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# ON $\mathscr{L}_{p,\lambda}$ SPACES FOR SMALL $\lambda$

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**It is shown for $1 \leq p \neq 2 < \infty$ that for each $\mathscr{L}_{p,\lambda}$ subspace $X$ of $L_p[0,1]$ with $\lambda$ sufficiently close to 1 there is a nonlinear projection from $L_p$ onto $X$ which factors through a linear projection on $L_1[0,1]$ with norm close to 1. Some additional results on representation of certain operators on $L_1$ are also proved.**

**Introduction.** It is well known that for $1 \leq p < \infty$ the range of a contractive projection on $L_p(\mu)$ is isometric to $L_p(\nu)$ and conversely if $X$ is a subspace of $L_p(\mu)$ and $X$ is isometric to $L_p(\nu)$ for some measure $\nu$ then there is a contractive projection from $L_p(\mu)$ onto $X$, [**8**]. If $X$ is a $\mathscr{L}_{p,\lambda}$ subspace of $L_p(\mu)$ and $\lambda$ is close to one then it is still true that $X$ is complemented in $L_p(\mu)$ and the projection may be chosen to have norm close to one, [**12**], [**2**]. However, it is an open question whether such a space $X$ must be isomorphic to $L_p(\nu)$ for some $\nu$.

In this paper we investigate this question and show (Proposition 2.10) that the question is equivalent for all $p$, $1 \leq p < \infty$, $p \neq 2$. We also reprove the theorems of Zippin [**13**] and Dor [**5**] on $\mathscr{L}_{p,\lambda}$ subspaces of $l_p$ and the result of [**2**]. We believe our proof clarifies the role of the atomic measure space in these results and the difficulty in the non atomic case.

The paper is divided into three sections. In the first section $w^*$ integral representations are considered for conditional expectations and isometries of $L_1$. In the second section the main results are proved and in the third section some ideas about directions for further work are described.

Throughout the paper we will use $L_p = L_p([0,1], \mathscr{B}, \lambda)$ where $\mathscr{B}$ is the Lebesgue measurable sets, $\lambda$ is Lebesgue measure and $p \in [1,\infty)$, $p \neq 2$. If $\mathscr{G}$ is a sub-$\sigma$-algebra of the Lebesgue measurable sets $\mathscr{E}(\cdot|\mathscr{G})$ will denote the conditional expectation operator with respect to $\mathscr{G}$. The following theorem summarizes the contractive case:

THEOREM 0.1. [**3**], [**8**].

(a) *If $P$ is a contractive projection on $L_p[0,1]$, $1 \leq p \neq 2 < \infty$, then the range of $P$ is of the form*

$$\left\{fh\colon f \in L_p\left([0,1], \mathscr{G}, |h|^p\, d\lambda\right)\right\}$$

*where $\mathscr{G}$ is a sub-$\sigma$-algebra of $\mathscr{B}$, $h \in L_p$, and $\mathscr{E}(|h|^p|\mathscr{G}) = 1_{\operatorname{supp} h}$.*

(b) *If $X$ is a subspace of $L_p$ isometric to $L_p(\nu)$ for some measure $\nu$ then $X$ is of the same form as above and the operator $Q$ defined by $Qf = \mathscr{E}(|h|^{p-2}hf|\mathscr{G})h$, for all $f \in L_p$ is a projection onto $X$, with $\|Q\| = 1$.*

We will also need some results about $\mathscr{L}_{p,\lambda}$ spaces for $\lambda$ close to one which are summarized in the following theorem:

THEOREM 0.2. *Let $p \in [1, \infty)$, $p \neq 2$.*

(a) [4] *If $\{x_i\colon i \in \mathbf{N}\}$ is contained in $L_p(\nu)$ for some measure $\nu$, such that for any set of scalars $\{a_i\colon i \in \mathbf{N}\}$, with finitely many nonzero,*

$$(1+\varepsilon)^{-1}\left(\sum |a_i|^p\right)^{1/p} \leq \left\|\sum a_i x_i\right\| \leq (1+\varepsilon)\left(\sum |a_i|^p\right)^{1/p},$$

*and if $\varepsilon$ is sufficiently small then there exists disjoint measurable sets $\{A_i\colon i \in \mathbf{N}\}$ such that $\|x_i|_{A_i^c}\| < a_1(p, \varepsilon)$, where $A_i^c$ denotes the complement of $A_i$. Moreover, $a_1(p, \varepsilon) \to 0$ as $\varepsilon \to 0$.*

(b) [12] *If $\{x_i\colon i \in \mathbf{N}\}$ is as in* (a) *and $\{A_i\colon i \in \mathbf{N}\}$ is a set of disjoint measurable sets such that*

$$\int_{A_i} |x_i|^p \, d\nu \geq c \quad \text{for all } i \in \mathbf{N}$$

*then*

$$\left\| \sum a_i x_i|_{A_i^c} \right\| \leq a(\varepsilon, c)\left(\sum |a_i|^p\right)^{1/p}$$

*for all sets of scalars $\{a_i\colon i \in \mathbf{N}\}$ with finitely many non-zero, and $a(\varepsilon, c) \to 0$ as $\varepsilon \to 0$ and $c \to 1$.*

(c) [9], [2]. *Suppose that $X$ is a $\mathscr{L}_{p,\lambda}$ subspace of $L_p$ and that $\{P_n\colon n \in \mathbf{N}\}$ is a sequence of projections from $L_p$ onto $X_n \subset X$ which on $X$ converge in the strong operator topology to the identity on $X$. Then if $\sup\|P_n\| = \lambda'$ is close enough to one there is a projection $P$ from $L_p$ onto $X$ with $\|P\| \leq K(p, \lambda')$ such that $K(p, \lambda') \to 1$ as $\lambda' \to 1$.*

We will use standard notation and facts from Banach space theory as may be found in the books of Lindenstrauss and Tzafriri [3]. We will make one particular abuse of notation which deserves comment. If $f \in L_1$ then $f$ acts a functional on $C[0,1]$, namely by $\langle f, g\rangle = \int gf\,d\lambda$ however we will at times simply write $f$ even when we are thinking of the measure $\nu(A) = \int_A f\,d\lambda$ as an element of $C[0,1]^*$. Also we will assume that the scalar field is the reals although simple modifications will give the same results in complex case. Finally let us mention a simple inequality for $L_p$, $p \geq 1$, which we will frequently use without comment

$$\left| \|z\|_p^p - \|x\|_p^p \right| \leq p\|z - x\|_p, \quad \max(\|x\|, \|z\|) \leq 1. \tag{0.3}$$

**1. Representations of operators.** In this section we present a way of looking at the isometric results (Theorem 0.1) that will motivate the approach we take to investigate $\mathscr{L}_{p,\lambda}$ for small $\lambda$. The representations that we get are essentially known for general classes of operators but our proofs use some special properties of the operators and allow us to conclude that the vector valued functions involved have special properties.

Our first proposition is about conditional expectation operators. The representation itself is the same as in [**6**], p. 499, (see also [**7**]).

PROPOSITION 1.1. *Suppose that $\mathscr{G}$ is a sub-$\sigma$-algebra of the Lebesgue measurable subsets, $\mathscr{B}$, of* $[0,1]$. *Then, there is a $w^*$ Borel measurable function $g$:* $[0,1] \to \mathscr{P}$ *where $\mathscr{P}$ denotes the probability measures on* $[0,1]$ *such that*

(a)

$$\mathscr{E}(f|\mathscr{G}) = \frac{d}{d\lambda}\int f(\omega)g(\omega)\,d\lambda$$

*where*

$$\left\langle \int f(\omega)g(\omega)\,d\lambda,\ h\right\rangle = \int f(\omega)\langle g(\omega),\ h\rangle\,d\lambda \quad \textit{for all } h \in C[0,1].$$

(b) range $g$ = uvb $l_1(\Gamma)$ (*usual unit vector basis of* $l_1(\Gamma)$).

(c)

$$\int_G g(\omega)\,d\lambda|_G = \int_G g(\omega)\,d\lambda \quad \textit{for all } G \in \mathscr{G}.$$

*Conversely if $g$ is a $\mathscr{G}$ to $w^*$ Borel measurable function from* $[0,1]$ *to $\mathscr{P}$ satisfying* (c) *and, for all $G \in \mathscr{G}$, $\int_G g(\omega)\,d\lambda$ is absolutely continuous with respect to $\lambda$ then, there is a function $h \in L_1$ such that*

$$\mathscr{E}(f|\mathscr{G})h = \frac{d}{d\lambda}\int f(\omega)g(\omega)\,d\lambda \quad \textit{for all } f \in L_1.$$

*Proof.* Let $\mathscr{G}_1 \subset \mathscr{G}_2 \subset \cdots \subset \mathscr{G}_n \cdots$ be a sequence of finite sub-$\sigma$-algebras of $\mathscr{G}$ which generate $\mathscr{G}$ and let $\{A_{nk}\}_{k=1}^{k(n)}$ be the atoms of $\mathscr{G}_n$. Define

$$g_n(\omega) = \sum_{k=1}^{k(n)} 1_{A_{nk}}(\omega)\lambda(A_{nk})^{-1}\lambda|_{A_{nk}}.$$

$(g_n)$ is a martingale valued in $\mathscr{P}$ and by the Martingale Convergence Theorem $(\langle g_n, h\rangle)$ converges a.e. for each $h \in C[0,1]$. Because $C[0,1]$ is separable it follows that the exceptional set may be chosen independent of $h$ and thus $(g_n)$ converges $w^*$ a.e. to a function $g$: $[0,1] \to \mathscr{P}$.

Clearly $g$ is $\mathscr{G}$ to $w^*$ Borel measurable and thus the operator

$$Tf = \int fg\,d\lambda$$

satisfies $Tf = T\mathscr{E}(f|\mathscr{G})$ for all $f \in L_1$. Now suppose that $f$ is a $\mathscr{G}_n$ measurable function,

$$f = \sum_{k=1}^{k(n)} a_k 1_{A_{nk}}.$$

So

$$Tf = \sum_{k=1}^{k(n)} a_k \int_{A_{nk}} g\,d\lambda = \sum_{k=1}^{k(n)} a_k \lambda|_{A_{nk}}$$

and hence

$$\frac{d\,Tf}{d\lambda} = \sum_{k=1}^{k(n)} a_k 1_{A_{nk}} = f.$$

Because such functions are dense in $L_1(\mathscr{G}, \lambda)$, (a) follows. (c) is an obvious consequence of (a).

To prove (b) we will show that for each $n$ there are sets $A'_{nk} \subset A_{nk}$ with $\lambda(A'_{nk}) = \lambda(A_{nk})$ such that if $w_k \in A'_{nk}$, $k = 1, 2, \ldots, k(n)$, $(g(w_k)) \sim$ uvb $l_1^{k(n)}$. Indeed for each $s \in \mathbf{N}$ choose $F_k^{s-1} \subset F_k^s \subset A_{nk}$, a closed subset such that $\lambda(A_{nk} \setminus F_k^s) < 2^{-2}\lambda(A_{nk})$ and let $(f_k^s)_{k=1}^{k(n)}$ be a partition of unity on $[0,1]$ such that

$$f_k^s(\omega) = \begin{cases} 1, & \omega \in F_k^s, \\ 0, & \omega \in F_{k'}^s,\ k' \neq k, \end{cases}$$

i.e., subordinate to $\{(\bigcup_{k' \neq k} F_{k'}^s)^c \colon k = 1, 2, \ldots, k(n)\}$. We have that

$$\left\langle \int_{A_{nk}} g\,d\lambda, f_k^s \right\rangle = \int_{A_{nk}} \langle g(\omega), f_k^s \rangle\, d\lambda(\omega) = \lim_l \int_{A_{nk}} \langle g_l(\omega), f_k^s \rangle\, d\lambda$$

$$= \lim_l \int_{A_{nk}} \sum_{r=1}^{k(l)} \lambda(A_{lr})^{-1} \int_{A_{lr}} f_k^s\, d\lambda (1_{A_{lr}})\, d\lambda = \lim_l \sum_{A_{lr} \subset A_{nk}} \int_{A_{lr}} f_k^s\, d\lambda$$

$$\geq \lim_l \sum_{A_{lr} \subset A_{nk}} \lambda(A_{lr} \cap F_k^s) = \lambda(F_k^s) \geq (1 - 2^{-s})\lambda(A_{nk}).$$

Hence

$$B_k^s = \left\{ \omega \colon \langle g(\omega), f_k^s \rangle > 1 - 2^{-s/2},\ \omega \in A_{nk} \right\}$$

is a subset of $A_{nk}$ of measure at least $(1 - 2^{-s/2})\lambda(A_{nk})$. If $\omega_k \in \liminf_s B_k^s$ for each $k$ then $\langle g(\omega_k), f_k^s\rangle > 1 - 2^{-s/2}$ and necessarily $\sum_{k' \neq k}\langle g(\omega_k), f_{k'}^s\rangle < 2^{-s/2}$ for all $s$. Thus $(g(\omega_k))$ is one equivalent to the usual unit vector basis of $l_1^{k(n)}$. Clearly $\lambda(\liminf_s B_k^s) = \lambda(A_{nk})$ for each $k$, proving our claim and (b) because $g(\omega) \neq g(\omega')$ only if there is an $n$ and a $k$ such that $\omega \in A_{nk}$ and $\omega' \notin A_{nk}$. (To be precise we may have to alter $g$ on a set of measure zero.)

For the converse let $h = (d/d\lambda)\int_\Omega g\,d\lambda$. Clearly the operator $Tf = (d/d\lambda)\int fg\,d\lambda$ is a norm one operator on $L_1$ and $Tf = T\mathscr{E}(f|\mathscr{G})$. Suppose that $f$ is $\mathscr{G}_n$ measurable and $f = \sum_{k=1}^{k(n)} a_k 1_{A_{nk}}$. Then

$$Tf = \sum_{k=1}^{k(n)} a_k \frac{d}{d\lambda}\int_{A_{nk}} g\,d\lambda = \sum_{k=1}^{k(n)} a_k h 1_{A_{nk}} \quad \text{(by (c))}$$
$$= \mathscr{E}\left(\sum_{k=1}^{k(n)} a_k 1_{A_{nk}} \middle| \mathscr{G}\right) h,$$

proving the result. □

Our interest in the contractive case and thus in conditional expectation operators arises from the results in **[1]**. There we showed that a subspace $X$ of $L_p$ which is close in the Banach Mazur distance to $L_p(\nu)$ is actually a perturbation of a subspace $Y$ of $L_p$ which is isometric to $L_p(\nu)$. Thus if it is true that a $\mathscr{L}_{p,\lambda}$ subspace of $L_p$ for small $\lambda$ is isomorphic to $L_p(\nu)$ and the distance goes to one as $\lambda$ goes to one, then there must be an isometric copy of $L_p(\nu)$ nearby. Moreover the standard projection as in Theorem 0.1(b) would then be an isomorphism on the $\mathscr{L}_{p,\lambda}$ subspace. Hence we are looking for a function $g$ as above and a change of measure and sign (the $h$ in Theorem 0.1(a)).

Our next result shows that functions like $g$ in the previous result play a role in isometries as well. Before stating the result we need to introduce a definition.

DEFINITION. Suppose $g$: $[0,1] \to C(K)^*$, $K$ compact metric, is $w^*$ Borel measurable and bounded. Then we will say that $g$ is *disjointness preserving* on a $\sigma$-algebra $\mathscr{G}$ with respect to a measure $\mu$ if for every $G_1$ and $G_2$ in $\mathscr{G}$, $G_1 \cap G_2 = \varnothing$ then

$$\int_{G_1} g\,d\mu \perp \int_{G_2} g\,d\mu.$$

Note that condition (c) of Proposition 1.1 guarantees that the function there is disjointness preserving on $\mathscr{G}$.

PROPOSITION 1.2. *Let $X$ be a subspace of $L_1$ isometric to $L_1(\nu)$ and suppose that $X = \{f \cdot h \mid f \in L_1(\mathcal{G}, |h|\,d\lambda)\}$ where $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{B}$ and $h \in L_1$ such that $\mathcal{E}(|h| \mid \mathcal{G}) = 1_{\operatorname{supp} h}$. Then an operator $T$: $X \to C(K)^*$, $K$ compact metric, is an isometry if and only if there is a $\mathcal{G}$ to $w^*$ Borel measurable function $g$: $[0,1] \to C(K)^*$ with $\|g(w)\| = 1$ a.e. $|h|\,d\lambda$ such that $g$ is disjointness preserving on $\mathcal{G}$ with respect to $|h|\,d\lambda$ and*

$$Tf = \int f(\operatorname{sgn} h) g \, d\lambda$$

*Moreover,* range $g = \operatorname{uvb} l_1(\Gamma)$, (*except on a set of measure zero*) *for some index set* $\Gamma$.

*Proof*. As in the proof of Proposition 1.1 let $(\mathcal{G}_n)$ be a nested sequence of finite sub-σ-algebras of $\mathcal{G}$ which generate $\mathcal{G}$ and let $(A_{nk})_{k=1}^{k(n)}$ be the atoms of $\mathcal{G}_n$.

Assume that $T$ is an isometry.

Define $g_n$: $[0,1] \to C(K)^*$ by

$$g_n(\omega) = \sum_{k=1}^{k(n)} \left[ \int_{A_{nk}} |h| \, d\lambda \right]^{-1} T(h 1_{A_{nk}}) 1_{A_{nk}}(\omega).$$

As before $(g_n)$ is a martingale but with respect to $|h|\,d\lambda$ and thus converges $w^*$ a.e. $|h|\,d\lambda$ to a function $g$: $[0,1] \to C(K)^*$.

Define $S$: $L_1 \to C(K)^*$ by

$$Sf = \int (\operatorname{sgn} h) f g \, d\lambda.$$

For each $n$ and $k$

$$Sh1_{A_{nk}} = \int_{A_{nk}} |h| g \, d\lambda = \int_{A_{nk}} |h| g_n \, d\lambda = T(h 1_{A_{nk}})$$

and thus $T = S|_X$.

Because

$$\|f \cdot h\| = \|T(f \cdot h)\| = \left\| \int fg(h) \, d\lambda \right\| \le \int |f| \, \|g\| \, |h| \, d\lambda,$$

$\|g(\omega)\| = 1$ a.e. $|h|\,d\lambda$($\|g(\omega)\| \le 1$ because $\|g_n(\omega)\| = 1$ for all $\omega \in [0,1]$). Finally it is well known that for an isometry $T$ of $L_1(\gamma)$ into $L_1(\mu)$, if $|f_1| \wedge |f_2| = 0$, $|Tf_1| \wedge |Tf_2| = 0$. Hence $g$ is disjointness preserving on $\mathcal{G}$ with respect to $|h|\,d\lambda$.

For the converse suppose that $g$ is given and

$$Tf = \int f(\operatorname{sgn} h) g \, d\lambda.$$

Because $g$ is disjointness preserving, if $f = \sum_{k=1}^{k(n)} a_n 1_{A_{nk}}$

$$\|Tf\| = \left\| \sum_{k=1}^{k(n)} a_k \int_{A_{nk}} g|h|\, d\lambda \right\| = \sum_{k=1}^{k(n)} |a_k| \left\| \int_{A_{nk}} g|h|\, d\lambda \right\|.$$

Thus $T$ will be an isometry if and only if $\|\int_{A_{nk}} g|h|\, d\lambda\| = \int_{A_{nk}} |h|\, d\lambda$, for all $n$ and $k$.

Let $g_n = \mathscr{E}(g|\mathscr{G}_n)$ (the expected value is with respect to $|h|\, d\lambda$.) $g_n \to g$ $w^*$ a.e. $|h|\, d\lambda$. Because $\|g(\omega)\| = 1$ a.e. $|h|\, d\lambda$, $\lim \|g_n(\omega)\| = 1$ a.e. $|h|\, d\lambda$.

Let $\varepsilon > 0$ and choose $l$ such that

$$\int_{\{\omega:\ \|g_l(\omega)\| \geq 1-\varepsilon\} \cap A_{nk}} |h|\, d\lambda \geq (1-\varepsilon) \int_{A_{nk}} |h|\, d\lambda.$$

Then

$$\begin{aligned}\left\| \int 1_{A_{nk}} g|h|\, d\lambda \right\| &= \sum_{A_{lr} \subset A_{nk}} \left\| \int_{A_{lr}} g|h|\, d\lambda \right\| \\ &= \sum_{A_{lr} \subset A_{nk}} \left\| \int_{A_{lr}} g_l|h|\, d\lambda \right\| = \sum_{A_{lr} \subset A_{nk}} \int_{A_{lr}} \|g_l\|\, |h|\, d\lambda \\ &= \int_{A_{nk}} \|g_l\|\, |h|\, d\lambda \geq (1-\varepsilon)^2 \int_{A_{nk}} |h|\, d\lambda.\end{aligned}$$

Because $\varepsilon$ was arbitrary $\|\int_{A_{nk}} g|h|\, d\lambda\| = \int_{A_{nk}} |h|\, d\lambda$, as required.

The "moreover" assertion is proved in a similar manner to the proof of (b) of Proposition 1.1 once we observe that the condition of preserving disjointness implies that there is a nested family $(\mathscr{G}'_n)$ of $\sigma$-algebras of subsets of $K$ and atoms $(A'_{nk})_{k=1}^{k(n)}$ of $\mathscr{G}'_n$ such that

$$\int_{A_{nk}} g|h|\, d\lambda|_{A'_{nk}} = \int_{A_{nk}} g|h|\, d\lambda$$

for each $n$, $k$. We omit the details. □

REMARK 1.3. The condition range $g = \text{uvb}\, l_1$ is not sufficient to guarantee that $g$ is disjointness preserving. Indeed let

$$g(t) = \begin{cases} \lambda, & t \in [0, \frac{1}{2}), \\ \dfrac{\delta_t + \delta_{t-1/2}}{2}, & t \in [\frac{1}{2}, 1]. \end{cases}$$

Then

$$\int_0^{1/2} g(t)\,d\lambda = \frac{\lambda}{2} \quad \text{and} \quad \int_{1/2}^1 g(t)\,d\lambda$$
$$= \frac{1}{2}\left[\int_{1/2}^1 \delta_t\,d\lambda + \int_{1/2}^1 \delta_{t-1/2}\,d\lambda\right] = \frac{\lambda}{2}\Big|_{[1/2,1]} + \frac{\lambda}{2}\Big|_{[0,1/2]} = \frac{\lambda}{2}.$$

REMARK 1.4. An "isomorphic" version of disjointness preserving would be that there exist $\delta > 0$ such that if $G_1, G_2, \ldots, G_n \in \mathscr{G}$, disjoint, then

$$\left\| \sum_{i=1}^n \frac{a_i}{\lambda[G_i]} \int_{G_i} g\,d\lambda \right\| \geq \delta \sum_{i=1}^n |a_i|$$

for all scalars $(a_i)$. It follows easily from this that $Tf = \int fg\,d\lambda$ is an isomorphism of $L_1(\mathscr{G}, \lambda)$ into $C(K)^*$.

Because we are searching for isometric copies of $L_p(\gamma)$ in $L_p$ which are perturbations of $\mathscr{L}_p$ subspaces in $L_p$ there is little difference between the case $p = 1$ and $p > 1$. More precisely

PROPOSITION 1.5. *Suppose $S$ is an isometry from the subspace $X = \{fh: f \in L_p(\mathscr{G}, |h|^p\,d\lambda)\}$, $\mathscr{E}(|h|^p|\mathscr{G}) = 1$, of $L_p$ into $L_p(\Omega, \mathscr{F}, \nu)$. Then,*

(i) *the nonlinear bijections*

$$V\colon L_p \to L_1 \text{ and } U\colon L_1(\Omega, \mathscr{F}, \nu) \to L_p(\Omega, \mathscr{F}, \nu) \text{ defined by}$$

$$(Vg)(\omega) = |g(\omega)^{p-1}|g(\omega) \quad \text{and}$$
$$(Ug)(\omega) = |g(\omega)|^{(1-p)/p} g(\omega)$$

*are uniformly continuous on bounded sets and*

(ii) *the operator*

$$T = U^{-1}SV^{-1} \text{ is a linear isometry from}$$
$$X^p = \left\{f\cdot|h|^{p-1}h\colon f \in L_1\left(\mathscr{G}, |h|^p\,d\mu\right)\right\} \text{ into } L_1(\Omega, \mathscr{F}, \nu).$$

(iii) $Qg = U(\mathscr{E}(V(g)\operatorname{sgn} h|\mathscr{G}))V(h)$ *is a (nonlinear) projection of $L_p$ onto $X$.*

We believe that this proposition is essentially known although we do not have a suitable reference. We will simply outline the proof:

The uniform continuity of $U$ follows from the simple inequalities

$$|s^p - t^p| \geq |s - t|^p \quad \text{for } s, t \geq 0,\ s, t \in \mathbf{R}$$

and

$$(|s| + |t|)^{1/p} \geq \frac{|s|^{1/p} + |t|^{1/p}}{2^{(p-1)/p}} \quad \text{for all } s, t \in \mathbf{R}.$$

For $V$ we use another inequality: for any $\varepsilon > 0$ there is a constant $K$ such that

$$K|s - t| \geq \left| |s|^{p-1}s - |t|^{p-1}t \right|^{1/p} \quad \text{for } s, t \in R$$

such that $s < (1 - \varepsilon)t$ or $t < (1 - \varepsilon)s$. This inequality can be proved with simple calculus.

To see that $T$ is a linear isometry one checks the result on function of the form $f \cdot |h|^{p-1}h$ where $f$ is simple and one uses the fact that $S$ must send disjoint functions to disjoint functions.

It is easy to verify (iii).

**2. $\mathscr{L}_{p,\lambda}$ descriptions.** In this section we will show that given a nested sequence of finite dimensional subspaces $X_n$, $n = 1, 2, \ldots,$ of a $\mathscr{L}_{p,\lambda}$ subspace $X$ of $L_p$ such that $\overline{\cup X_n} = X$ and $d(X_n, l_p^{\dim X_n}) < \lambda'$ with $\lambda'$ close to 1 that we can find a sequence of finite dimensional subspaces $Y_k$, $k = 1, 2, \ldots,$ of $X$ which are almost nested, $[Y_k\colon k \in \mathbf{N}] = X$, and the $Y_k$'s have nicer properties than the $X_n$'s. The intuition for the arguments is most apparent for the case $p = 1$, so the reader may find it helpful to assume that $p = 1$. Let us also note that we will not keep track of the actual values of the estimates which occur in the arguments and we have made no effort to keep them smaller than necessary to achieve qualitative results.

Let us begin by fixing a $\mathscr{L}_p$ description of $X$ which we will use throughout this section. Thus we have for each $n$ a subspace $X_n$ of $X$ with $\dim X_n = k(n)$ and $(x_k^n)_{k=1}^{k(n)} \subset X_n$ such that the operator $T_n\colon l_p^{k(n)} \to X_n$ defined by

$$T_n e_k^n = x_k^n \quad \text{where } e_k^n = (0, 0, \ldots, 0, 1, 0, \ldots, 0) \text{ (“1” in the } k\text{th place)}$$

is an isomorphism with $\|T_n\| < \lambda_1$, $\|T_n^{-1}\| < \lambda_1$, $\|x_k^n\| = 1$, for each $k$ and $n$, $X_n \subset X_{n+1}$, and $\overline{\cup X_n} = X$.

Our first lemma shows that there are natural descendents of elements on the upper levels on all lower levels. (We will speak of downward as the direction of greater dimension, i.e., $(x_k^{n-1})$ is above $(x_k^n)$.) The reader should think of the normalized (in $L_p$) indicator functions of dyadic intervals in $[0,1]$ as the ideal case. In that case the descendents are normalized restrictions of upper level functions to smaller sets.

We will adopt the notational convention that indicator functions will be written as the set alone, i.e., $f1_A = fA$ and to prevent confusion we will use capital Roman letters from the first part of the alphabet to denote subsets of $[0,1]$.

LEMMA 2.1. *For each* $n \in \mathbf{N}$ *and* $k \in \{1,2,\ldots,k(n)\}$ *there is a measurable subset* $A_k^n \subset [0,1]$ *and for each* $l > n$ *there is a subset* $\mathscr{A}_k^{nl}$ *of* $\{1,2,\ldots,k(l)\}$ *such that*

(a) $\inf\{|x_k^n(\omega)|\colon \omega \in A_k^n\} > 0$

(b) *for each* $s \in \mathscr{A}_k^{nl}$, *there is* $\sigma = \pm 1$ *such that*

$$\left\|x_k^n(A_k^n \cap A_s^l) - \sigma x_s^l \left\|x_k^n(A_k^n \cap A_n^l)\right\|\right\| < a_2(\lambda_1)\left\|x_k^n(A_k^n \cap A_s^l)\right\|$$

*and*

$$\left\|x_k^n(A_k^n \cap A_s^l)\right\| \neq 0.$$

(c) $\|x_k^n(A_k^n \cap \bigcup\{A_s^l\colon s \in \mathscr{A}_k^{nl}\})\| > 1 - a_2(\lambda_1)$

(d) $a_2(\lambda_1) \to 0$ *as* $\lambda_1 \to 1$.

*Proof*. By Theorem 0.2 for each $n \in \mathbf{N}$ there are disjoint measurable sets $A_k^n \subset [0,1]$, $k = 1,2,\ldots,k(n)$ such that

$$\left\|x_k^n(A_k^n)^c\right\| < a_1(p,\lambda_1 - 1).$$

Clearly we may assume that $\inf|x_k^n(\omega)|\colon \omega \in A_k^n\} > 0$, by passing to slightly smaller sets.

Suppose $x_k^n = \sum a_s x_s^l$, $l > n$. By Theorem 0.2(b)

$$\begin{aligned}\left\|\sum a_s x_s^l(A_s)^c\right\| &\le \left(\sum |a_s|^p\right)^{1/p} a\left(\lambda_1 - 1, 1 - a_1(p,\lambda_1 - 1)^p\right)\\ &\le \lambda_1 a\left(\lambda_1 - 1, 1 - a_1(p,\lambda_1 - 1)^p\right)\end{aligned}$$

and consequently

$$\begin{aligned}&\left\|x_k^n(A_k^n) - \sum a_s x_s^l(A_s^l)\right\|\\ &\qquad \le a_1(p,\lambda_1 - 1) + \lambda_1 a\left(\lambda_1 - 1, \left(1 - a_1(p,\lambda_1 - 1)^p\right)\right)\\ &\qquad = b_1(p,\lambda_1) - \lambda_1 a_1(p,\lambda_1 - 1).\end{aligned}$$

Also

$$\begin{aligned}
&\Big\| x_k^n(A_k^n) - \sum_s a_s x_s^l(A_s^l) \Big\| \\
&\quad \geq \left( \sum_s \big\| x_k^n(A_k^n \cap A_s^l) - a_s x_s^l(A_s^l) \big\|^p \right)^{1/p} \\
&\quad \geq \left( \sum_s \Big| \operatorname{sgn} a_s \big\| x_k^n(A_k^n \cap A_s^l) \big\| - a_s \Big|^p \right)^{1/p} - \lambda_1 a_1(p, \lambda_1 - 1) \\
&\quad \geq \Big\| \sum_s \operatorname{sgn} a_s \big\| x_k^n(A_k^n \cap A_s^l) \big\| x_s^l(A_s^l) - \sum a_s x_s^l(A_s^l) \Big\| \\
&\qquad - \lambda_1 a_1(p, \lambda_1 - 1).
\end{aligned}$$

Let $\rho > 0$ and let

$$\begin{aligned}
\mathscr{B}_\rho = \Big\{ s \colon \big\| x_k^n(A_k^n \cap A_s^l) - (\operatorname{sgn} a_s) x_s^l \| x_k^n(A_k^n \cap A_s^l) \| \big\| \\
> \rho \big\| x_k^n(A_k^n \cap A_s^l) \big\| \Big\}
\end{aligned}$$

and

$$\mathscr{A}_k^{nl} = \Big\{ s \colon \big\| x_k^n(A_k^n \cap A_s^l) \big\| \neq 0 \text{ and } s \notin \mathscr{B}_\rho \Big\}.$$

We have that

$$\begin{aligned}
2b_1(p, \lambda_1) &\geq \Big\| x_k^n(A_k^n) - \sum_s \operatorname{sgn} a_s \big\| x_k^n(A_k^n \cap A_s^l) \big\| x_s^l(A_s^l) \Big\| \\
&\geq \left( \sum_s \Big\| x_k^n(A_k^n \cap A_s^l) - \operatorname{sgn} a_s \big\| x_k^n(A_k^n \cap A_s^l) \big\| x_s^l(A_s^l) \Big\|^p \right)^{1/p} \\
&\geq \rho \left( \sum_{s \in \mathscr{B}_\rho} \big\| x_k^n(A_k^n \cap A_s^l) \big\|^p \right)^{1/p} \\
&\quad - \left( \sum_{s \in \mathscr{B}_\rho} \big\| x_k^n(A_k^n \cap A_s^l) \big\|^p \big\| x_s^l(A_s^l)^c \big\|^p \right)^{1/p} \\
&\geq \rho \left( \sum_{s \in \mathscr{B}_\rho} \big\| x_k^n(A_k^n \cap A_s^l) \big\|^p \right)^{1/p} - a_1(p, \lambda_1 - 1).
\end{aligned}$$

Thus

$$\left\|x_k^n\left(A_k^n \cap \bigcup\{A_s^l: s \in \mathscr{A}_k^{nl}\}\right)\right\| = \left(\sum_{s \in \mathscr{A}_k^{nl}} \|x_k^n(A_k^n \cap A_s^l)\|^p\right)^{1/p}$$

$$\geq \left(\sum_s \|x_k^n(A_k^n \cap A_s^l)\|^p\right)^{1/p} - \left(\sum_{s \in \mathscr{B}_\rho} \|x_k^n(A_k^n \cap A_s^l)\|^p\right)^{1/p}$$

$$\geq \left\|x_k^n\left(A_k^n \cap \bigcup\{A_s^l: s = 1,2,\ldots,k(l)\}\right)\right\| - \frac{(2b_1(p,\lambda_1) + a_1(p,\lambda_1 - 1))}{\rho}$$

$$\geq \|x_k^n\| - \left(\left\|x_k^n - \sum_s a_s x_s^l(A_s^l)\right\| + \left\|\sum_s a_s x_s^l(A_s^l) - x_k^n\left(A_k^n \cap \bigcup\{A_s^l: s = 1,2,\ldots,k(l)\}\right)\right\|\right) - \frac{(2b_1(p,\lambda_1) + a_1(p,\lambda_1 - 1))}{\rho}$$

$$\geq 1 - 2b_1(p,\lambda_1) - \frac{(2b_1(p,\lambda_1) + a_1(p,\lambda_1 - 1))}{\rho}.$$

Hence if $\rho = \sqrt{2b_1(p,\lambda_1) + a_1(p,\lambda_1 - 1)}$ and $a_2(\lambda_1) = 2b_1(p,\lambda_1) + \sqrt{2b_1(p,\lambda_1) + a_1(p,\lambda_1 - 1)}$, (b), (c) and (d) will be satisfied. □

While this lemma gives us for each level $n$ a nearby isometric copy of $l_p^{k(n)}$ there is no apparent relationship between the sets chosen for one level and those for lower levels. To help understand this difficulty consider the following simple example.

EXAMPLE 2.2. Fix $\varepsilon > 0$ and let $(B_n)$ be an infinite sequence of (stochastically) independent subsets of $[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon)$ with normalized Lebesgue measure and let $X_1 = [1_{[0,\frac{1}{2}+\varepsilon)}, 1_{[\frac{1}{2}-\varepsilon,1)}]$. If we apply Dor's result (Proposition 0.2(a)) to $x_1 = 1_{[0,\frac{1}{2}+\varepsilon)}$, $x_2 = 1_{[\frac{1}{2}-\varepsilon,1)}$ any of the pairs of sets $C_1^n = [0, \frac{1}{2} - \varepsilon) \cup B_n$, $C_2^n = [\frac{1}{2} - \varepsilon, 1) \setminus B_n$ satisfy the conclusion. Now suppose that $X_1 \subset X_2 \subset \cdots$ is the first of the spaces in the $\mathscr{L}_{p,\lambda}$ description of $X$. It is conceivable that in choosing sets $A_k^l$ for the space $X_l$, $\bigcup\{A_k^l: k \in \mathscr{A}_1^{1l}\} = C_1^l$ and $\bigcup\{A_k^l: k \in \mathscr{A}_2^{1l}\} = C_2^l$. Thus there is no consistency in the choice of sets.

The next proposition is actually the key step in our refinement of the $\mathscr{L}_p$ description of $X$. It will allow us to fix the shape (change of measure) for $X$. Thus we will be able to determine a function like $h$ in Theorem 0.1(a). The notation in this lemma is the same as in the previous lemma.

PROPOSITION 2.3. *There exist a sequence $z_i = x_{k(i)}^{n(i)}$, $i = 1, 2, \ldots,$ an infinite subset $M \subset \mathbf{N}$ and for each $i \in \mathbf{N}$ and $l \in M$, $l \geq l(i)$, there is a set $\mathscr{B}_i^l \subset \{1, 2, \ldots, k(l)\}$ such that*

(a) $\mathscr{B}_i^l \cap \mathscr{B}_j^l = \varnothing$ *for all* $i \neq j$, $l \in M$

(b) $\|z_i(B_i \cap \bigcup\{A_j^l\colon j \in \mathscr{B}_i^l\})\| > 1 - a_3(\lambda_1)$ *for each* $i \in N$ *and* $l \in M$, $l \geq l(i)$ *where* $B_i = A_{k(i)}^{n(i)}$

(c) *for all* $f \in X$

$$\overline{\lim_{l \in M}}\, d\left(f, \left[x_j^l\colon j \in \mathscr{B}_i^l, i = 1, 2, \ldots\right]\right) < a_3(\lambda_1)\|f\|$$

(d) *if* $s \in \mathscr{B}_i^l$ *there is a set* $B_s^l \subset A_s^l$ *such that* $\|z_i(B_i \cap B_s^l)\| \neq 0$ *and*

$$\left\|z_i\left(B_i \cap B_s^l\right) - \sigma\left\|z_i\left(B_i \cap B_s^l\right)\right\| x_s^l\left(A_s^l\right)\right\| < a_3(\lambda_1)\left\|z_i\left(B_i \cap B_s^l\right)\right\|$$

*where* $\sigma = \pm 1$.

(e) $a_3(\lambda_1) = a_3(\lambda_1, p) \to 0$ *as* $\lambda_1 \to 1$.

*Proof*. We will determine the sequence $(z_i)$ by examining each $x_k^n$ in the order $(k, n) < (k', n')$ if and only if $n < n'$, or $k < k'$ and $n = n'$. Let $\varepsilon > 0$. (We will place conditions on $\varepsilon$ later.)

To begin the process let $z_1 = x_1^1$, $B_1 = A_1^1$ and $\mathscr{C}_1^l = \mathscr{A}_1^{1l}$ for $l = 1, 2, \ldots,$ where $\mathscr{A}_1^{1l}$ is as in Lemma 2.1, and $M_1 = \mathbf{N}$. Now suppose that we have chosen $z_i$ for $i = 1, 2, \ldots, j$, $B_i = A_{k(i)}^{n(i)}$, $M_i \subset M_{i-1} \subset \mathbf{N}$ infinite, and $\mathscr{C}_i^l = \mathscr{A}_{k(i)}^{n(i)l}$ for $l \geq l(i)$, $l \in M_i$.

We next consider each $x_k^n$, $(k(j), n(j)) < (k, n)$ until we find the first $(k, n)$ such that

$$(1)\quad \overline{\lim_{l \in M_j}} \inf\left\{\left\|x_k^n - \sum_{i=1}^{j} \sum_{s \in \mathscr{C}_i^l} \sigma_s \left\|x_k^n\left(A_s^l \cap A_k^n\right)\right\| x_s^l\left(A_s^l \cap B_i\right)\right\|\colon \sigma_s = \pm 1\right\} > 1 - \varepsilon.$$

Let $z_{j+1} = x_k^n$ $(k = k(j+1),\ n = n(j+1))$.

If no such $x_k^n$ exists then the sequence $(z_i)$ terminates at $j$. Assuming we have $z_{j+1}$ let

$$B_{j+1} = A_{k(j+1)}^{n(j+1)}, \quad \mathscr{C}_{j+1} = \mathscr{A}_{k(j+1)}^{n(j+1)l} \setminus \bigcup\left\{\mathscr{C}_i^l\colon i \leq j\right\}$$

and let $M_{j+1} \subset M_j$ so that the limit in (1) over $M_{j+1}$ exists and equals the limit superior.

Thus we have defined the sequence $(z_j)$ and by a simple diagonalization argument we can find an infinite subset $M$ of $N$ so that $M \setminus M_j$ is finite for each $j$. Note that the sets $\mathscr{C}_j^l$, $j = 1, 2, \ldots,$ are disjoint for each $l$. We will first prove

$$\text{(b}'\text{)} \qquad \left\| z_i\left(B_i \cap \bigcup\left\{A_j^l \colon j \in \mathscr{C}_i^l\right\}\right)\right\| > 1 - b_3(\lambda_1)$$

for each $i$, and $l \in M$, $l$ sufficiently large, where

$$b_3(\lambda_1) \to 0 \quad \text{as } \lambda_1 \to 1.$$

First we estimate

$$\left\| z_i - \sum_{s \in \mathscr{A}_{k(i)}^{n(i)l}} \sigma_s \left\| z_i\left(B_i \cap A_s^l\right)\right\| x_s^l\left(B_i \cap A_s^l\right)\right\|$$

where $\sigma_s = \pm 1$ is chosen as in Lemma 2.1(b).

$$\begin{aligned}
&\left\| z_i - \sum_{s \in \mathscr{A}_{k(i)}^{n(i)}} \sigma_s \left\| z_i\left(B_i \cap A_s^l\right)\right\| x_s^l\left(B_i \cap A_s^l\right)\right\| \\
&\quad \le \left\| z_i - z_i\left(\bigcup\left\{A_s^l \colon s \in \mathscr{A}_{k(i)}^{n(i)l}\right\} \cap B_i\right)\right\| \\
&\qquad + \left( \sum_{s \in \mathscr{A}_{k(i)}^{n(i)l}} \left\| z_i\left(B_i \cap A_s^l\right) - \sigma_s \left\| z_i\left(B_i \cap A_s^l\right)\right\| x_s^l\left(A_s^l\right)\right\|^p \right)^{1/p} \\
&\quad \le \left(1 - (1 - a_2(\lambda_1))^p\right)^{1/p} + \left( \sum_{s \in \mathscr{A}_{k(i)}^{n(i)l}} a_2(\lambda_1)^p \left\| z_i\left(B_i \cap A_s^l\right)\right\|^p \right)^{1/p} \\
&\hspace{20em} \text{(by Lemma 2.1)} \\
&\quad \le \left(1 - (1 - a_2(\lambda_1))^p\right)^{1/p} + a_2(\lambda_1) \le (p+1) a_2(\lambda_1)^{1/p} \\
&\hspace{20em} \left(\text{assuming } a_2(\lambda_1) < 1\right).
\end{aligned}$$

Thus

$$\text{(2)} \quad \left\| z_i - \sum_{s \in \mathscr{A}_{k(i)}^{n(i)l}} \sigma_s \left\| z_i\left(B_i \cap A_s^l\right)\right\| x_s^l\left(B_i \cap A_s^l\right)\right\| \le (p+1) a_2(\lambda_1)^{1/p}$$

and a minor change in the above argument gives

$$\text{(2a)} \quad \left\| z_i(B_i) - \sum_{s \in \mathscr{A}_{k(i)}^{n(i)l}} \sigma_s \left\| z_i\left(B_i \cap A_s^l\right)\right\| x_s^l\left(A_s^l\right)\right\| \le (p+1) a_2(\lambda_1)^{1/p}.$$

Also note that these estimates hold for any $x_k^n$, $A_k^n$ in place of $z_i$, $B_i$.

Next suppose that $j \in \mathscr{A}_{k(i)}^{n(i)l} \cap \bigcup_{s=1}^{i-1} \mathscr{C}_s^l$. Then for some $s \in \{1, 2, \ldots, i-1\}$, $k = s$ and $i$, $\sigma_k = \pm 1$,

$$\left\| z_k\left(B_k \cap A_j^l\right) - \sigma_k \left\| z_k\left(B_k \cap A_j^l\right) \right\| x_j^l\left(B_k \cap A_j^l\right) \right\| < a_2(\lambda_1) \left\| z_k\left(B_k \cap A_j^l\right) \right\| \tag{3}$$

and consequently

$$\begin{aligned} \left\| x_j^l\left(B_k \cap A_j^l\right) - x_j^l \right\|^p &= 1 - \left\| x_j^l\left(B_k \cap A_j^l\right) \right\|^p \\ &\leq p \left\| \frac{z_k\left(B_k \cap A_j^l\right)}{\left\| z_k\left(B_k \cap A_j^l\right) \right\|} - \sigma_k x_j^l\left(B_k \cap A_j^l\right) \right\| \leq p a_2(\lambda_1) \end{aligned} \tag{4}$$

(by inequality 0.3).

Hence

$$\begin{aligned} &\left\| \left\| z_i\left(B_i \cap A_j^l\right) \right\| x_j^l\left(B_i \cap A_j^l\right) - \left\| z_i\left(B_s \cap A_j^l\right) \right\| x_j^l\left(B_s \cap A_j^l\right) \right\| \\ &\leq \left\| z_i\left(B_i \cap A_j^l\right) \right\| \left\| x_j^l\left(B_i \cap A_j^l\right) - x_j^l \right\| \\ &\quad + \left| \left\| z_i\left(B_i \cap A_j^l\right) \right\| - \left\| z_i\left(B_s \cap A_j^l\right) \right\| \right| \\ &\quad + \left\| z_i\left(B_s \cap A_j^l\right) \right\| \left\| x_j^l - x_j^l\left(B_s \cap A_j^l\right) \right\| \qquad \left( \left\| x_j^l \right\| = 1 \right) \\ &\leq p a_2(\lambda_1)^{1/p} \left\| z_i\left(A_j^l\right) \right\| + \left| \left\| z_i\left(B_i \cap A_j^l\right) \right\| - \left\| z_i\left(B_i \cap A_j^l \cap B_s\right) \right\| \right| \\ &\quad + \left| \left\| z_i\left(B_i \cap A_j^l \cap B_s\right) \right\| - \left\| z_i\left(B_s \cap A_j^l\right) \right\| \right| + p a_2(\lambda_1)^{1/p} \left\| z_i\left(A_j^l\right) \right\| \\ &\leq 2 p a_2(\lambda_1)^{1/p} \left\| z_i\left(A_j^l\right) \right\| + \left| \left\| z_i\left(B_i \cap A_j^l\right) \right\| \left( \left\| x_j^l \right\| - \left\| x_j^l\left(B_i \cap A_j^l \cap B_s\right) \right\| \right) \right| \\ &\quad + \left\| \left\| z_i\left(B_i \cap A_j^l\right) \right\| x_j^l\left(B_i \cap A_j^l \cap B_s\right) - \sigma_i z_i\left(B_i \cap A_j^l \cap B_s\right) \right\| \\ &\quad + \left\| z_i\left(B_s \cap A_j^l \cap B_i^c\right) \right\| \qquad \left( \left\| x_j^l \right\| = 1 \text{ and the triangle inequality} \right) \\ &\leq 2 p a_2(\lambda_1)^{1/p} \left\| z_i\left(A_j^l\right) \right\| + 2 p a_2(\lambda_1)^{1/p} \left\| z_i\left(B_i \cap A_j^l\right) \right\| \\ &\quad + a_2(\lambda_1) \left\| z_i\left(B_i \cap A_j^l\right) \right\| + \left\| z_i\left(B_s \cap A_j^l \cap B_i^c\right) \right\| \quad \text{(by (3) and (4))}. \end{aligned}$$

Thus we have

$$\begin{aligned} &\left\| \left\| z_i\left(B_i \cap A_j^l\right) \right\| x_j^l\left(B_i \cap A_j^l\right) - \left\| z_i\left(B_s \cap A_j^l\right) \right\| x_j^l\left(B_s \cap A_j^l\right) \right\| \\ &\qquad \leq 5 p a_2(\lambda_1)^{1/p} \left\| z_i\left(A_j^l\right) \right\| + \left\| z_i\left(B_s \cap A_j^l \cap B_i^c\right) \right\|. \end{aligned} \tag{5}$$

Now we are ready to estimate

$$\left\| z_i\left(B_i \cap \bigcup\left\{A_j^l \colon j \in \mathscr{C}_i^l\right\}\right) \right\|.$$

For large $l \in M_i \cap M$

$$\begin{aligned}
1 - \varepsilon &< \left\| z_i - \sum_{j=1}^{i-1} \sum_{s \in \mathscr{C}_j^l} \sigma_s \left\| z_i\left(A_s^l \cap B_j\right) \right\| x_s^l\left(A_s^l \cap B_j\right) \right\| \\
&\le \left\| z_i - \sum_{j=1}^{i-1} \sum_{s \in \mathscr{C}_j^l \cap \mathscr{A}_{k(i)}^{n(i)l}} \sigma_s \left\| z_i\left(A_s^l \cap B_i\right) \right\| x_s^l\left(A_s^l \cap B_i\right) \right\| \\
&\quad + \left\| \sum_{j=1}^{i-1} \sum_{s \in \mathscr{C}_j^l \cap \mathscr{A}_{k(i)}^{n(i)l}} \sigma_s \left\| z_i\left(A_s^l \cap B_i\right) \right\| x_s^l\left(A_s^l \cap B_i\right) \right. \\
&\qquad\qquad \left. - \sum_{j=1}^{i-1} \sum_{s \in \mathscr{C}_j^l \cap \mathscr{A}_{k(i)}^{n(i)l}} \sigma_s \left\| z_i\left(A_s^l \cap B_j\right) \right\| x_s^l\left(A_s^l \cap B_j\right) \right\| \\
&\quad + \left\| \sum_{j=1}^{i-1} \sum_{s \in \mathscr{C}_j^l \setminus \mathscr{A}_{k(i)}^{n(i)l}} \left\| z_i\left(A_s^l \cap B_j\right) \right\| x_s^l\left(A_s^l \cap B_j\right) \right\| \\
&\le \left\| z_i - \sum_{s \in \mathscr{A}_{k(i)}^{n(i)l}} \sigma_s \left\| z_i\left(A_s^l \cap B_i\right) \right\| x_s^l\left(A_s^l \cap B_i\right) \right\| \\
&\quad + \left\| \sum_{s \in \mathscr{A}_{k(i)}^{n(i)l} \setminus \bigcup\{\mathscr{C}_j^l \colon j < i\}} \sigma_s \left\| z_i\left(A_s^l \cap B_i\right) \right\| x_s^l\left(A_s^l \cap B_i\right) \right\| \\
&\quad + \left( \sum_{j=1}^{i-1} \sum_{s \in \mathscr{C}_j^l} \left\| \left\| z_i\left(A_s^l \cap B_i\right) \right\| x_s^l\left(A_s^l \cap B_i\right) \right. \right. \\
&\qquad\qquad \left. \left. - \left\| z_i\left(A_s^l \cap B_j\right) \right\| x_s^l\left(A_s^l \cap B_j\right) \right\|^p \right)^{1/p} \\
&\quad + \left( \sum_{j=1}^{i-1} \sum_{s \in \mathscr{C}_j^l \setminus \mathscr{A}_{k(l)}^{n(i)l}} \left\| z_i\left(A_s^l \cap B_j\right) \right\|^p \right)^{1/p}
\end{aligned}$$

(by the triangle inequality and the disjointness of the sets $A_s^l$)

$$\le (p+1)a_2(\lambda_1)^{1/p} + \Big\| \sum_{s \in \mathscr{A}_{k(i)}^{n(i)l} \setminus \bigcup\{\mathscr{C}_j^l:\ j<i\}} \sigma_s \| z_i(A_s^l \cap B_i) \| x_s^l(A_s^l \cap B_i) \Big\|$$

$$+ \left( \sum_{j=1}^{i-1} \sum_{s \in \mathscr{C}_j^l} \left( 5pa_2(\lambda_1)^{1/p} \| z_i(A_s^l) \| + \| z_i(B_j \cap A_s^l \cap B_i^c) \| \right)^p \right)^{1/p}$$

$$+ \left\| z_i \left( \bigcup_{j=1}^{i-1} \bigcup_{s \in \mathscr{C}_j^l \setminus \mathscr{A}_{k(i)}^{n(i)l}} A_s^l \cap B_j \right) \right\| \quad \text{(by (2) and (5))}$$

$$\le (p+1)a_2(\lambda_1)^{1/p} + \Big\| \sum_{s \in \mathscr{A}_{k(i)}^{n(i)l} \setminus \bigcup\{\mathscr{C}_j^l:\ j<i\}} \sigma_s \| z_i(A_s^l \cap B_i) \| x_s^l(A_s^l \cap B_i) \Big\|$$

$$+ 5pa_2(\lambda_1)^{1/p} + \| z_i(B_i^c) \| + (p+1)a_2(\lambda_1)^{1/p}$$

(we have used (2) and Lemma 2.1(c))

Hence if $b(\lambda_1) = (7p+3)a_2(\lambda_1)^{1/p}$

$$(6)\quad 1 - \varepsilon - b(\lambda_1) < \Big\| \sum_{s \in \mathscr{A}_{k(i)}^{n(i)l} \setminus \bigcup\{\mathscr{C}_j^l:\ j<i\}} \sigma_s \| z_i(A_s^l \cap B_i) \| x_s^l(A_s^l \cap B_i) \Big\|$$

$$\left( \| z_i(B_i^c) \| < a_1(p, \lambda_1 - 1) \right).$$

From this we see that

$$(7)\quad \left\| z_i\left(B_i \cap \bigcup\{A_s^l:\ s \in \mathscr{C}_i^l\}\right) \right\|$$

$$\ge \Big\| \sum_{s \in \mathscr{C}_i^l} \sigma_s \| z_i(A_s^l \cap B_i) \| x_s^l(A_s^l \cap B_i) \Big\|$$

$$- \Big\| \Big( z_i - \sum_{s \in \mathscr{C}_i^l} \sigma_s \| z_i(A_s^l \cap B_i) \| x_s^l(A_s^l \cap B_i) \Big) \cdot \left(B_i \cap \bigcup\{A_s^l:\ s \in \mathscr{C}_i^l\}\right) \Big\|$$

$$\ge 1 - \varepsilon - b(\lambda_1) - \Big\| z_i - \sum_{s \in \mathscr{A}_{k(i)}^{n(i)l}} \sigma_s \| z_i(A_s^l \cap B_i) \| x_s^l(A_s^l \cap B_i) \Big\|$$

$$\ge 1 - \varepsilon - b(\lambda_1) - (p+1)a_2(\lambda_1)^{1/p}.$$

Thus we will need

$$(*)\qquad a_3(\lambda_1) > \varepsilon + b(\lambda_1) + (p+1)a_2(\lambda_1)^{1/p}.$$

The sets $\mathscr{C}_i^l$ are not large enough to guarantee (c). But $\bigcup \mathscr{C}_i^l$ is large enough to have dense linear span as we will next show:

$$\inf\left\{ \|x - y\|: y \in \left[ x_k^l: k \in \bigcup_{i=1}^{\infty} \mathscr{C}_i^l, l \in M, l \geq n \right] \right\} = 0 \tag{8}$$

for all $x \in X$ and $n \in \mathbf{N}$.

In order to actually satisfy (c) we will need to enlarge the sets $\mathscr{C}_i^l$.

To prove (8) note that we have for each $k \leq k(n)$, $n \in \mathbf{N}$, and $\delta > 0$

$$\left\| x_k^n - \sum_{j=1}^{\infty} \sum_{s \in \mathscr{C}_j^l} a_s x_s^l \left( A_s^l \cap B_j \right) \right\| \leq 1 - \varepsilon + \delta$$

for some $(a_s)$ and for $l$ sufficiently large, by (1) or (2), provided

$$(p + 1) a_2 (\lambda_1)^{1/p} \leq 1 - \varepsilon.$$

Because $(x_k^n)_{k=1}^{k(n)}$ is equivalent to the unit vector basis of $l_p^{k(n)}$ Theorem 0.2(a) gives us disjoint sets $\mathscr{D}_k \subset \{1, 2, \ldots, k(l)\}$ such that $\|T_l^{-1} x_k^n (\mathscr{D}_k^c)\| < a_1(p, \lambda_1^2 - 1)$. Thus if $x = \sum b_k x_k^n$ then

$$\begin{aligned}
\left\| x - \sum_k b_k \sum_{j=1}^{\infty} \sum_{s \in \mathscr{C}_j^l \cap \mathscr{D}_k} a_s x_s^l \right\| &= \left\| \sum b_k \left( x_k^n - \sum_j \sum_{s \in \mathscr{C}_j^l \cap \mathscr{D}_k} a_s x_s^l \right) \right\| \\
&\leq \|T_l\| \left\| \sum b_k \left( T_l^{-1} x_k^n - \sum_j \sum_{s \in \mathscr{C}_j^l \cap \mathscr{D}_k} a_s e_s^l \right) \right\| \\
&\leq \lambda_1 \left[ \left\| \sum b_k \left( T_l^{-1} x_k^n - T_l^{-1} x_k^n (\mathscr{D}_k) \right) \right\| \right. \\
&\qquad \left. + \left\| \sum b_k \left( T_l^{-1} x_k^n (\mathscr{D}_k) - \sum_j \sum_{s \in \mathscr{C}_j^l \cap \mathscr{D}_k} a_s e_s^l \right) \right\| \right] \\
&\leq \lambda_1 a\left( \lambda_1^2 - 1, 1 - a_1(p, \lambda_1^2 - 1)^p \right) \left( \sum |b_k|^p \right)^{1/p} \\
&\qquad + \lambda_1 \left( \sum |b_k|^p \right)^{1/p} \max_k \left\| \left( T_l^{-1} x_k^n - \sum_j \sum_{s \in \mathscr{C}_j^l} a_s e_s \right) (\mathscr{D}_k) \right\| \\
&\leq \lambda_1^2 a\left( \lambda_1^2 - 1, 1 - a_1(p, \lambda_1^2 - 1)^p \right) \|x\| + \lambda_1^3 \|x\| (1 - \varepsilon + \delta).
\end{aligned}$$

Consequently if

$$(**)\quad 1 > \gamma = \lambda_1^2\big(a\big(\lambda_1^2 - 1, 1 - a_1\big(p, \lambda_1^2 - 1\big)^p\big) + (1 - \varepsilon + \delta)\lambda_1\big)$$

the standard proof of the Open Mapping Theorem, e.g., **[11]**, shows (8).

We have yet to define the sets $\mathscr{B}_i^l$ but to do so we will first introduce sets $\mathscr{D}_i^l$ which will not be disjoint for fixed $l$ but will be related to the sets $\mathscr{B}_i^l$ by $\mathscr{B}_i^l \subset \mathscr{D}_i^l$ for each $i$ and $l$ and $\bigcup_i \mathscr{B}_i^l = \bigcup_i \mathscr{D}_i^l$.

Note that for each $k \in \mathscr{C}_i^n$ we have

$$\big\|x_k^n\big(A_k^n \cap A_s^l\big) - \sigma_s \big\|x_k^n\big(A_k^n \cap A_s^l\big)\big\| x_s^l\big(A_s^l\big)\big\| < a_2(\lambda_1)\big\|x_k^n\big(A_k^n \cap A_s^l\big)\big\| \quad \text{for all } s \in \mathscr{A}_k^{nl}$$

and

$$\big\|z_i\big(A_k^n \cap B_i\big) - \sigma\big\|z_i\big(A_k^n \cap B_i\big)\big\| x_k^n\big(A_k^n\big)\big\| < a_2(\lambda_1)\big\|z_i\big(A_k^n \cap B_i\big)\big\|.$$

Let

$$\begin{aligned}\mathscr{D}_i^l = \big\{s\colon & \text{there exist } A \subset A_s^l \text{ such that} \\ & \big\|z_i(A \cap B_i) - \sigma\big\|z_i(A \cap B_i)\big\| x_s^l\big(A_s^l\big)\big\| \\ & \leq (p+2)a_2(\lambda_1)^{1/2p}\big\|z_i(A \cap B_i)\big\|, \\ & \text{for some } \sigma = \pm 1 \text{ and } \big\|z_i(A \cap B_i)\big\| \neq 0\big\}.\end{aligned}$$

(Note that $\mathscr{D}_i^l \supset \mathscr{A}_{k(i)}^{n(i)l}$.)

The two inequalities we sighted above suggest that $\mathscr{D}_i^l$ will contain much of $\bigcup_{k \in \mathscr{C}_i^n} \mathscr{A}_k^{nl}$ and our next argument will make this more precise.

$$\begin{aligned} a_2(\lambda_1)\big\|z_i\big(A_k^n \cap B_i\big)\big\| &\geq \big\|z_i\big(A_k^n \cap B_i\big) - \sigma\big\|z_i\big(A_k^n \cap B_i\big)\big\| x_k^n\big(A_k^n\big)\big\| \\ &\geq \Big\|z_i\big(A_k^n \cap B_i\big) - \sigma\big\|z_i\big(A_k^n \cap B_i\big)\big\| \sum_{s \in \mathscr{A}_k^{nl}} \sigma_s \big\|x_k^n\big(A_s^l \cap A_k^n\big)\big\| x_s^l\big(A_s^l\big)\Big\| \\ &\quad - \big\|z_i\big(A_k^n \cap B_i\big)\big\| \Big\|x_k^n\big(A_k^n\big) - \sum_{s \in \mathscr{A}_k^{nl}} \sigma_s\big\|x_k^n\big(A_s^l \cap A_k^n\big)\big\| x_s^l\big(A_s^l\big)\Big\| \\ &\geq \Bigg(\sum_{s \in \mathscr{A}_k^{nl}} \big\|z_i\big(A_k^n \cap B_i \cap A_s^l\big) - \sigma\sigma_s\big\|z_i\big(A_k^n \cap B_i\big)\big\| \\ &\qquad\qquad \times \big\|x_k^n\big(A_s^l \cap A_k^n\big)\big\| x_s^l\big(A_s^l\big)\big\|^p\Bigg)^{1/p} \\ &\quad - \big\|z_i\big(A_k^n \cap B_i\big)\big\| a_2(\lambda_1)^{1/p}(p+1) \end{aligned}$$

(by (2a) but for $x_k^n$ rather than $z_i$).

Hence

$$a_2(\lambda_1)^{1/p}(p+2)\|z_i(A_k^n \cap B_i)\|$$

$$\geq \Bigg( \sum_{s \in \mathscr{A}_k^{nl}} \|z_i(A_k^n \cap B_i \cap A_s^l) - \sigma\sigma_s \|z_i(A_k^n \cap B_i)\| \, \|x_k^n(A_s^l \cap A_k^n)\| x_s^l(A_s^l)\|^p \Bigg)^{1/p}$$

$$\geq \Bigg( \sum_{s \in \mathscr{A}_k^{nl} \setminus \mathscr{D}_i^l} (p+2)^p a_2(\lambda_1)^{1/2} \|z_i(A_k^n \cap B_i \cap A_s^l)\|^p \Bigg)^{1/p}$$

$$- \Bigg( \sum_{s \in \mathscr{A}_k^{nl} \setminus \mathscr{D}_i^l} \Big| \, \|z_i(A_k^n \cap B_i \cap A_s^l)\| - \|z_i(A_k^n \cap B_i)\| \, \|x_k^n(A_s^l \cap A_k^n)\| \, \Big|^p \Bigg)^{1/p}$$

$\big($definition of $\mathscr{D}_i^l$ with $A = A_k^n \cap A_s^l\big)$

$$\geq (p+2)a_2(\lambda_1)^{1/2p}\Big\|z_i\Big(\bigcup\{A_k^n \cap B_i \cap A_s^l \colon s \in \mathscr{A}_k^{nl} \setminus \mathscr{D}_i^l\}\Big)\Big\|$$

$$- \Big\|\big[z_i(A_k^n \cap B_i) - \sigma\|z_i(A_k^n \cap B_i)\|x_k^n(A_k^n)\big] \cup \{A_s^l \colon s \in \mathscr{A}_k^{nl}\}\Big\|.$$

Rearranging and estimating the last term by $a_2(\lambda_1)^{1/p}\|z_i(A_k^n \cap B_i)\|$ we have

$$2a_2(\lambda_1)^{1/2p}\|z_i(A_i^n \cap B_i)\| \geq \Big\|z_i\Big(\bigcup\{A_k^n \cap B_i \cap A_s^l \colon s \in \mathscr{A}_k^{nl} \setminus \mathscr{D}_i^l\}\Big)\Big\|. \tag{9}$$

Above we used only those $s$ for which the inequality in the definitions of $\mathscr{D}_k^l$ fails for $A = A_k^n \cap A_s^l$. Denote by $\tilde{\mathscr{D}}_i$ the subset of $\mathscr{D}_i^l$ for which this inequality holds for $\sigma = \sigma_s$ and $A = A_k^n \cap A_s^l$. Next we will see that $x_k^n$ is close to $[x_s^l(A_s^l) \colon s \in \mathscr{A}_k^{nl} \cap \tilde{\mathscr{D}}_i^l]$ and thus is close to $[x_s^l(A_s^l) \colon s \in \mathscr{A}_k^{nl} \cap \mathscr{D}_i^l]$.

$$\Bigg\| x_k^n(A_k^n) - \sum_{s \in \mathscr{A}_k^{nl} \cap \tilde{\mathscr{D}}_i^l} \sigma_s \|x_k^n(A_s^l \cap A_k^n)\| x_s^l(A_s^l) \Bigg\|$$

$$\leq \Bigg\| x_k^n(A_k^n) - \sum_{s \in \mathscr{A}_k^{nl}} \sigma_s \|x_k^n(A_k^n \cap A_s^l)\| x_s^l(A_s^l) \Bigg\|$$

$$+ \Bigg\| \sum_{s \in \mathscr{A}_k^{nl} \setminus \tilde{\mathscr{D}}_i^l} \sigma_s \|x_k^n(A_k^n \cap A_s^l)\| x_s^l(A_s^l) \Bigg\|$$

$$\leq (p+1)a_2(\lambda_1)^{1/p} + \left( \sum_{s \in \mathscr{A}_k^{nl} \setminus \tilde{\mathscr{D}}_i^l} \left\| x_k^n\left(A_k^n \cap A_s^l\right) \right\|^p \right)^{1/p}$$

(by (2a) applied to $x_k^n$ rather than $z_i$)

$$= (p+1)a_2(\lambda_1)^{1/p} + \left\| x_k^n\left(A_k^n \cap \bigcup\{A_s^l: s \in \mathscr{A}_k^{nl} \setminus \tilde{\mathscr{D}}_i^l\}\right) \right\|$$

$$\leq (p+1)a_2(\lambda_1)^{1/p}$$

$$+ \left\| \left( x_k^n(A_k^n) - \frac{\sigma z_i(A_k^n \cap B_i)}{\|z_i(A_k^n \cap B_i)\|} \right) \left( \bigcup\{A_s^l: s \in \mathscr{A}_k^n \setminus \tilde{\mathscr{D}}_i^l\} \right) \right\|$$

$$+ \frac{\left\| z_i\left(A_k^n \cap B_i \cap \bigcup\{A_s^l: s \in \mathscr{A}_k^{nl} \setminus \tilde{\mathscr{D}}_i^l\}\right) \right\|}{\|z_i(A_k^n \cap B_i)\|}$$

$$\leq (p+1)a_2(\lambda_1)^{1/p} + a_2(\lambda_1) + 2a_2(\lambda_1)^{1/2p}$$

(because $k \in \mathscr{C}_i^n$ and by (9) and the remark following (9)).

So if $k \in \mathscr{C}_i^n$

$$(10) \quad d\left(x_k^n(A_k^n), \left[x_s^l(A_s^l): s \in \mathscr{A}_k^{nl} \cap \mathscr{D}_i^l\right]\right) \leq (p+4)a_2(\lambda_1)^{1/2p}.$$

Now we will prove (c). Let $f \in X$. By the proof of (8) for any $n \in \mathbf{N}$ there are integers $l(i) \in M$, $n \leq l(1) \leq l(2) \leq \cdots$ and

$$x_i \in \left[x_k^{l(i)}: k \in \mathscr{C}_j^{l(i)}, \ j = 1, 2, \ldots\right]$$

such that $\|f - \sum_{i=1}^N x_i\| < \gamma^N \|f\|$ and $\|x_i\| \leq (\gamma^{i-1} + \gamma^i)\|f\|$.

Suppose that $x_i = \sum_{m \in \bigcup \mathscr{C}_j^{l(i)}} a_m^i x_m^{l(i)}$. Then by (10) and Theorem 0.2 for any $l$ sufficiently large there is an element

$$y_m^i \in \left[x_k^l(A_k^l): k \in \mathscr{D}_m^l \cap \mathscr{A}_m^{l(i)l}\right]$$

such that $\|x_m^{l(i)}(A_m^{l(i)}) - y_m^i\| < (p+5)a_2(\lambda_1)^{1/2p}$ for each $m \in \bigcup \mathscr{C}_j^{l(i)}$ and the $y_m^i$'s are disjointly supported. Thus

$$\left\| x_i - \sum a_m^i y_m^i \right\|$$
$$\leq \left[a\left(\lambda_1 - 1, 1 - a_1(p, \lambda_1 - 1)^p\right) + (p+5)a_2(\lambda_1)^{1/2p}\right] \lambda_1 \|x_i\|.$$

Let

$$\alpha = \lambda_1\left[a\left(\lambda_1 - 1, 1 - a_1(p, \lambda_1 - 1)^p\right) + (p+5)a_2(\lambda_1)^{1/2p}\right].$$

Then

$$\left\| f - \sum_{i=1}^{N} \sum_m a_m^i y_m^i \right\| \le \left\| f - \sum_{i=1}^{N} x_i \right\| + \sum_{i=1}^{N} \left\| x_i - \sum_m a_m^i y_m^i \right\|$$

$$\le \gamma^N \|f\| + \alpha \sum_{i=1}^{N} (\gamma^{i-1} + \gamma^i) \|f\| \le \left( \gamma^N + \alpha \frac{1+\gamma}{1-\gamma} \right) \|f\|.$$

Hence if

$$(***) \qquad a_3(\lambda_1) > \alpha \left( \frac{1+\gamma}{1-\gamma} \right)$$

(c) will be satisfied.

Finally to choose the sets $\mathscr{B}_i^l$ note that $\mathscr{D}_i^l \supset \mathscr{A}_{k(i)}^{n(i)l} \supset \mathscr{C}_i^l$ and the $\mathscr{C}_i^l$ are disjoint and satisfy (7). So to satisfy (b) we need only choose $\mathscr{B}_i^l \supset \mathscr{C}_i^l$ by allocating $\bigcup \mathscr{D}_i^l \setminus \bigcup \mathscr{C}_i^l$ among the $\mathscr{B}_i^l$ so that $\mathscr{D}_i^l \supset \mathscr{B}_i^l$.

It is easy to see that $\varepsilon$ and $a_3(\lambda_1)$ can be chosen to satisfy (*), (**) and (***), completing the proof. □

REMARK 2.4. The condition (b) in the statement of Proposition 2.3 also can be made to hold for $\bigcup\{B_j^l: j \in \mathscr{B}_i^l\}$ in place of $\bigcup\{A_j^l: j \in \mathscr{B}_i^l\}$ with perhaps a slight change of $a_1(\lambda_1)$ to $\tilde{a}_3(\lambda_1)$ which also tends to zero as $\lambda_1$ goes to one. Indeed if $j \in \mathscr{B}_i^l$, (d) and Lemma 2.1(b) show that

$$\left\| z_i\left(B_i \cap \bigcup\{A_j^l: j \in \mathscr{B}_i^l\}\right) - z_i\left(B_i \cap \bigcup\{B_j^l: j \in \mathscr{B}_i^l\}\right) \right\|$$

$$\le \left( \sum_{j \in \mathscr{B}_i^l} \left\| z_i(B_i \cap A_j^l) - z_i(B_i \cap B_j^l) \right\|^p \right)^{1/p}$$

$$\le \left( \sum_{j \in \mathscr{B}_i^l} [a_3(\lambda_1) + a_2(\lambda_1)]^p \left\| z_i(B_i \cap A_j^l) \right\|^p \right)^{1/p}$$

$$\le a_3(\lambda_1) + a_2(\lambda_1).$$

From this point on we can use the elements

$$z_i(B_i \cap B_s^l) / \|z_i(B_i \cap B_s^l)\|, \qquad l \in M,\ s \in \mathscr{B}_i^l$$

in place of the original $(x_k^n)$. Indeed Proposition 2.3(d) guarantees that $x_s^l \sim \sigma z_i(B_i \cap B_s^l) \|z_i(B_i \cap B_s^l)\|^{-1}$ and (c) guarantees that the closed spans are close together. Because for each $n \in M$, each $k \le k(n)$ belongs to at most one $\mathscr{B}_i^n$ we will without loss of generality assume that the sign $\sigma$ in Proposition 2.3(d) is 1. One could produce a projection on $X$ by observing that a weak operator limit ($p > 1$) of projections on

$$\left[ z_i(B_i \cap B_s^l) / \|z_i(B_i \cap B_s^l)\| : s \in \mathscr{B}_i^l,\ i = 1, 2, \ldots \right]$$

is an isomorphism from $X$ onto its range. However this does not give us any new information. We will instead produce natural associated operators on $L_1$.

First we need to refine our family of functions a little more. For each $i$ consider the sequence $(a_{il})_{l\in M}$ where $a_{il} = \lambda(B_i \cap \bigcup\{B_s\colon s \in \mathscr{B}_i^l\})$. By passing to a subsequence we may assume that $\lim_{l\to\infty} \sum_{j=i}^{\infty} a_{jl} = a_i$ exists for $i = 1, 2, \ldots$.

Note that

$$\lim_l a_{il} = a_i - a_{i+1} > 0 \tag{11}$$

because $\|z_i(B_i \cap \bigcup\{B_s^l\colon s \in \mathscr{B}_i^l\})\| > 1 - \tilde{a}_3(\lambda_1)$ by Remark 2.4.

Define $f_l\colon [0,1] \to C[0,1]^*$ by

$$f_l(\omega) = \sum_{i=1}^{\infty} \sum_{j\in\mathscr{B}_i^l} \frac{V(z_i)(B_i \cap B_j^l)}{\|V(z_i)(B_i \cap B_j^l)\|} 1_{B_i \cap B_j^l}(\omega) \operatorname{sgn} z_i(\omega).$$

(We have omitted the $d\lambda$.) Where $Vf = |f|^{p-1}f$ as in Proposition 1.5. Because $\mathscr{B}_i^l \neq \varnothing$ for at most finitely many $i$, $f_l$ is a simple function.

LEMMA 2.5. *If $I$ is a measurable subset of* $[0,1]$, $(\int_I f_l\, d\lambda)$ *is uniformly absolutely continuous with respect to Lebesgue measure. Consequently the sequence of operators* $(T_l)$, $T_l f = \int f f_l\, d\lambda$, $f \in L_1$ *is relatively compact in the weak operator topology.*

*Proof*. Let $\varepsilon > 0$. choose $i(o)$ such that $a_{i(o)} < \varepsilon/2$. For each $i < i(o)$ there is a $\delta_i > 0$ such that if $A \subset [0,1]$ is measurable and $\lambda(A) < \delta_i$, $\int |z_i|^p(A)\, d\lambda < (\rho_i^p)\varepsilon/2i(o)$ where $\rho_i = \inf\{z_i(w)\colon w \in B_i\} > 0$, (by Lemma 2.1(a)). Now let $\delta = \min \delta_i$. If $A \subset [0,1]$ and $\lambda(A) < \delta$ then

$$\left| \left[ \int_I f_l\, d\lambda \right](A) \right| \le \sum_{i=1}^{\infty} \sum_{j\in\mathscr{B}_i^l} \frac{\lambda(B_i \cap B_j^l \cap I) \int_A |V(z_i)|(B_i \cap B_j^l)\, d\lambda}{\|V(z_i)(B_i \cap B_j^l)\|}$$

$$\le \sum_{i=i(o)}^{\infty} \lambda\left(B_i \cap \bigcup\left\{B_j^l\colon j \in \mathscr{B}_i^l\right\} \cap I\right)$$

$$+ \sum_{i=1}^{i(o)-1} \sum_{j\in\mathscr{B}_i^l} \frac{\lambda(B_i \cap B_j^l \cap I)}{\rho_i^p \lambda(B_i \cap B_j^l)} \int_{A\cap B_i \cap B_j^l} |V(z_i)|\, d\lambda$$

$$\le \sum_{i=i(o)}^{\infty} a_{il} + \sum_{i=1}^{i(o)-1} \rho_i^{-p} \int_A |V(z_i)|\, d\lambda \le \frac{\varepsilon}{2} + \sum_{i=1}^{i(o)-1} \rho_i^{-p}\left(\rho_i^p \frac{\varepsilon}{2i(o)}\right) < \varepsilon$$

for large enough $l$, proving the lemma. □

We are now in a position to prove a result of this author and W. B. Johnson [2].

COROLLARY 2.6. *There is a constant $\lambda_0$ such that if $\lambda < \lambda_0$ and $X$ is a $\mathscr{L}_{1,\lambda}$ subspace of $L_1$ then $X$ is complemented in $L_1$.*

*Proof.* With the notation above we have a sequence of operators $T_l$ on $L_1$ defined by $T_l f = \int f f_l \, d\lambda$ with $\|(T_l - I)|_{[x_k^l:\ k \in \cup \mathscr{B}_i^l]}\| \leq b(\lambda)$ where $b(\lambda) \to 0$ as $\lambda \to 1$. Indeed

$$T_l x_k^l = \sum_{i=1}^{\infty} \sum_{j \in \mathscr{B}_i^l} \int_{B_i \cap B_j^l} (\operatorname{sgn} z_i) x_k^l \, d\lambda \frac{z_i(B_i \cap B_j^l)}{\|z_i(B_i \cap B_j^l)\|}.$$

Because

$$\left\| x_k^l(A_k^l) - \frac{z_s(B_s \cap B_k^l)}{\|z_s(B_s \cap B_k^l)\|} \right\| \leq a_3(\lambda_1) \quad \text{for some } s,$$

$$\left| \int_{B_s \cap B_k^l \cap A_k^l} (\operatorname{sgn} z_s) x_k^l - 1 \right| \leq a_3(\lambda_1) \qquad (B_k^l \subset A_k^l).$$

Hence

$$\left\| T_l x_k^l - \frac{z_s(B_s \cap B_k^l)}{\|z_s(B_s \cap B_k^l)\|} \right\| \leq a_3(\lambda_1) + \left\| x_k^l(B_s \cap B_k^l)^c \right\| \leq 2a_3(\lambda_1)$$

(by Proposition 2.3(d))

and thus $\|T_l x_k^l - x_k^l\| \leq 4a_3(\lambda_1)$, proving our assertion. ($(x_k^l)$ is equivalent to the uvb $l_1^d$.)

The previous lemma shows that $(T_l)$ has a weak operator limit point $T$. If $x \in X$, Proposition 2.3(c) implies that for each $l$ sufficiently large there is an element $y_l \in [x_k^l:\ k \in \cup \mathscr{B}_i^l]$ such that $\|x - y_l\| \leq 2a_3(\lambda_1)$. Therefore

$$\begin{aligned} \|Tx - x\| &\leq \overline{\lim_l} \left[ \|T_l x - T_l y_l\| + \|T_l y_l - y_l\| + \|y_l - x\| \right] \\ &\leq 8\lambda_1 a_3(\lambda_1). \end{aligned}$$

Also if $z \in \operatorname{range} T$ then $z = w \lim T_l y$ for some $y \in L_1$. Then $T_l y \in [z_i(B_i \cap B_j^l):\ j \in \cup \mathscr{B}_i^l]$ and thus $d(T_l(y), X) \leq 2\lambda_1 a_3(\lambda_1)$. Because $z \in \operatorname{co} T_l y$, $d(z, X) \leq 2\lambda_1 a_3(\lambda_1)$ as well.

It follows that if $\lambda_1$ is close enough to one $T$ is an isomorphism of $X$ onto the range of $T$ and that $Q = (T|_X)^{-1} T$ is a projection onto $X$.

Our next lemma states that an operator $T$ which is a weak operator limit of operators $T_l$ on $L_1$ is representable in the sense of Proposition 1.1 and that the vector-valued function is an average of the $f_l$.

LEMMA 2.7. *Suppose that $(T_l)$ is a sequence of operators on $L_1$ of the form $T_l f = \int f f_l \, d\lambda$, where $f_l$: $[0,1] \to C[0,1]^*$, which converges in the weak operator topology to an operator $T$. Then there is a function $g$: $[0,1] \to C[0,1]^*$ such that $Tf = \int fg \, d\lambda$ for all $f \in L_1$ and necessarily*

$$\int_A g \, d\lambda = w \lim \int_A f \, d\lambda, \quad \textit{for all measurable sets } A.$$

We omit the proof of this lemma which follows from general representation theorems, [**6**]. Let us remark that it is also possible to prove this lemma by considering the functions $g_j = \sum_{k=1}^{2^j} [\lim_l \int_{I_{jk}} f_l \, d\lambda] 1_{I_{jk}}$ where $I_{jk}$ is the $k$th dyadic interval of length $2^{-j}$, and then using the Martingale Convergence Theorem as we did in the proof of Proposition 1.1.

Next we will show that if $X$ is a $\mathscr{L}_{p,\lambda}$ space for $p > 1$ the arguments used in Corollary 2.6 also apply to $V(X)$.

PROPOSITION 2.7. *Let $T_l f = \int f f_l \, d\lambda$ for all $f \in L_1$, $l \in M$, and let $T$ be a weak operator limit of $T_l$. Then there is an infinite subset $K \subset M$ such that*

(i) *for each $l \in K$*

$$\left\| (I - T)|_{[V(x_s^m): s \in \cup\{\mathscr{B}_j^m: j=1,2,\ldots\}]} \right\| < a_4(\lambda_1) \quad \textit{for all } m \in K,\ m \le l$$

*and $T_l|_{[V(x_s^l): s \in \cup\{\mathscr{B}_j^l: j=1,2,\ldots\}]}$ is an isomorphism onto* range $T_l$ *with $\|T_l^{-1}\| \le 1 + a_4(\lambda_1)$, if $\lambda_1$ is close enough to one.*

(ii) $\|(I - T)|_{\text{range}\, T}\| < a_4(\lambda_1)$

(iii) *for all $l \in K$,*

$$\left\| (I - T)|_{[V(x_s^l): s \in \cup\{\mathscr{B}_j^l: j=1,2,\ldots\}]} \right\| < a_4(\lambda_1)$$

(iv) $a_4(\lambda_1) \to 0$ *as* $\lambda_1 \to 1$.

*Proof*. Fix $l \in M$ and note that

$$\frac{V(x_s^l)(B_s^l \cap B_j)}{\left\| V(x_s^l)(B_s^l \cap B_j) \right\|}, \qquad s \in \cup \mathscr{B}_j^l$$

is 1-equivalent to uvb $l_1^r$ where $r$ is the cardinality of $\cup_j \mathscr{B}_j^l$ and thus

$(V(x_s^l))$ is also equivalent to uvb $l_1^r$. Suppose that $s \in \mathscr{B}_t^l$. (Here $\|\cdot\|$ is the $L_1$ norm.)

$$\begin{aligned}
\|V(x_s^l) - T_l V(x_s^l)\| &= \left\| V(x_s^l) - \int f_l V(x_s^l)\, d\lambda \right\| \\
&\leq \left\| V(x_s^l) - \frac{V(z_t)(B_t \cap B_s^l)}{\|V(z_t)(B_t \cap B_s^l)\|} \int_{B_t \cap B_s^l} V(x_s^l) \operatorname{sgn} z_t \, d\lambda \right\| \\
&\quad + \sum_{i \neq t} \sum_{j \in \mathscr{B}_i^l} \|V(x_s^l)(B_i \cap B_j^l)\| + \sum_{\substack{j \in \mathscr{B}_t^l \\ j \neq s}} \|V(x_s^l)(B_i \cap B_j^l)\| \\
&\leq \left\| V(x_s^l) - \frac{V(z_t)(B_t \cap B_s^l)}{\|V(z_t)(B_t \cap B_s^l)\|} \right\| \\
&\quad + \left| 1 - \int_{B_t \cap B_s^l} V(x_s^l) \operatorname{sgn} z_t \, d\lambda \right| + \|V(x_s^l)(B_t \cap B_s^l)^c\| \\
&\leq 4\rho(a_3(\lambda_1))
\end{aligned}$$

where $\rho$ is the modulus of continuity of $V$ on $B_{L_p}$.

Because $V(x_s^l)$ is $(1 - \rho(a_3(\lambda_1)))^{-1}$ equivalent to the uvb $l_1^m$ we have that

$$\|y - T_l(y)\| \leq (1 - \rho(a_3(\lambda_1)))^{-1} \rho(a_3(\lambda_1)) \|y\|$$
$$\text{for any } y \in \left[ V(x_s^l) \colon s \in \bigcup \{ \mathscr{B}_j^l,\ j = 1, 2, \dots \} \right].$$

Thus if $\lambda_1$ is close enough to one $T_l|_{[V(x_s^l)\colon s \in \bigcup \mathscr{B}_j^l]}$ is an isomorphism. The other assertion in (i) follows from Proposition 2.3(c) and the uniform continuity of $V$ if we choose a suitable subset $K$ of $M$. Indeed, if $x_s^m = \sum a_j x_j^l$

$$\begin{aligned}
\|V(x_s^m) - \textstyle\sum a_j V(x_u^l)\| &\leq \rho\left( \|x_s^m - \textstyle\sum a_j x_j^l(A_j^l)\| \right) \\
&\quad + \left\| \textstyle\sum a_j \left[ V(x_j^l) - V(x_j^l(A_j^l)) \right] \right\|.
\end{aligned}$$

Now use (c) to make the first term small. The inequality for the span follows from the fact that $(V(x_s^m))$ is equivalent to uvb $l_1^q$ where $q$ is the cardinality of $\bigcup \mathscr{B}_j^m$.

The proofs of (ii) and (iii) are similar to the last part of the proof of Corollary 2.6 and we leave the details to the reader. $\square$

We are now ready to prove our nonlinear factorization theorem.

THEOREM 2.8. *For each* $p$, $1 < p \neq 2 < \infty$, *there is a* $\lambda_p$ *such that if* $X$ *is a* $\mathscr{L}_{p,\lambda}$ *subspace of* $L_p$, $\lambda < \lambda_p$, *then there is a linear projection* $Q$ *defined on* $L_1$ *and a continuous nonlinear projection* $W$ *from* $L_p$ *onto* $X$ *such that*

(i) $\|UQVx - x\| < a_5(\lambda)\|x\|$ *for all* $x \in B_X$, $a_5(\lambda) \to 0$ *as* $\lambda \to 1$

(ii) $W = PUQV(PUQV|_X)^{-1}$

*where*

$$Uf = |f|^{(1-p)/p}f, \qquad Vf = |f|^{p-1}f,$$

*and* $P$ *is a bounded linear projection of* $L_p$ *onto* $X$.

(iii) $\|P\| < 1 + a_6(\lambda)$, $\|Q\| < 1 + a_6(\lambda)$ *where* $a_6(\lambda) \to 0$ *as* $\lambda \to 1$.

*Proof*. Proposition 2.7(ii) implies that if $Y = \operatorname{range} T$, $Q = (T|_Y)^{-1}T$ is a projection onto $Y$, if $\lambda$ is close enough to 1.

If $x \in X$, $\|x\| = 1$ then for $l \in K$ sufficiently large there is a $z \in [V(x_s^l)\colon s \in \bigcup \mathscr{B}_j^l]$, $\|z\| = 1$ such that $\|V(x) - z\| < 4(1 - \rho(a_3(\lambda_1)))^{-1} = b(\lambda_1)$.

By Proposition 2.7(iii),

$$\begin{aligned}\|V(x) - QV(x)\| &\leq \|V(x) - z\| + \|z - Qz\| + \|Qz - QV(x)\| \\ &\leq b(\lambda_1) + \|z - Tz\| + \left\|Tz - \left(T_{|_Y}^{-1}\right)Tz\right\| + \|Q\|b(\lambda_1) \\ &\leq b(\lambda_1) + a_4(\lambda_1) + \frac{1}{1 - a_4(\lambda_1)}(a_4(\lambda_1) + b(\lambda_1)) = b_1(\lambda_1)\end{aligned}$$

and thus

$$\|x - UQVx\| = \|UV(x) - UQVx\| \leq \delta(b_1(\lambda_1))$$

where $\delta$ is the modulus of continuity of $U$ on $B_{L_1}$. Now observe that $UQV$ is positive homogeneous. Indeed, if $\gamma > 0$, $UQV\gamma x = UQ\gamma^p Vx = U\gamma^p QVx = \gamma UQVx$. Hence for any $x \in X$

$$\|x - UQVx\| \leq \delta(b_1(\lambda_1))\|x\|.$$

Because $X$ is a $\mathscr{L}_{p,\lambda}$ subspace of $L_p$ and $\lambda$ is close to one there is a projection $P$ of $L_p$ onto $X$ with $\|P\| \leq 1 + b_2(\lambda)$ where $b_2(\lambda) \to 0$ as $\lambda \to 1$.

Hence

$$\|x - PUQVx\| = \|P(x - UQVx)\| \leq (1 + b_2(\lambda))\delta(b_1(\lambda_1))\|x\|$$

and $I - PUQV$ maps $X$ to itself. Thus if $\lambda$ (and thus $\lambda_1$) is close enough to one, $PUQV$ has a continuous inverse $(\sum_{n=0}^{\infty}(I - PUQV)^n = (PUQV)^{-1}$converges uniformly on bounded sets.) □

Next we want to observe that the results of Zippin **[13]** and Dor **[5]** follow from our results.

COROLLARY 2.9. *For each $p$, $1 \leq p < \infty$, $p \neq 2$, there is a constant $\lambda_p$ such that if $X$ is a $\mathcal{L}_{p,\lambda}$ subspace of $l_p$, $\lambda < \lambda_p$, then $X$ is isomorphic to $l_p$ and $d(X, l_p) \to 1$ as $\lambda \to 1$.*

*Proof*. We may assume that $X \subset L_p(\mathscr{B}_1)$ where $\mathscr{B}_1$ is a purely atomic sub $\sigma$-algebra of $\mathscr{B}$. We may also assume that the sets in the conclusion of Proposition 2.3 are all $\mathscr{B}_1$ measurable and thus the operators $T_l$ have range in $L_1(\mathscr{B}_1) = l_1$. By Lemma 2.5, $(T_l)$ has a $w$ operator limit $T$. Because $l_1$ has the Schur property **[6]**, p. 295, $T$ is also a strong operator limit. Each of the operators $T_l$ is a contractive projection and therefore $T$ is a contractive projection as well ($(T_l^2)$ converges to $T^2$). Hence by Theorem 0.1 the range of $T$ is isometric to $l_1$ and is of the form

$$\{f \cdot h\colon f \in L_1(\mathscr{B}_2, |h|\, d\lambda)\} \quad \text{where } \mathscr{B}_2 \subset \mathscr{B}_1 \text{ and } \mathscr{E}(\,|h|\,|\mathscr{B}_2) = 1_{\operatorname{supp} h}.$$

Now observe that $z = U(\text{range}\, T)$ is a closed subspace of $L_p(\mathscr{B}_1, \lambda)$ which is isometric to $l_p$. Moreover the projection $P$ as in the previous theorem must be an isomorphism of $Z$ onto $X$. □

Our final result is that the solution of our problem for all $p$ would be consequence of a solution for any $p$.

PROPOSITION 2.10. *Suppose that for some $p$, $1 \leq p < \infty$, $p \neq 2$, there is a function $\rho(\lambda)$: $[1, 1 + \varepsilon] \to \mathbf{R}$, $\lim_{\lambda \to 1} \rho(\lambda) = 1$, such that if $X$ is a $\mathcal{L}_{p,\lambda}$ subspace of $L_p$, $\lambda \leq 1 + \varepsilon$, then there is a measure $\nu$ such that $d(X, L_p(\nu)) < \rho(\lambda)$. Then for each $r$, $1 \leq r < \infty$, $r \neq 2$, the same result holds.*

*Proof*. We will show first that if the hypothesis holds for $p > 1$, the result holds for $r = 1$. Let $X$ be a $\mathcal{L}_{1,\lambda}$ subspace of $L_1$. Let $(z_i)$, $(B_i)$ and $(B_s^l)$ be as in the conclusion of Proposition 2.3. Consider the projections on $L_p$ defined by

$$S_l f = \sum_{i=1}^{\infty} \sum_{s \in \mathscr{B}_i^l} \int_{B_i \cap B_s^l} |z_i|^{p-2} z_i f \, d\lambda \frac{z_i(B_i \cap B_s^l)}{\| z_i(B_i \cap B_s^l) \|_p}.$$

Each $S_l$ is a contractive projection and because $p > 1$ we may assume that $(S_l)$ converges in the weak operator topology to an operator $S$ on $L_p$.

We claim that $\|(S - I)|_{\operatorname{range} S}\| < c(\lambda)$ where $c(\lambda) \to 0$ as $\lambda \to 1$. We will omit the proof of this claim because the proof is similar to that of Proposition 2.7.

It now follows that if $c(\lambda)$ is sufficiently small that $Q = (S|_{\operatorname{range} S})^{-1} S$ is a projection onto $Y = \operatorname{range} S$ and that $Y$ is a $\mathscr{L}_{p,\alpha}$ subspace of $L_p$ and $\alpha = \alpha(\lambda) \to 1$.

If $\lambda$ is close enough to one then $d(Y, L_p(\nu)) < \rho(\alpha)$ for some $\nu$. Moreover there is a subspace $Z$ of $L_p$ such that $Z$ is isometric to $L_p(\nu)$ and $Z$ is a perturbation of $Y$, [**1**]. Hence $V(Z)$ is isometric to $L_1(\nu)$ and is a perturbation of $X$.

Now assume that the hypothesis holds for $p = 1$ and that $X$ is a $\mathscr{L}_{r,\lambda}$ subspace of $L_r$. By Theorem 2.8 there is a projection $Q$ on $L_1$ with $\operatorname{range} Q$ a $\mathscr{L}_{1,\alpha(\lambda)}$ subspace of $L_1$ where $\alpha(\lambda) \to 1$ as $\lambda \to 1$. If $\alpha(\lambda)$ is small enough there is a subspace $Z$ of $L_1$ which is isometric to $L_1(\nu)$ for some measure $\nu$ and which is a perturbation of $\operatorname{range} Q$. It follows that $U(Z)$ is isometric to $L_r(\nu)$ and is a perturbation of $X$. □

**3. Final remarks and open questions.** The results of the last section provide some evidence for the conjecture that $\mathscr{L}_{p,\lambda}$ spaces are $L_p$ spaces for $\lambda$ small enough and suggest that $\mathscr{L}_{1,\lambda}$ spaces may be the easiest point to begin. The results of section one indicate that it may be possible to obtain results in this case by examining the representing functions.

*Question* 1. Suppose $(E_i)$ is a sequence of conditional expectation operators, i.e., $E_i = \mathscr{E}(\cdot|\mathscr{G}_i)$, and $\text{w.op}\lim_i E_i = T$. What conditions on $(E_i)$ will guarantee that there is a conditional expectation operator $E$ with $\|E - T\|$ small? In particular, what if $\|T - P\|$ is small for a projection $P$?

This is a special case of what we need to solve the general problem. However, it may be that an extreme point argument, applied to the representing functions would be an approach.

*Question* 2. If $X$ is a $\mathscr{L}_{1,\lambda}$ space, $\lambda$ near one, is there a sequence of conditional expectation operators $(E_i)$ such that $X$ is isomorphic to $\operatorname{range} w \text{ op} \lim E_i$, i.e., can the problem be reduced to Question 1?

Proposition 2.3 almost proves this. We have there reduced the "shapes" to that of the $z_i$'s but we would need either a single function $z$ or equivalently to have *disjoint* sets $B_i$ as in Proposition 2.3 to obtain this result. At first glance it might seem that

$$z = w \lim \sum_{i=1}^{\infty} \sum_{s \in \mathscr{B}_i^l} \lambda\left(B_i \cap B_s^l\right) \frac{z_i\left(B_i \cap B_s^l\right)}{\left\| z_i\left(B_i \cap B_s^l\right) \right\|}$$

would be satisfactory. However trivial sign changes could result in $z$ vanishing on important sets and even the weak limit of the absolute values does not seem to guarantee that the span of $(z(B_i \cap B_s^l))$ is appropriate. Again what we need is not an average but an extremal element. Another equivalent viewpoint on this question and Proposition 2.3, is that we need to decompose the $\mathscr{L}_{1,\lambda}$ space into an $l_1$ sum of subspaces each with an appropriate $z_i$.

Finally let us note that it may be possible to obtain at least a qualitative result by showing that if $X$ is not isomorphic to $l_1$ then $[z_i(B_i \cap B_s^l)\colon s \in \mathscr{B}_i^l,\ l = 1, 2, \ldots]$ must contain an isomorph of $L_1$ for some $i$ and thereby avoid the decomposition difficulty.

*Added in proof*. Proposition 1.5 is essentially due to Mazur. S. Mazur, *Une remarque sur l'homéomorphism de champs fonctionnels*, Studia Math., **1** (1930), 83–85.

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