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CLASSIFICATION OF THE STABLE HOMOTOPY TYPES OF STUNTED REAL PROJECTIVE SPACES

DONALD M. DAVIS AND MARK MAHOWALD

Let P_n^{n+k} denote the stunted real projective space RP^{n+k}/RP^{n-1}. In this paper we complete the classification of stable homotopy type of stunted real projective spaces begun by Feder, Gitler, and Mahowald in 1977.

Let $\phi(m,n)$ denote the number of integers j satisfying (a) $j \equiv 0$, 1, 2, or 4 mod 8 and (b) $m \le j \le n$ if $m \not\equiv 0\ (4)$, or $m < j \le n$ if $m \equiv 0\ (4)$. Let $\phi(n) = \phi(1,n)$, and let $\nu(n)$ denote the exponent of 2 in n.

THEOREM 1. *P_n^{n+k} and P_m^{m+k} have the same stable homotopy type ($\sim$) if and only if*
$\nu(n-m)$

$$\ge \begin{cases} \phi(k) & \text{if } k \le 8 \text{ or } \nu(n) \ge \phi(k)-1 \text{ or } \nu(n+k+1) \ge \phi(k)-1 \\ \max\{\phi(n,n+k)-1,\ \phi(-n-k-1,-n-1)-1\} & \text{otherwise.} \end{cases}$$

This max *will equal either $\phi(k)$ or $\phi(k)-1$. A mindless restatement of Theorem* 1 *is given by the following result.*

THEOREM 1′. *$P_m^{m+k} \sim P_n^{n+k}$ iff $\nu(n-m) \ge \phi(k) - \varepsilon(m,k)$, where $\varepsilon(m,k)$ is either* 0 *or* 1. *In the following cases $\varepsilon(m,k)$ is* 0; *otherwise it is* 1.

(i) $k \le 8$

(ii) $\nu(m) \ge \phi(k) - 1$

(iii) $\nu(m+k+1) \ge \phi(k) - 1$.

(iv) *The congruence classes of* m mod 4 *and of* k mod 8 *are such that a $*$ occurs in the following table.*

$m\equiv$ \ $k\equiv$	0	1	2	3	4	5	6	7
0								
1	*	*		*				*
2	*						*	*
3				*		*	*	*

Readers familiar with the spectrum $J(=\text{fibre}(bo \to \Sigma^4 bsp))$ will appreciate the following corollary.

COROLLARY 2. *Stunted real projective spaces have the same stable homotopy type if and only if their J-homology and J-cohomology groups are isomorphic as graded abelian groups (with dimension shift).*

This can be established by comparing Theorem 1 with the calculations in **[11]** of $J_*(P_n^{n+k})$. Here we use Whitehead's theorem (**[13]**) that $h_*(X) \approx h^{-*}(DX)$, where DX is the S-dual of X, to express $J_*(P_{-n-k-1}^{-n-1})$ as $J^{-*}(P_n^{n+k})$.

[7] had conjectured Theorem 1 and had established it in certain congruences. (Their table has a misprint when n is odd and $k \equiv 2$.) Our proof follows the approach of **[7]**.

That $\nu(n-m) \geq \phi(k)$ is sufficient for $P_m^{m+k} \sim P_n^{n+k}$ is clear from **[3]** since the spaces are then Thom complexes of J-equivalent vector bundles. That $\nu(n-m) \geq \phi(k)$ when $k \leq 8$ is necessary is easily deduced from the action of the Steenrod algebra. That $\nu(n-m) \geq \max$ is necessary was proved using Adams operations in **[9]** and **[7**; 2.1**]**. That $\nu(n-m) \geq \phi(k)$ when $\nu(n) \geq \phi(k)-1$ or $\nu(n+k+1) \geq \phi(k)-1$ is necessary was stated in **[7**; 1.1**]** and proved using Adams operations in **[8]**.

We will prove that

(i) $P_{8a}^{8a+8b+3} \sim P_{8a'}^{8a'+8b+3}$ if $\nu(8a) < 4b+1 \leq \nu(8a-8a')$, and

(ii) $P_{4a}^{4a+8b+7} \sim P_{4a'}^{4a'+8b+7}$ if $\max\{\nu(4a), \nu(4a+8b+8)\} < 4b+2 \leq \nu(4a-4a')$.

Of course the equivalence of (ii) also implies equivalences if cells are removed from the bottom or top. This, and an application of S-duality to (i), then handles all cases unresolved in **[7]**.

We begin with (i). Write $8a' = 8a + c \cdot 2^{4b+1}$ and let L be such that $\nu(L) \geq 4b+2$ and $L > 8a'+8b+4$. The multiple $(L-8a-8b-4)\xi_{8b+3}$ of the Hopf bundle is classified by the composite

$$P^{8b+3} \xrightarrow{\Delta} P^{8b+3} \times P^{8b+3} \xrightarrow{c\cdot 2^{4b+1}\xi \times d\xi} BO_{c\cdot 2^{4b+1}} \times BO_d \to BO_{L-8a-8b-4},$$

where $d = L - 8a' - 8b - 4$. Since $2^{4b+1}\xi_{8b+1}$ is trivial and $\pi_{8b+3}(BO) = 0$, the first factor of the middle map can be factored as

$$P^{8b+3} \xrightarrow{c} P_{8b+2}^{8b+3} = S^{8b+2} \vee S^{8b+3} \xrightarrow{c} S^{8b+2} \xrightarrow{\theta} BO_{c\cdot 2^{4b+1}}$$

with θ the nonzero element of $\pi_{8b+2}(BO_{c\cdot 2^{4b+1}})$. The induced map of Thom spaces is

$$P^{L-8a-1}_{L-8a-8b-4} \to \left(S^{c\cdot 2^{4b+1}} \cup_\beta e^{c\cdot 2^{4b+1}+8b+2}\right) \wedge P^{L-8a'-1}_{L-8a'-8b-4},$$

where β generates the image of J in the $(8b+1)$-stem. If the map is made skeletal, we obtain

$$(*) \quad P^{L-8a-1}_{L-8a-8b-4} \to \Sigma^{c\cdot 2^{4b+1}}\left(P^{L-8a'-1}_{L-8a'-8b-4} \cup_{\beta\vee\beta} \left(e^{L-8a'-2} \vee e^{L-8a'-1}\right)\right),$$

where the attaching map is

$$S^{L-8a'-3} \vee S^{L-8a'-2} \xrightarrow{\beta\vee\beta} S^{L-a'-8b-4} \vee S^{L-8a'-8b-3}$$
$$= P^{L-8a'-8b-3}_{L-8a'-8b-4} \hookrightarrow P^{L-8a'-1}_{L-8a'-8b-4}.$$

We will prove later

PROPOSITION 3. *If $\nu(4A+8b+4) < 4b+1$, then the composite*

$$S^{4A+8b+2} \xrightarrow{\beta} S^{4A+1} \hookrightarrow S^{4A} \vee S^{4A+1} = P^{4A+1}_{4A} \hookrightarrow P^{4A+8b+2}_{4A}$$

is null-homotopic, where β generates the image of J.

The hypotheses of (i) imply $\nu(L-8a') < 4b+1$, so that Proposition 3 implies that the $e^{L-8a'-1}$ of the target in $(*)$ splits off, giving us a map

$$P^{L-8a-1}_{L-8a-8b-4} \xrightarrow{g} \Sigma^{c\cdot 2^{4b+1}}\left(P^{L-8a'-1}_{L-8a'-8b-4} \cup_\beta e^{L-8a'-2}\right).$$

That this map is injective in $H^*(\ ;\mathbf{Z})$ on the projective space is clear for all except the top cell; that it is also true on the top cell follows from the argument of [7; 3.1]. This is proved by using the diagram

$$\begin{tikzcd}[column sep=small]
& \Sigma^{c\cdot 2^{4b+1}}\left(P_d^{L-8a'-2} \cup e^{L-8a'-2} \cup e^{L-8a'-1}\right) \arrow[r, "r"] \arrow[d, hook] & \Sigma^{c\cdot 2^{4b+1}}\left(P_d^{L-8a'-2} \cup e^{L-8a'-2}\right) \arrow[d, hook] \\
P^{L-8a-1}_{L-8a-8b-4} \arrow[r, "g'"] \arrow[rr, bend right, "g"'] & \Sigma^{c\cdot 2^{4b+1}}\left(P_d^{L-8a'-1} \cup e^{L-8a'-2} \cup e^{L-8a'-1}\right) \arrow[r, "\hat{r}"] & \Sigma^{c\cdot 2^{4b+1}}\left(P_d^{L-8a'-1} \cup e^{L-8a'-2}\right),
\end{tikzcd}$$

where r and $\hat{r}$ are retractions for the splitting, and r exists because the dimension of the target in Proposition 3 is one less than one might have naively required. g'^* sends both classes in $H^{L-8a-1}(\ ;\mathbf{Z})$ nontrivially, but $\hat{r}^*(\ )$ hits only the first.

The S-dual of g is

$$(**)\qquad \left(S^{8a'+1} \vee P_{8a'}^{8a'+8b+2}\right) \cup_{\beta \vee \alpha} e^{8a'+8b+3} \xrightarrow{f} \Sigma^{c \cdot 2^{4b+1}} P_{8a}^{8a+8b+3},$$

where β as before generates the image of J and α is the attaching map for the top cell of $P_{8a'}^{8a'+8b+3}$. Now

$$\pi_{8a'+1}\left(\Sigma^{c \cdot 2^{4b+1}} P_{8a}^{8a+8b+3}\right) \approx \pi_{8a+1}\left(P_{8a}^{8a+8b+3}\right) \approx \mathbf{Z}_2 \oplus \mathbf{Z}_2,$$

with one generator η on the bottom cell and the other of degree 1 on the next-to-bottom cell. By Proposition 3, β composed with the latter class is 0, while β composed with η is 0. Thus the restriction of the map f of $(**)$ to $S^{8a'+1}$ preceded by β is trivial. The existence of f says that

$$\left(f \mid S^{8a'+1} \vee P_{8a'}^{8a'+8b+2}\right) \circ (\beta \vee \alpha) = 0.$$

Subtracting, we deduce that $f \mid P_{8a'}^{8a'+8b+2}$ extends over $P_{8a'}^{8a'+8b+3}$ to give the desired map, which is certainly an equivalence below the top cell and at the odd primes, and has odd degree on the top cell by Sq^4.

The argument for (ii) is similar, but slightly more elaborate. Let $4a' = 4a + c \cdot 2^{4b+2}$ and $L > 4a' + 8b + 8$ with $\nu(L) \geq 4b + 3$. A map

$$P_{L-4a-8b-8}^{L-4a-1} \to \left(S^{c \cdot 2^{4b+2}} \cup_\beta e^{c \cdot 2^{4b+2}+8b+4}\right) \wedge P_{L-4a'-8b-8}^{L-4a'-1},$$

with β a generator of the image of J in the $(8b+3)$-stem, is constructed as before. This yields

$$P_{L-4a-8b-8}^{L-4a-1} \xrightarrow{\Psi} \Sigma^{c \cdot 2^{4b+2}}\left(P_{L-4a'-8b-8}^{L-4a'-1} \cup_{\beta \vee \beta \vee \beta} \left(e^{L-4a'-4} \vee CM \vee e^{L-4a'-1}\right)\right)$$

where CM is the cone on $M = S^{L-4a'-4} \cup_2 e^{L-4a'-3}$. We prove later

PROPOSITION 4. *If β generates the image of J in the $(8b+3)$-stem, the composite*

$$S^{4A+8b+4} \cup_2 e^{4A+8b+5} \xrightarrow{\beta \wedge 1} P_{4A+1}^{4A+2} \to S^{4A} \vee P_{4A+1}^{4A+2} = P_{4A}^{4A+2} \hookrightarrow P_{4A}^{4A+8b+6}$$

is null-homotopic.

PROPOSITION 5. *If $\nu(4A+8b+8) < 4b+2$, the composite*

$$S^{4A+8b+6} \xrightarrow{\beta} S^{4A+3} \hookrightarrow S^{4A} \vee P_{4A+1}^{4A+2} \vee S^{4A+3} = P_{4A}^{4A+3} \hookrightarrow P_{4A}^{4A+8b+6}$$

is null-homotopic.

These propositions imply that in the map Ψ above the top 2 parts of the 3-part wedge split off, and as before the resulting map is injective in $H^*(\ ;\mathbf{Z})$ on the projective space. The dual of the resulting map is analogous to $(**)$:

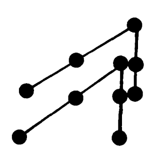
$$\left(S^{4a'+3} \vee P_{4a'}^{4a'+8b+6}\right) \cup_{\beta \vee \alpha} e^{4a'+8b+7} \xrightarrow{f} \Sigma^{c\cdot 2^{4b+1}} P_{4a}^{4a+8b+7}.$$

An argument similar to that applied to $(**)$, using Proposition 5, shows that $f \mid P_{4a'}^{4a'+8b+6}$ extends over $P_{4a'}^{4a'+8b+7}$, as desired. This extension induces an isomorphism in the top cohomology group because f^* does, while the extension

$$S^{4a'+3} \cup_\beta e^{4a'+8b+7} \to \Sigma^{c\cdot 2^{4b+2}} P_{4a}^{4a+8b+7} \quad \text{of } f \mid S^{4a'+3}$$

does not since by Proposition 5 the null-homotopy can be factored through $\Sigma^{c\cdot 2^{4b+2}} P_{4a}^{4a+8b+6}$.

Proof of Proposition 3. We let $N = 4A + 8b + 4$. If $A = 2\bar{A}$ is even, then the argument is straightforward: β has filtration $4b$ by [**11**; 8.2] and is annihilated by η. But $\pi_{8\bar{A}+8b+2}(P_{8\bar{A}}^{8\bar{A}+8b+2})$ in filtration $\geq 4b$ is acted on injectively by η. To see this, we use charts of [**12**] to get



$\leftarrow s = 4b$

$\uparrow$

$t - s = 8\bar{A} + 8b + 2$

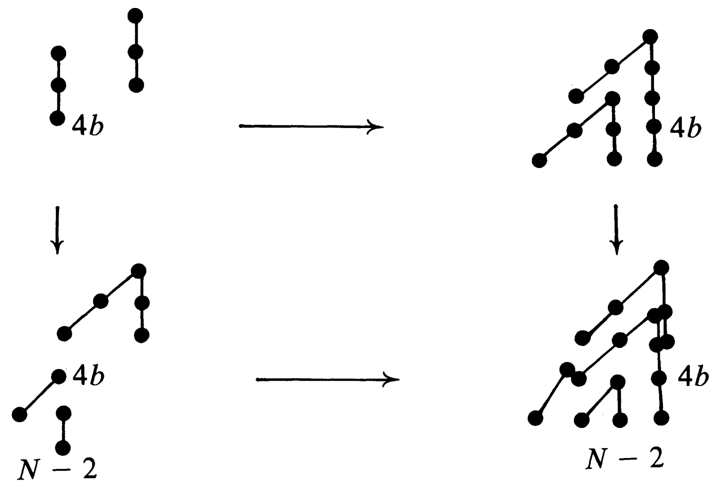
and follow into $J_*(\ )$ to see that none of these classes is hit by a differential.

Thus we may assume $N \equiv 0\ (8)$. In the diagram

$$\begin{array}{ccc} S^{N-8b-5} \cup_\eta e^{N-8b-3} & \overset{i_1}{\hookrightarrow} & P_{N-8b-5}^{N-2} \\ c\downarrow & & c\downarrow \\ S^{N-8b-3} & \overset{i_2}{\hookrightarrow} & P_{N-8b-4}^{N-2} \end{array}$$

the charts for $\mathrm{Ext}_A^{s,t}(H^*(\ ), \mathbf{Z}_2)$ in $s \geq 4b - 1$, $N - 3 \leq t - s \leq N$, are

(Cl)



Here the maps denoted c are collapse maps, and the horizontal inclusions maps are of cells near the bottom of the stunted projective spaces. The Ext charts are taken from the tables of [**12**]. That they are as claimed can be proved by the Adams periodicity theorem ([**1**]). The chart in the upper left is obtained from $\mathrm{Ext}_A^{*,*}(\mathbf{Z}_2, \mathbf{Z}_2) + \mathrm{Ext}_A^{*,*+2}(\mathbf{Z}_2, \mathbf{Z}_2)$ by inserting differentials between classes related by h_1.

In these charts, all homomorphisms in $(t-s, s) = (N-2, 4b)$ are isomorphisms. This can be deduced from [**12**] or by low-level calculation and periodicity. It is proved in [**11**; 8.2] that the generator of the image of J in $\pi_{N-2}(S^{N-8b-3})$ is represented by the element in $(N-2, 4b)$ in the chart in the lower left. Thus the composite of Proposition 3 is represented by the element in $(N-2, 4b)$ in the chart in the lower right. The following lemma implies that this class must be 0 in $\pi_*(P_{N-8b-4}^{N-2})$, proving Proposition 3. (The desired class cannot be nonzero in higher filtration since it is annihilated by η, but the higher-filtration classes are not.)

LEMMA 6. *The image-of-J generator of $\pi_{N-2}(S^{N-8b-3})$ coextends to a filtration-$4b$ element of $\pi_{N-2}(S^{N-8b-5} \cup_\eta e^{N-8b-3})$, and if $3 \le \nu(N) \le 4b$, then i_{1*} sends this class to 0 in $\pi_{N-2}(P_{N-8b-5}^{N-2})$.*

Proof. The first part is equivalent to the dual statement—that there is a map $S^{8b+1} \cup_\eta e^{8b+3} \to S^0$ of filtration $4b$ whose restriction to S^{8b+1} generates the image of J. By [**11**; 8.2] β_{8b+1} lifts to E_{4b}, the $4b$th stage of an Adams resolution of S^0. Since $\pi_{8b+2}(SO) = 0$, $\eta\beta_{8b+1} = 0$. The only way that the composite

$$S^{8b+2} \xrightarrow{\eta} S^{8b+1} \to E_{4b}$$

might fail to be 0 is if in the Adams spectral sequence (ASS) for $\pi_*(S^0)$ there was a nonzero differential $E_r^{s,s+8b+3} \to E_r^{s+r,s+r+8b+2}$ with $s < 4b < s + r$. Since the only possible target element ($s + r = 4b + 2$) survives to a homotopy class, the first half of the lemma is established.

We will prove

LEMMA 7. *If* $3 \le 4c - 3 \le \nu(N) \le 4c$, *then*

$$\pi_{N-2}\left(P_{N-8c-5}^{N-2}\right) \xrightarrow{\eta_*} \pi_{N-1}\left(P_{N-8c-5}^{N-2}\right)$$

is injective on elements of filtration $\ge 4c$, *or equivalently* (*by the previous* ASS *chart* (C1)) *the element in* $(N - 2, 4c)$ *is hit by a differential in the* ASS.

From this we now easily deduce the second half of Lemma 6, and hence Proposition 3. The hypothesis of Lemma 6 says that $\nu(N)$ can be written as in Lemma 7 for some positive $c \le b$. Let

$$P_{N-8c-5}^{N-2} \to P_{N-8b-5}^{N-2}$$

be the composite of $b - c$ of the filtration 4 maps considered in [**10**], [**4**], or [**5**]. As these induce an isomorphism in Ext near the upper edge, we use Lemma 7 to deduce that the element in $(N - 2, 4b)$ of ASS(P_{N-8b-5}^{N-2}) is hit by a differential, implying the second half of Lemma 6.

The proof of Lemma 7 has two main parts. The first is to prove that if $\nu(N) = 4c - \varepsilon$, then there is a commutative diagram as below with β a generator of the image of J and $i\tilde{\beta} \simeq *$. The case $\varepsilon = 2$ is representative,

$\underline{\varepsilon = 0}$

$$\begin{tikzcd} & P_{N-8c-3}^{N-8c-2} \arrow[r, "i"] \arrow[d] & P_{N-8c-3} \arrow[d] \\ S^{N-2} \arrow[ur, "\tilde{\beta}"] \arrow[r, "\beta"] & S^{N-8c-2} \arrow[r] & P_{N-8c-2} \end{tikzcd}$$

$\underline{\varepsilon = 1}$

$$\begin{tikzcd} & P_{N-8c-2}^{N-8c-1} \arrow[r, "i"] \arrow[d] & P_{N-8c-2} \arrow[d] \\ S^{N-2} \arrow[ur, "\tilde{\beta}"] \arrow[r, "\beta"] & S^{N-8c-1} \arrow[r] & P_{N-8c-1} \end{tikzcd}$$

$\underline{\varepsilon = 2}$

$$\begin{tikzcd} & P_{N-8c-1}^{N-8c+3} \arrow[r, "i"] \arrow[d] & P_{N-8c-1} \arrow[d] \\ S^{N-2} \arrow[ur, "\tilde{\beta}"] \arrow[r, "\beta"] & S^{N-8c+3} \arrow[r] & P_{N-8c+3} \end{tikzcd}$$

$\underline{\varepsilon = 0}$

$$\begin{tikzcd} & P_{N-8c+3}^{N-8c+5} \arrow[r, "i"] \arrow[d] & P_{N-8c+3} \arrow[d] \\ S^{N-2} \arrow[ur, "\tilde{\beta}"] \arrow[r, "\beta"] & S^{N-8c+5} \arrow[r] & P_{N-8c+5} \end{tikzcd}$$

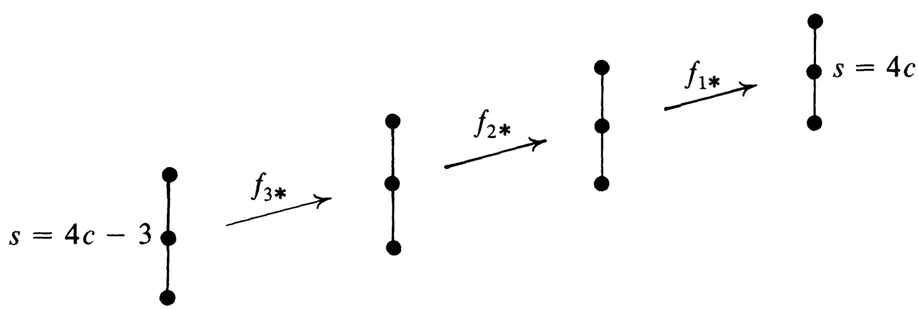
and so we prove it. We show that there exists a coextension $\tilde{\beta}$ so that the map i extends over the mapping cone of $\tilde{\beta}$, by constructing the dual map $P_M^{M+8c} \xrightarrow{d} S^M \cup C\Sigma^{-1}P_{M+8c-4}^{M+8c}$, where the attaching map $\Sigma^{-1}P_{M+8c-4}^{M+8c} \to S^M$ is an extension of a generator of the image of J, and $\nu(M) = 4c - 2$. The desired map d is the map of Thom spaces $T(M\xi_{8c}) \to T(\theta)$, where θ is a bundle over P_{8c-4}^{8c} which pulls back to $M\xi_{8c}$. The bundle θ exists since $\widetilde{KO}(P^{8c-5}) \approx \mathbf{Z}/2^{4c-2}$. The Thom space $T(\theta)$ is as stated by [**6**; 2.2] and [**2**; 10.1].

The second part of the proof of Lemma 7 utilizes the filtration 1 maps of [**11**; 7.15]. We illustrate again with the case $\varepsilon = 2$; the cases $\varepsilon = 0$ and 1 are easier, while $\varepsilon = 3$ is slightly harder because it requires the spaces C of [**11**; 7.15]. In the diagram

$$(***) \qquad \begin{tikzcd}
 & S^{N-2}\cup_2 e^{N-1} \arrow[r, "g"] \arrow[d, "\alpha"] & S^{N-2}\cup_2 e^{N-1} \arrow[r, "\alpha"] \arrow[d, "\alpha"'] & P_{N-8c-3}^{N-2} \arrow[d, "2"] \arrow[dl, "c"'] & \\
S^{N-2} \arrow[r, "b"] \arrow[ur, dashed, "\tilde{b}"] & P_{N-8c-1}^{N-2} \arrow[r, "f_3"] \arrow[d, "i_1"] & P_{N-8c-2}^{N-2} \arrow[r, "f_2"] \arrow[d] & P_{N-8c-3}^{N-2} \arrow[r, "f_1"] & P_{N-8c-5}^{N-2} \\
 & P_{N-8c-1}^{N} \arrow[r, "f_3'"] & P_{N-8c-2}^{N} & &
\end{tikzcd}$$

the maps f_i are filtration 1 maps constructed as in [**11**; 7.15], and α are attaching maps for cells of projective spaces. f_3 may be viewed as the restriction of f_3', and g is the induced map of cofibres, also of filtration 1. The map b is the above $\tilde{\beta}$ followed by the inclusion, and since $i\tilde{\beta}$ is null-homotopic, so is $i_1 b$, giving the map $\tilde{b}$. Since $2g \simeq *$, we have $2 \circ \alpha \circ g \circ \tilde{b} \simeq *$, and hence $f_1 f_2 f_3 b \simeq *$.

In $\mathrm{Ext}_A^{*,*+N-2}(H^*(\ ), \mathbf{Z}_2)$ there are isomorphisms

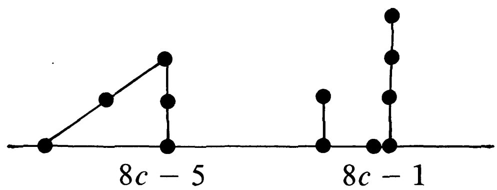


induced by the above diagram. We will be done once we note that the map b is detected in filtration $4c - 3$. To do this, we go back to the preceding $\varepsilon = 2$ diagram, and show that $\tilde{\beta}$ is detected in filtration $4c - 3$, from which Ext charts show that the same is true of b.

This analysis of the filtration of $\tilde{\beta}$ is similar to the proof of the first half of Lemma 6. We prove the dual result—if E_{4c-3} is the indicated stage of a minimal Adams resolution of S^0, then the lifting to E_{4c-3} of the generator of the image of J extends to γ as indicated

$$\begin{tikzcd}
\Sigma^{-1}P^{8c}_{8c-4} \arrow[rr, dashed, "\gamma"] & & E_{4c-3} \arrow[d] \\
S^{8c-5} \arrow[u, hook] \arrow[urr] \arrow[rr, "\beta"] & & S^0.
\end{tikzcd}$$

This is proved by noting that a chart for $\pi_*(E_{4c-3})$ begins



and so the composite

$$S^{8c-3} \cup_2 e^{8c-2} \xrightarrow{c} S^{8c-2} \xrightarrow{\nu} S^{8c-5} \to E_{4c-3}$$

is trivial. □

The proof of Proposition 4 is much easier. We just need to note that $\beta \wedge 1$ has filtration $\geq 4b + 1$ by [**11**; 8.2] and that $\mathrm{Ext}_A^{s,t}(H * P_{4A}, \mathbf{Z}_2) = 0$ if $4A + 8b + 4 \leq t - s \leq 4A + 8b + 5$ and $s \geq 4b$ by the tables of [**12**] and Adams periodicity.

We complete the paper by giving the proof of Proposition 5, which is very similar to that of Proposition 3. The following result, which implies Proposition 5, is analogous to Lemma 6. As before, $N \equiv 0\ (8)$.

LEMMA 8. *The image-of-J generator of* $\pi_{N-2}(S^{N-8b-5})$ *coextends to a filtration* $(4b + 1)$ *element of* $\pi_{N-2}(P^{N-8b-5}_{N-8b-9})$, *and if* $\nu(N) \leq 4b + 1$ *then this class goes to* 0 *in* $\pi_{N-2}(P^{N-2}_{N-8b-9})$.

The first half of Lemma 8 was proved above, in the consideration of $\tilde{\beta}$ when $\varepsilon = 2$. The second half of Lemma 8 is proved as before by applying filtration 4 maps to the following analogue of Lemma 7.

LEMMA 9. *If* $4c-2 \le \nu(N) \le 4c+1$, *then the elements in* $\pi_{N-2}(P^{N-2}_{N-8c-9})$ *of filtration* $\ge 4c+1$ *are hit by differentials in the* ASS.

The first step of the proof of Lemma 9 is to show that if $\nu(N) = 4c + \varepsilon$, then there is a commutative diagram as below with β a generator of the image of J and $i\tilde{\beta} \simeq *$. The proof mimics that of the analogous previous result.

$$\begin{array}{ccccc} & & P^{N-8c-k}_{N-8c-l} & \xrightarrow{i_1} & P_{N-8c-l} \\ & \tilde{\beta} \nearrow & \downarrow & & \downarrow \\ S^{N-2} & \xrightarrow{\beta} & S^{N-8c-k} & \rightarrow & P_{N-8c-k} \end{array}$$

ε	k	l
1	3	7
0	2	6
-1	1	5
-2	-3	1

The second half of the proof of Lemma 9 mimics that of Lemma 7. We shall consider the case $\varepsilon = -1$, which contains one subtle point that should have been addressed in the case $\varepsilon = 1$ of the proof of Lemma 7. The filtration of β_{8c-1} is better behaved in the Moore space than in the sphere. Therefore we modify the above diagram in this case to

$$\begin{array}{ccccc} & & P^{N-8c}_{N-8c-5} & \xrightarrow{i_1} & P_{N-8c-5} \\ & \tilde{\beta} \nearrow & \downarrow & & \downarrow \\ S^{N-2} & \xrightarrow{\beta} & S^{N-8c}_{N-8c-1} & \hookrightarrow & P_{N-8c-1} \end{array}$$

Once we have shown that $\tilde{\beta}$ is nontrivial in filtration $4c-2$, then the analogue of (***) can be used to complete the proof. It follows from [**11**; 8.3] that β has filtration exactly $4c-2$; to show the same is true of $\tilde{\beta}$, we dualize, so that we must show $\Sigma^{-1}P^{8c}_{8c-1} \to E_{4c-2}$ extends over $\Sigma^{-1}P^{8c+4}_{8c-1}$, or equivalently that $\Sigma^{-2}P^{8c+4}_{8c+1} \xrightarrow{\alpha} \Sigma^{-1}P^{8c}_{8c-1} \to E_{4c-2}$ is trivial. We calculate that all elements of $[\Sigma^{-2}P^{8c+4}_{8c+1}, E_{4c-2}]$ of positive filtration map nontrivially to $[\Sigma^{-2}P^{8c+4}_{8c+1}, S^0]$, but here our map is certainly trivial since it is the obstruction to existence of an existing map, namely the dual of $\tilde{\beta}$.

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LEHIGH UNIVERSITY
BETHLEHEM, PA 18015

AND

NORTHWESTERN UNIVERSITY
EVANSTON, IL 60201