr{U}$. The main results of this paper are a singularities theorem (13) which puts an upper bound on the pathology of the contact cones if X is not ruled, and an algebraization theorem (17) which says roughly that if X is a hypersurface whose contact cones resemble those of an algebraic hypersurface of low degree then X is algebraic. Hypersurfaces are the simplest case—in a future paper we show that in general hypersurfaces are determined up to projective equivalence by the projective moduli of the third contact cone with a little help from the ideal of the fourth.

The contact cones have a scheme structure defined in terms of the functor of principal parts (jets) $\mathscr{P}^r_{J/\mathbf{G}}$ [5, §16]. Let $\mathscr{F}$ be a sheaf of $\mathscr{O}_J$-modules. Form the fiber product $J \times_{\mathbf{G}} J$. Let $\mathscr{I}_\Delta$ be the ideal sheaf of the diagonal, and $J^r \xrightarrow{\Delta^r} J \times_{\mathbf{G}} J$ the subscheme defined by $\mathscr{I}_\Delta^{r+1}$. One has a commutative diagram

$$\begin{array}{ccccc} & & J^r & & \\ & p^r \swarrow & \downarrow \Delta^r & \searrow q^r & \\ J & \overset{p}{\leftarrow} & J \times_{\mathbf{G}} J & \overset{q}{\rightarrow} & J \end{array}$$

where p, q are the projections. Then

$$\mathscr{P}^r_{J/\mathbf{G}}\mathscr{F} = q^r_* p^{r*}\mathscr{F} \cong \mathscr{F} \otimes_{\mathscr{O}_J} \mathscr{P}^r_{J/\mathbf{G}}\mathscr{O}_J.$$

$\mathscr{P}^r_{J/\mathbf{G}}\mathscr{O}_J$ is a locally free sheaf of rank $r+1$ consisting of relative r jets of sections of $\mathscr{O}_J$.

Let $\mathscr{I}_X \subset \mathscr{O}_{\mathscr{U}}$ be the ideal sheaf of X.

1. Definition. C^r, $0 \le r < \infty$, is the zero scheme of the sheaf of sections $\mathscr{P}^r_{J/\mathbf{G}}(\pi^*\mathscr{I}_X) \subset \mathscr{P}^r_{J/\mathbf{G}}\mathscr{O}_J \,|\, \mathscr{U}$. $C^\infty = \bigcap_{r=0}^\infty C^r$ is the intersection scheme. $C^r_p = \pi^{-1}(p) \cap C^r$, $0 \le r \le \infty$, is the fiber over $p \in \mathscr{U}$.

Since $J \times_{\mathbf{G}} J \xrightarrow{\pi\times\pi} \mathbf{P}^n \times \mathbf{P}^n$ is the blow up of $\mathbf{P}^n \times \mathbf{P}^n$ along the diagonal the exceptional divisor J is naturally isomorphic to the projectivized tangent space $PT\mathbf{P} \to \mathbf{P}^n$, via the relation "$v$ is tangent to l". In particular the relative cotangent sheaf $\Omega^1_{J/\mathbf{G}} \cong \mathscr{I}_\Delta/\mathscr{I}^2_\Delta$ of J is just the dual $\mathscr{O}_T(1)$ of the universal subbundle $\mathscr{O}_T(-1)$ of $\pi^*T\mathbf{P}$ over $PT\mathbf{P}$. $J \cong \operatorname{Proj}(S^{\boldsymbol{\cdot}}\Omega^1_{J/\mathbf{G}}\mathbf{P})$ where $S^{\boldsymbol{\cdot}}\Omega^1_{J/\mathbf{G}}$ is the sheaf of graded rings

$$S^{\boldsymbol{\cdot}}\Omega^1_{J/\mathbf{G}} \cong \mathscr{O}_J \oplus \mathscr{I}_\Delta/\mathscr{I}^2_\Delta \oplus \mathscr{I}^2_\Delta/\mathscr{I}^3_\Delta \oplus \cdots.$$

There is an (additive) sheaf homomorphism $d^r_{J/\mathbf{G}}\colon \mathscr{O}_J \to \mathscr{P}^r_{J/\mathbf{G}}\mathscr{O}_J$ induced by the corresponding map on sections [5, p. 16]. One has a commutative diagram

$$\begin{array}{ccccccc}
 & & d^r_{J/\mathbf{G}}(\pi^*\mathscr{I}_X) & \to & d^{r-1}_{J/\mathbf{G}}(\pi^*\mathscr{I}_X) & \to & 0 \\
 & & \downarrow & & \downarrow & & \\
0 \to \mathscr{I}^r_\Delta/\mathscr{I}^{r+1}_\Delta & \xrightarrow{\iota} & \mathscr{P}^r_{J/\mathbf{G}}\mathscr{O}_J & \xrightarrow{\rho} & \mathscr{P}^{r-1}_{J/\mathbf{G}}\mathscr{O}_J & \to & 0
\end{array}$$

over $\pi^{-1}\mathscr{U}$ arising directly from the definition. Define contact ideal sheaves $\mathscr{J}^r_X \subset S^{\boldsymbol{\cdot}}\Omega^1_{J/\mathbf{G}}\,|_{\pi^{-1}\mathscr{U}}$ inductively by

$$\mathscr{J}^0_X = \pi^*\mathscr{I}_X \otimes_{\mathscr{O}_J} S^{\boldsymbol{\cdot}}\Omega^1_{J/\mathbf{G}}$$

$$\mathscr{J}^r_X = \mathscr{J}^{r-1}_X + \iota^{-1}\big(d^r_{J/\mathbf{G}}(\pi^*\mathscr{I}_X) + \ker\rho\big) \otimes_{\mathscr{O}_J} S^{\boldsymbol{\cdot}}\Omega^1_{J/\mathbf{G}}$$

$$\mathscr{J}^\infty_X = \sum_{r=0}^\infty \mathscr{J}^r_X$$

on $\pi^{-1}\mathscr{U}$. $\mathscr{J}^r_X$ is the ideal sheaf of C^r in $S^{\boldsymbol{\cdot}}\Omega^1_{J/\mathbf{G}}$.

This leads to a convenient version in coordinates. Let $x = (x_1, \ldots, x_n)$ be an *affine* coordinate system on $\mathscr{U}$, $dx = (dx_1, \ldots, dx_n)$, $p \in \mathscr{U}$, $g \in \mathscr{O}_{p,\mathscr{U}}$. Expand in a power series

$$g(x(p) + t) = g^0(x) + g^1(x;t) + g^2(x;t) + \cdots \tag{9}$$

where $t = (t_1, \ldots, t_n)$ are indeterminants and $g^r(x; t)$ is the rth order term. Replacing t by dx,

$$\mathscr{I}_X^r = \left(g^s(x; dx)\,|\,0 \le s \le r,\ g \in \mathscr{I}_X\right) \tag{10}$$

in coordinates.

The geometry of the contact cones is controlled by the "derivative relation":

$$\frac{\partial g^r(x;t)}{\partial x_i} = \frac{\partial g^{r+1}(x;t)}{\partial t_i}, \qquad i = 1, \ldots, n, \tag{11}$$

in coordinates (see [**5**, p. 43] for a coordinate free version).

12. Proposition. *If $\mathscr{I}_X^r = \mathscr{I}_X^{r+1}$ for some r then $\mathscr{I}_X^r = \mathscr{I}_X^\infty$, so $\pi C^r \subset X$ is ruled (by line segments).*

Proof. Let $g_1, \ldots, g_m$ generate $\mathscr{I}_X$ over $\mathscr{U}$. By hypothesis there exists a local relation

$$g_i^{r+1}(x;t) = \sum_{j=1}^{m} \sum_{s=0}^{r} a_{ijs}(x;t) g_j^s(x;t), \qquad i = 1, \ldots, m.$$

Differentiate with respect to x_λ and apply the derivative relation (11):

$$\frac{\partial g_i^{r+2}}{\partial t_\lambda} \equiv \sum_{j=1}^{m} \sum_{s=0}^{r} a_{ijs} \frac{\partial g_j^{s+1}}{\partial t_\lambda} \quad \operatorname{mod} \mathscr{I}_X^r,$$

$\lambda = 1, \ldots, n$. Multiply by t_λ, sum over λ, and apply the Euler relation:

$$(r+2) g^{r+2} \equiv 0 \quad \operatorname{mod} \mathscr{I}_X^{r+1}.$$

Continue inductively. Since πC^∞ is obviously ruled we are done. (Of course C^r may be empty.)

Example. If $X \subset \mathbf{P}^n$ is algebraic of degree d then $\mathscr{I}_X^d = \mathscr{I}_X^\infty$. A partial converse is Theorem (17).

The tangent cone $TY \subset TJ\,|_Y$ of a subscheme $Y \subset J$ is the locus of tangent vectors annihilating the ideal of Y. In particular

$$TC_p^r = (\ker \pi_*) \cap TC^r\,|_{C_p^r}$$

where $C_p^r = \pi^{-1}(p) \cap C^r$ and $\pi_*\colon TJ \to \pi^* T\mathbf{P}$ is the differential.

$\pi C_p^r \subset \mathscr{U}$ is a cone with vertex at p. Identify $T_{p,l}C_p^r$ with the corresponding plane in $\mathbf{P}^n$ tangent to πC_p^r along l. In local coordinates (9) $T_{p,l}C_p^r$ is the plane through $x(p)$ cut out by the hyperplanes

$$\sum \frac{\partial g^s(x(p);t)}{\partial t_i}(x_i - x(p)) = 0, \qquad s = 1, \ldots, r,\ g \in \mathscr{I}_X.$$

Over a dense open subset $\mathscr{W} \subset C^r$ the fibers $T_{p,l}C_p^r$ will have locally constant dimension. We shall say that $T_{p,l}C_p^r$ is locally constant along l if $\lambda^{-1}(l) \cap \mathscr{W}$ is nonempty and $T_{p,l}C_p^r$ is locally constant as a plane in $\mathbf{P}^n$ along $\lambda^{-1}(l) \cap \mathscr{W}$.

It is easy to write down the condition for this to happen, using the coordinates of (9) (for a coordinate-free method, see [**1**, p. 10] "second fundamental form"). Let $(\bar{x}, \bar{t})$ represent $(p, l) \in \lambda^{-1}(l) \cap \mathscr{W}$. $\lambda^{-1}(l)$ is locally parametrized by $s \mapsto (\bar{x} + s\bar{t}, \bar{t})$ for s near 0. Regarding $T_{p,l}C_p^r$ as a subspace of $\mathbf{C}^{n+1}$ we have a vector bundle $T_{\cdot,l}C_\cdot^r$ over $\pi^{-1}(l) \cap \mathscr{W}$. Let $v(s) = \sum a_i(s)\partial/\partial t_i$ be a local holomorphic section, so that

$$0 \equiv \sum_i a_i(s)\frac{\partial g^\nu}{\partial t_i}(\bar{x} + s\bar{t}, \bar{t}), \quad \text{for all } g \in \mathscr{I}_X,\ \nu = 1, \ldots, r,$$

identically in s. $T_{p,l}C_p^r$ is locally constant along l iff for all such sections v the derivative

$$0 \equiv \sum_i a_i'(s)\frac{\partial g^\nu}{\partial t_i}(\bar{x} + s\bar{t}, \bar{t}), \qquad \nu = 1, \ldots, r,\ g \in \mathscr{I}_X,$$

also vanishes identically.

13. THEOREM. *Fix* $r \geq 1$. *Suppose*

$$Z \subset \left\{(p, l) \in C^r \mid T_{p,l}C_p^{r-1} = T_{p,l}C_p^r\right\}$$

is a nonempty subscheme, and πZ *contains an irreducible component of* πC^{r-1} *as a subscheme. Then*

(i) $Z \subset C^\infty$,

(ii) $T_{p,l}C_p^{r-1}$ *is locally constant along the rulings* l *for generic* $(p, l) \in Z$.

Proof. We work in the coordinates (9). Let $(\bar{x}, \bar{t}) = (p, l) \in Z$, $v = \sum a_i\partial/\partial x_i + \sum b_i\partial/\partial t_i$. Then $v \in T_{p,l}C^{r-1}$ iff for all $g \in \mathscr{I}_{p,X}$, $\nu = 0, \ldots, r-1$,

$$0 = dg^\nu(v) = \sum_i a_i\frac{\partial g^\nu}{\partial x_i} + \sum_i b_i\frac{\partial g^\nu}{\partial t_i} = \sum_i a_i\frac{\partial g^{\nu+1}}{\partial t_i} + \sum_i b_i\frac{\partial g^\nu}{\partial t_i}.$$

By the Euler relation, $\nu g^\nu(\bar{x}, \bar{t}) = \sum_i \bar{t}_i\partial g^\nu/\partial t_i$. Since $(p, l) \in C^r$, $T_{p,l}C^{r-1}$ contains $w_x = \sum \bar{t}_i\partial/\partial x_i$ (11).

Since πZ contains a component of πC^{r-1}, and $Z \subset C^r \subset C^{r-1}$, it follows that at a generic point $(p, l) \in Z$ the differential π_*: $T_{p,l}C^r \to \pi_* T_{p,l}C^{r-1}$ is surjective. Its kernel is $T_{p,l}C_p^r$. But $T_{p,l}C_p^r = T_{p,l}C_p^{r-1}$, so $T_{p,l}C^r = T_{p,l}C^{r-1}$. In particular $w_x \in TC^r$, so $0 = \sum \bar{t}_i\partial g^r/\partial x_i = \sum \bar{t}_i\partial g^{r+1}/\partial t_i = (r+1)g^{r+1}(\bar{x}, \bar{t})$. Hence $Z \subset C^{r+1}$.

Now let $v_t = \sum b_i \partial/\partial t_i \in T_{p,l}C_p^{r-1}$ be any vector and set $v_x = \sum b_i \partial/\partial x_i$. Then $v_t \in T_{p,l}C_p^r$, hence for all $g \in \mathscr{I}_{p,X}$, $0 = \sum b_i \partial g^\nu/\partial t_i$, $\nu = 1, \ldots, r$. Thus $v_x \in T_{p,l}C^{r-1}$. But $T_{p,l}C^{r-1} = T_{p,l}C^r$, so $v_x \in T_{p,l}C^r$, thus $v_t \in T_{p,l}C_p^{r+1}$. Therefore $T_{p,l}C_p^r = T_{p,l}C_p^{r+1}$ and (i) follows by induction.

As for (ii), if $s \mapsto (\bar{x} + s\bar{t}, \bar{t})$ is a local parametrization of $\lambda^{-1}(l)$ and $\sum a_i(s)\partial/\partial t_i$ is a local holomorphic section of $T_{\cdot,l}C_\cdot^{r-1}$ over $\lambda^{-1}(l)$ then

$$0 \equiv \sum a_i(s) \frac{\partial g^\nu}{\partial t_i}(\bar{x} + s\bar{t}, \bar{t}), \quad \text{hence}$$

$$0 \equiv \sum_i a_i' \frac{\partial g^\nu}{\partial t_i} + \sum_{ij} a_i \bar{t}_j \frac{\partial^2 g^\nu}{\partial t_i \partial x_j}, \qquad \nu = 1, \ldots, r-1,$$

but $\partial^2 g^\nu/\partial t_i \partial x_j = \partial^2 g^{\nu+1}/\partial t_i \partial t_j$, so the second term vanishes by the Euler relation since $T_{p,l}C_p^{r-1} = T_{p,l}C_p^r$.

REMARK. If X is ruled then the hypotheses of (13) are satisfied for some r.

EXAMPLE. *Fundamental Forms.* (See [**4**, p. 373].) In affine coordinates, the rth osculating space $T_p^r X \subset \mathbf{P}^n$ is the span of p and the derivatives $\sigma'(p), \ldots, \sigma^{(r)}(p)$ of all open curves $\sigma \subset X$ through p. Let $p \mapsto \gamma^r(p) = T_p^r X$ be the associated rth order Gauss map. There is a natural way of representing its derivative at a generic point p by an element

$$d\gamma^r(p) \in H^0(\mathbf{P}T_pX, \mathcal{O}(r+1)) \otimes N_p(T_p^r X)$$

where $N(T_p^r X) = T_p\mathbf{P}^n / T_p(T_p^r X)$ is the normal space. $d\gamma^r(p)$ is the $r + 1$st fundamental form of X at p.

Let $v = \sum a_{i,\sigma}\sigma^{(i)}$ be any local section of the associated bundle $T^r X$ (with fiber $(T^r X)_q = T_q^r X$) defined near p. Then

$$v'(p) \equiv \sum a_{r,\sigma}\sigma^{(r+1)} \quad \operatorname{mod} T_p(T_p^r X)$$

in coordinates. So define $d\gamma^r$ by

$$\left[d\gamma^r\left(\sigma'(p)^{\otimes r+1}\right)\right] \lrcorner\, dg = (g \circ \sigma)^{(r+1)}(p), \quad \text{for all } g \in \mathscr{I}_{T_p^r X, p}.$$

(This does not depend on any choices.)

The associated linear system

$$L^{r+1} = \left\{ d\gamma^r \lrcorner\, \theta \mid \theta \in N_p^*(T_p^r X) \right\} \subset H^0(\mathbf{P}T_pX, \mathcal{O}(r+1))$$

is contained in the ideal of C_p^{r+1} (viewed as a subvariety of $\mathbf{P}T_pX$). (Since p is a generic point we may represent X as a graph $y_j = f_j(x)$, $j = 1, \ldots, k$, $x = (x_1, \ldots, x_m)$ in affine coordinates near p. If $g = \sum a_j y_j$ vanishes on

$T_p^r X$ then $d\gamma^r \lrcorner dg = \Sigma a_j f_j^{r+1}(x(p); dx)$. For $r \geq 1$ this is the $r+1$st order part of an element, $\Sigma a_j(f_j(x) - y_j)$, of $\mathscr{I}_X$). Geometrically the reason is that, if $(p, l) \in C_p^{r+1}$ then choose a curve $\sigma \subset X$ through p which meets l through order $r+1$. $\sigma'(p), \ldots, \sigma^{(r+1)}(p)$ lie along $l \subset T_p^1 X \subset T_p^r X$, so if g vanishes on $T_p^r X$ then $(g \circ \sigma)^{(r+1)}(p) = 0$.

At a generic p, L^2 generates the ideal of C_p^2 in $\mathbf{P}T_p X$, but this is not in general true of the higher L^r's. For example, if X is a hypersurface, not a hyperplane, then $T_p^2 X = \mathbf{P}^n$ so $L^3 = \{0\}$. But $\mathscr{I}_X^3 \neq \mathscr{I}_X^2$ unless X is ruled (12). A less trivial example is the following, due to Mark Green:

EXAMPLE. (Green [3].) Consider the surface $X \subset \mathbf{P}^4$ parametrized by

$$p(s,t) = (t, s^2t^2, s^6t^3, s^{12}t^4)$$

in affine coordinates. Then

$$\frac{\partial^2 p}{\partial t^2} = \frac{s}{t^2}\frac{\partial p}{\partial s},$$

so every $Q \in L^2$ vanishes on $(\partial p/\partial t)^{\otimes 2}$. In fact

$$L^2 = \operatorname{span}\{ds^2, ds \cdot dt\}.$$

By a result of Griffiths and Harris [4, p. 373], the Jacobian system of L^{r+1} is contained in L^r, $r = 2, 3, \ldots$. It follows that

$$L^r \equiv 0 \mod \{ds^r, ds^{r-1} \cdot dt\}, \qquad r = 2, 3, \ldots.$$

Griffiths and Harris conjectured that any surface with such L^r's ought to be ruled [4, p. 377]. But X is not ruled. In particular, by (12), the L^r's cannot generate the ideal of C_p^r if $r \geq 3$ at a generic p.

EXAMPLE. [4, p. 387]. The second fundamental form represents the derivative of the Gauss map $\gamma = \gamma^1$. $\ker d\gamma_p$ (projectivized) is the common singular locus in $\mathbf{P}T_p X$ of all the quadrics in L^2.

Conversely if, at a generic $p \in X$, all the quadrics in L^2 have a common singular locus Z_p, then the hypotheses of (13) are satisfied with $r = 2$: take $Z = \bigcup Z_p$. Then X is ruled by the planes πZ_p, which are the fibers of γ (locally).

Examples of such X are cones and developable varieties. Recently F. Zak [7, p. 540 see [2] for a proof] proved that if X is a *smooth* algebraic variety of degree ≥ 2 then the fibers of γ are finite (zero dimensional).

14. COROLLARY. *Let $X \subset \mathscr{U}$ be an irreducible variety. If X is not ruled then over a generic $p \in X$ the dimensions* $\dim T_{p,l} C_p^r$, $r = 0, 1, 2, \ldots,$ *are strictly decreasing to zero for all* $(p, l) \in \pi^{-1}(p)$.

Proof. Let $Z^r = \{(p,l) \in C^r \mid T_{p,l}C_p^r = T_{p,l}C_p^{r-1}\}$. Z^r is an analytic variety. Since π is proper, πZ^r is an analytic subvariety of X. If X is the countable union $X = \bigcup_{r=1}^{\infty} \pi Z^r$ then one of the Z^r's, say Z^r, must map dominantly to X. Restricting to an open subset one may assume Z^r is surjective. Then $X = \pi Z^r = \pi C^{r-1}$. Apply (13).

The following answers a question in Griffiths and Harris [**4**, p. 450].

15. COROLLARY. *Let $X \subset \mathscr{U}$ be an irreducible hypersurface, $p \in X$ a generic point. Then for each $r = 1, \ldots, n = \dim \mathbf{P}^n$, if C_p^s is* not *a smooth complete intersection of type $(1, 2, \ldots, s)$ in $\mathbf{P}(T_p\mathbf{P}^n)$ for* all *$s = 1, \ldots, r$ (if $s = n$ this means C_p^n is not empty) then X is ruled, and C_p^r is singular or has codimension $< r$ in $\mathbf{P}(T_p\mathbf{P}^n)$.*

Proof. Let g be a local generator for $\mathscr{I}_X$. Then $C_p^r \cong \{t \mid g^1(x(p); t) = \cdots = g^r(x(p); t) = 0\}$ in $\mathbf{P}(T_p\mathbf{P}^n)$. Let $1 \leq r \leq n$ be the least integer such that C_p^r is not a smooth complete intersection of type $(1, \ldots, r)$. Then C_p^r is singular or $C_p^r = C_p^{r-1}$. Since C_p^r has codimension at most 1 in C_p^{r-1} it follows that for some $(p,l) \in C_p^r$, $T_{p,l}C_p^r = T_{p,l}C_p^{r-1}$. Apply (14).

If X is ruled then say the rulings are in general position if $(\operatorname{span} C_p^{\infty}) = \mathbf{P}T_pX$ at a generic $p \in X$.

16. LEMMA. *Let $\mathscr{U} \subset \mathbf{P}^n$ be an open set, $X \subset \mathscr{U}$ an irreducible, ruled variety whose rulings are in general position. Then X is piecewise linearly connected i.e. given $p, q \in X$ there exists a finite sequence l_i, $i = 0, \ldots, m$, of line segments in X such that $p \in l_0$, $q \in l_m$ and l_i meets l_{i+1} for each i.*

Proof. Let $Y \subset X$ be the locus of points $p \in X$ such that C_p^{∞} spans T_pX. Y is a dense open subset. Let $\mathscr{U}' \subset \mathscr{U}$ be a convex open subset such that $\mathscr{U}' \cap X \subset Y$ is nonempty. Let $X' \subset \mathscr{U}'$ be an irreducible component of $\mathscr{U}' \cap X$, and let $C^{\infty\prime}$ be the ∞ contact cone of X' in $\pi^{-1}\mathscr{U}'$. Let $p' \in X'$.

Since $\pi\colon \pi^{-1}\mathscr{U}' \to \mathscr{U}'$ is a proper map one can define a sequence of analytic subvarieties of X by

$$C_{p'}^{\infty\prime}(1) = \pi C_{p'}^{\infty\prime}, \qquad C_{p'}^{\infty\prime}(k+1) = \pi\pi^{-1}C_{p'}^{\infty\prime}(k), \quad k = 1, 2, 3, \ldots.$$

Clearly $C_{p'}^{\infty\prime}(k+1)$ consists of all points in X' connected to points in $C_{p'}^{\infty\prime}(k)$ by line segments in X'. Eventually the dimension of $C_{p'}^{\infty\prime}(k)$ will reach a maximum. Then a generic smooth point q' of $C_{p'}^{\infty\prime}(k)$ is also a smooth point of $C_{p'}^{\infty\prime}(k+1)$. But $C_{p'}^{\infty\prime}(k+1)$ contains all the lines in X'

through q'. Since the rulings are in general position, $\dim C_{p'}^{\infty\prime}(k+1) = \dim X'$. Since X' is irreducible, $C_{p'}^{\infty\prime}(k+1) = X'$.

Now replace X' by X, $\mathcal{U}'$ by $\mathcal{U}$. Going through the same construction, construct $C_{p'}^{\infty\prime}(k+1)$. Then $C_{p'}^{\infty\prime}(k+1) \subset C_{p'}^{\infty}(k+1)$; since X is irreducible, $C_{p'}^{\infty}(k+1) = X$. So every point $p \in X$ can be connected to p' by at most $k+1$ line segments, hence any two points can be connected to each other by at most $2k+2$ line segments.

17. THEOREM. *Let $X \subset \mathcal{U} \subset \mathbf{P}^n$ be an irreducible analytic hypersurface, $p \in X$, $g \in \mathcal{I}_{p,X}$ a generator. Assume*

(i) $\mathcal{I}_X^d = \mathcal{I}_X^{d+1}$ *for some $d \leq n-1$.*

(ii) $g^1(x(p); t), \ldots, g^d(x(p); t)$ *are a regular sequence of polynomials*

(iii) C_p^d *is reduced.*

Then X is algebraic—there is a polynomial $f(x_1, \ldots, x_n)$ of degree $\leq d$ (in affine coordinates) vanishing on X.

Proof. Recall some consequences of (i), (ii), (iii):

18. $C_p^d = \{t \in \mathbf{P}T_p\mathbf{P}^n \mid g^1(x(p); t) = \cdots = g^d(x(p); d) = 0\}$, $g \in \mathcal{I}_{X,p}$ a generator, is nonempty (since $d \leq n-1$), smooth on a dense open subset (by (iii)), and $C_p^d = C_p^\infty$ (by (12)).

19. Every homogeneous polynomial vanishing identically on C_p^d is in the homogeneous ideal generated by $g^1, \ldots, g^d$.

20. Every homogeneous relation $\Sigma_{r=1}^d a^r g^r = 0$ is of the form $a^r = \Sigma_s Q_{rs} g^s$ where Q_{rs} is an antisymmetric matrix of polynomials (19, 20 follow from (ii), (iii); use a Koszul complex).

21. If $a^i(t)$, $i = 1, \ldots, d$, are homogeneous polynomials satisfying the identity

$$0 \equiv \sum_{i=1}^{d} a^i \frac{\partial g^i}{\partial t_\lambda} \bmod g^1, \ldots, g^d, \quad \text{for all } \lambda = 1, \ldots, n,$$

then $a^i \equiv 0 \bmod g^1, \ldots, g^d$, for all i.

Proof of 21. If $\Sigma_{i=1}^d a^i dg^i \equiv 0 \bmod g^1, \ldots, g^d$ then $0 \equiv \Sigma_{i=1}^d a^i dg^i \wedge dg^1 \wedge \cdots \wedge \widehat{dg^j} \wedge \cdots \wedge dg^d \equiv \pm a^j dg^1 \wedge \cdots \wedge dg^d \bmod g^1, \ldots, g^d$. By 18, $dg^1 \wedge \cdots \wedge dg^d \neq 0$ on a dense open subset of C_p^d, so $a^j \equiv 0$ on C_p^d. Apply 19.

22. The points of C_p^d are in general position in the hyperplane $g^1(t) = 0$ (by 19, since $\deg g^i = i$).

Proof of theorem. We may assume g generates $\mathscr{I}_X$ on $\mathscr{U}$. Taken together (ii), (iii) are open conditions—assume they are satisfied everywhere on $\mathscr{U}$. We shall work in the ring $\mathscr{O}_{\mathscr{U}}[t]$ of polynomials in t with holomorphic coefficients. All polynomials are homogeneous. Degree means degree as a polynomial in t.

Set $e = d + 1$. As in the proof of (12) one has local relations on $\pi^{-1}\mathscr{U}$:

$$0 \equiv \sum_{i=0}^{e} a^{e-i}(x;t)g^i(x;t), \tag{23}$$

$$0 \equiv \sum_{i=0}^{e+1} b^{e+1-i}(x;t)g^i(x;t). \tag{24}$$

$\deg a^i = \deg b^i = i$ for all i, and $a^0, b^0 \neq 0$. The idea is this: if $f(x) = g(x)h(x)$ were a polynomial of degree $< e$ (in x) vanishing on X then, expanding as a power series, one has $0 \equiv f^e = \sum h^{e-i}g^i$. So one can hope to recover f from (23).

One may replace b^i by $b^i(a^0/b^0) + a^{i-1}(a^1/a^0 - b^1/b^0)$, $i = 0, \ldots, e+1$, (set $a^{-1} = 0$). Then

$$a^0 = b^0, \quad a^1 = b^1.$$

Differentiate (23) with respect to x_λ and (24) with respect to t_λ:

$$0 \equiv \sum_{i=0}^{e} \frac{\partial a^{e-i}}{\partial x_\lambda} g^i + \sum_{i=0}^{i} a^{e-i}\frac{\partial g^i}{\partial x_\lambda} \equiv \sum_{i=0}^{e} \frac{\partial a^{e-i}}{\partial x_\lambda} g^i + \sum_{i=1}^{e+1} a^{e+1-i}\frac{\partial g^i}{\partial t_\lambda}$$

$$0 \equiv \sum_{i=0}^{e} \frac{\partial b^{e+1-i}}{\partial t_\lambda} g^i + \sum_{i=1}^{e+1} b^{e+1-i}\frac{\partial g^i}{\partial t_\lambda}$$

for all $\lambda = 1, \ldots, n$, since $\deg g^0 = \deg b^0 = 0$. Subtract:

$$0 \equiv \sum_{i=0}^{e}\left(\frac{\partial a^{e-i}}{\partial x_\lambda} - \frac{\partial b^{e+1-i}}{\partial t_\lambda}\right)g^i + \sum_{i=1}^{e-1}\left(a^{e+1-i} - b^{e+1-i}\right)\frac{\partial g^i}{\partial t_\lambda}. \tag{25}$$

Since $g^1, \ldots, g^{e-1}$ is a regular sequence it follows (21) that $a^{e+1-i} \equiv b^{e+1-i} \bmod g^0, \ldots, g^{e-1}$ for all $i = 1, \ldots, e+1$. Define $a^{e+1} = b^{e+1}$. Write

$$b^{e+1-i} = a^{e+1-i} + \sum_{j=0}^{e+1} P_{ij}g^j, \qquad i = 0, \ldots, e+1, \tag{26}$$

where P_{ij} has degree $e+1-i-j$ when $0 \le i, j, i+j \le e+1$ and vanishes for i, j outside this range. Set

$$A_{ij} = \sum_{r=0}^{j+1} P_{i+1+r,j-r} - \sum_{r=0}^{i+1} P_{j+1+r,i-r}.$$

Then $A_{ij} = -A_{ji}$, $\deg A_{ij} = e - i - j$ for all i, j, and

$$A_{i-1,j} - A_{i,j-1} = P_{ij} + P_{ji} \quad \text{for all } i, j = 0, \ldots, e+1.$$

Define B_{ij} by

$$A_{i-1,j} + A_{i,j-1} - 2B_{ij} = P_{ij} - P_{ji}, \qquad i, j = 0, \ldots, e+1.$$

Then $B_{ij} = -B_{ji}$, $\deg B_{ij} = e + 1 - i - j$ for all i, j. Set

$$\bar{a}^{e-i} = a^{e-i} + \sum_{j=0}^{e+1} A_{ij} g^j, \qquad i = -1, \ldots, e,$$

$$\bar{b}^{e+1-i} = b^{e+1-i} + \sum_{j=0}^{e+1} B_{ij} g^j, \qquad i = 0, \ldots, e+1.$$

Since A_{ij}, B_{ij} are antisymmetric the $\bar{a}^i$, $\bar{b}^i$ satisfy (23, 24). Moreover they have the right degree, and $\bar{a}^0 = a^0 + A_{eo} g^0$ does not vanish near the locus ($g^0 = 0$). Finally, one may check using (26), that

$$\bar{a}^i = \bar{b}^i, \qquad i = 0, \ldots, e+1.$$

Replace the a, b's by the $\bar{a}$, $\bar{b}$'s in (23, 24). Then (25) becomes

$$0 = \sum_{i=0}^{e} \left(\frac{\partial a^{e-i}}{\partial x_\lambda} - \frac{\partial a^{e+1-i}}{\partial t_\lambda} \right) g^i, \qquad \lambda = 1, \ldots, n.$$

Subtract $(\partial a^0/\partial x_\lambda - \partial a^1/\partial t_\lambda)/a^0$ times eq. (23) from this and get

$$0 \equiv \sum_{i=0}^{e-1} \left\{ \frac{\partial a^{e-1}}{\partial x_\lambda} - \frac{\partial a^{e+1-i}}{\partial t_\lambda} - \frac{a^{e-i}}{a^0} \left(\frac{\partial a^0}{\partial x_\lambda} - \frac{\partial a^1}{\partial t_\lambda} \right) \right\} g^i$$

a homogeneous relation among the g^i's. Reducing mod g^0 one can apply (20), then by adding an appropriate multiple of g^0 one has

$$\text{(27)} \qquad \frac{\partial a^{e-i}}{\partial x_\lambda} = \frac{\partial a^{e+1-i}}{\partial t_\lambda} + \frac{a^{e-i}}{a^0} \left(\frac{\partial a^0}{\partial x_\lambda} - \frac{\partial a^1}{\partial t_\lambda} \right) + \sum_{j=0}^{e-1} Q_{ij}^\lambda g^j$$

$i = 0, \ldots, e-1$, $\lambda = 1, \ldots, n$, $\deg Q_{ij}^\lambda = e - i - j$ where Q_{ij}^λ is an antisymmetric matrix of polynomials.

Multiplying (23) by $1/a^0$ we may assume $a^0 \equiv 1$. Then for $i = e - 1$ (27) becomes

$$\text{(28)} \qquad \frac{\partial a^1}{\partial x_\lambda} = \frac{\partial a^2}{\partial t_\lambda} - a \frac{\partial a^1}{\partial t_\lambda} + Q_{e-1,0}^\lambda g^0 + Q_{e-1,1}^\lambda g^1.$$

Consider the form

$$\Phi = \sum_\mu \frac{\partial a^1}{\partial t_\mu} dx_\mu, \qquad d\Phi = \sum_{\lambda\mu} \frac{\partial^2 a^1}{\partial x_\lambda dt_\mu} dx_\lambda \wedge dx_\mu.$$

Applying (28),

$$d\Phi = g^0 \sum_{\lambda\mu} \frac{\partial Q^\lambda_{e-1,0}}{\partial t_\mu} dx_\lambda \wedge dx_\mu + \sum_{\lambda\mu} Q^\lambda_{e-1,1} \frac{\partial g^1}{\partial t_\mu} dx_\lambda \wedge dx_\mu.$$

Since $\Sigma(\partial g^1/\partial t_\mu)\, dx_\mu = dg^0$, Φ is closed along ($g^0 = 0$). So locally along ($g^0 = 0$) one can solve the equation $d \log h(x) = \Phi$. Multiply the a^i's by $h(x)$. Then

$$\sum_\lambda \frac{\partial a^0}{\partial x_\lambda} dx_\lambda = \sum_\lambda \frac{\partial a^1}{\partial t_\lambda} dx_\lambda \mod g^0,\ dg^0. \tag{29}$$

Let $\bar{x} \in X$. Define a polynomial $f(x)$ of degree $\leq e - 1$ by

$$f(x) = \sum_{i=0}^{e-1} \sum_{j=0}^{i} a^{i-j}(\bar{x}, x - \bar{x}) g^j(\bar{x}, x - \bar{x}). \tag{30}$$

It remains to show that f vanishes on X. Clearly

$$\begin{aligned} (31)\quad & 0 = f^0(\bar{x}) = a^0(\bar{x}) g^0(\bar{x}), \\ & f^r(\bar{x}; t) = \sum_{j=0}^{r} a^{r-j}(\bar{x}; t) g^j(\bar{x}; t), \qquad r = 0, \dots, e-1, \quad \text{and} \\ & f^e(x; t) \equiv 0. \end{aligned}$$

Define functions

$$f^r_\lambda(x; t) = \sum_{j=0}^{r-1} \frac{\partial a^{r-j}}{\partial t_\lambda}(x; t) g^j(x; t) + \sum_{j=1}^{r} a^{r-j}(x; t) \frac{\partial g^j}{\partial t_\lambda}(x; t),$$

$$r = 1, \dots, e,\ \lambda = 1, \dots, n.$$

In particular $f^e_\lambda \equiv 0$ by (23), and f^r_λ is homogeneous of degree $r - 1$. Differentiate:

$$\frac{\partial f^r_\mu}{\partial x_\lambda} - \frac{\partial f^{r+1}_\mu}{\partial t_\lambda} \equiv \sum_{j=1}^{r} \left(\frac{\partial a^{r-j}}{\partial x_\lambda} - \frac{\partial a^{r+1-j}}{\partial t_\lambda} \right) \frac{\partial g^j}{\partial t_\mu} \mod g^0, \dots, g^{e-1}.$$

Then substituting in (27, 29) this becomes

$$\sum_\lambda \left(\frac{\partial f^r_\mu}{\partial x_\lambda} - \frac{\partial f^{r+1}_\mu}{\partial t_\lambda} \right) dx_\lambda \equiv 0 \mod g^0, \dots, g^{e-1}, dg^0. \tag{32}$$

Let $l(s) = (x + st, t)$ be a line in C^{e-1}. Then $g^0, \ldots, g^{e-1}$ vanish on l. So by (32)

$$(33) \qquad \frac{d}{ds}\bigg|_0 f_\mu^r(x + st, t) = \sum_\lambda t_\lambda \frac{\partial f_\mu^r}{\partial x_\lambda}(x, t) = r f_\mu^{r+1}(x, t),$$

along l. Now it is easy to show that functions f_μ^r, homogeneous of degree $r - 1$ in t, satisfying (33) and the condition $f_\mu^e \equiv 0$ are uniquely determined along a line by their values at a single point.

On the other hand the functions $(\partial f^r/\partial t_\mu)(x; t)$ derived from the polynomial (30) also satisfy these relations, moreover they agree with the f_μ^r's at any point $(\bar{x}, \bar{t})$ lying on a line in C^{e-1} through $\bar{x}$ (differentiate (31) at $\bar{x}$). By (22) the rulings of X are in general position, so by (12), (16) $f_\mu^r = \partial f^r/\partial t_\mu$ everywhere on C^{e-1}.

In particular $\partial f^1/\partial t_\lambda = f_\lambda^1$ on C^{e-1}. But $df^0 = \Sigma(\partial f^1/\partial t_\lambda)\, dx_\lambda$ and

$$\sum f_\lambda^1\, dx_\lambda = \sum \left(\frac{\partial a^1}{\partial t_\lambda} g^0 + a^0 \frac{\partial g^1}{\partial t_\lambda} \right) dx_\lambda \equiv 0 \mod g^0, dg^0.$$

Hence f^0 is constant $= f^0(\bar{x}) = 0$ on C^{e-1}. Since $\pi C^{e-1} = X$ (18), f vanishes on X.

Example. If C_p^r is not reduced then the conclusion of (17) may not hold.

Let $X \subset \mathbf{P}^3$ be the cylinder

$$X = \{(x_1, x_2, x_3) | g(x_1, x_2) = 0\}$$

in affine coordinates. X may not be algebraic (if g is not).

$$g^1(x; dx) = g_1\, dx_1 + g_2\, dx_2$$

$$g^2(x; dx) = \tfrac{1}{2}(g_{11}\, dx_1^2 + 2g_{12}\, dx_1\, dx_2 + g_{22}\, dx_2^2)$$

etc., where $g_i = \partial g/\partial x_i$. If $g^1(x(p); dx) \neq 0$ and $g^1(x(p); dx)$ does not divide $g^2(x(p); dx)$ then g^1, g^2 are a regular sequence generating any homogeneous cubic in dx_1, dx_2. In particular $g^3 \equiv 0 \mod g^1, g^2$. C_p^2 is supported on the point $[dx_1, dx_2, dx_3] = [0, 0, 1]$ but it is not reduced, since $\{ dx_1, dx_2,\} \not\subset \mathcal{I}_X^2$.

I would like to thank my teachers at UCLA, especially Mark Green, for their invaluable help and encouragement.

References

[1] A. Altman and S. Kleiman, *Introduction to Grothendieck Duality Theory*, Springer Lecture Notes 146, 1970.

[2] W. Fulton and R. Lazersfeld, *Connectivity and its applications in algebraic geometry*, (preprint), Brown University.

[3] Mark Green (Lecture given at UCLA, 1983).
[4] P. Griffiths and J. Harris, *Algebraic geometry and local differential geometry*, Ann. Scient. Éc. Norm. Sup. 4^e série, t. **12** (1979), 355–432.
[5] A. Grothedieck and J. Dieudonné, *Eléments de Géométrie Algébrique IV*, Publ. Math. I.H.E.S., **32** (1967).
[6] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Math., Springer-Verlag, 1977.
[7] F. Zak, *Projection of algebraic varieties*, Math. USSR Sbornik, **44** (1983), 535–544.

Received June 3, 1985 and in revised form October 20, 1985.

UNIVERSITY OF WASHINGTON
SEATTLE, WA 98195