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APPROXIMATING CODIMENSION TWO EMBEDDINGS OF CELLS

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It is shown that every topological embedding of a k-cell into a piecewise linear $(k+2)$-manifold can be arbitrarily closely approximated by locally flat piecewise linear embeddings. The new ingredient in the proof is an ε-controlled piping lemma.

Introduction. The main result of this paper is the following.

THEOREM 1. *Let $h: D^k \to M^{k+2}$ be a topological embedding of the k-cell D^k into a piecewise linear $(k+2)$-manifold M^{k+2}. Then for every $\varepsilon > 0$ there exists a locally flat piecewise linear embedding $g: D^k \to M^{k+2}$ such that $d(h(x), g(x)) < \varepsilon$ for each $x \in D^k$.*

Approximation theorems of this kind are already known for embeddings of cells in codimensions ≥ 3 [**2**] and in case $k = 2$ [**4**]. In addition, the case $k = 3$ was announced in [**6**]. Here the techniques of [**2**] and [**4**] are combined to obtain Theorem 1. By [**4**, Corollary 1], it is enough to consider only the case $M = R^{k+2}$ and so that is the only case which will be mentioned in the remainder of this paper.

The proof given in this paper is modelled on Miller's proof in [**2**]. We must add a new ingredient to push the technique up to codimension two, but our proof does not give a new proof of Miller's result; it would reduce to exactly his proof in codimension three. The new proof is somewhat different from that in [**4**] in the case $k = 2$.

The new ingredient needed to win the extra dimension is an ε-controlled piping lemma. It is commonly known that piping (in the sense of Zeeman [**7**, Lemma 48]) requires a global rather than local move; e.g., see the Remark in [**7**], Chapter VII, p. 45. However, in the presence of the topological embedding h, we are able to achieve that global modification by means of a homotopy which moves each point only a small distance. That requires more elaborate geometric constructions than those used by Zeeman. The geometry of the two kinds of piping is described at the beginning of §2.

There are actually two piping lemmas in this paper. The first is an ε-controlled version of Zeeman's Lemmas 48 and 49. The second involves piping more like that used by Zeeman in the proof of his unknotting theorem (see [7], Chapter VIII, Lemma 65). Since the point of these lemmas is that ε-control can be added to Zeeman's piping technique, we spell out very carefully in the proof of Lemma 1 just how all the ε's and δ's work.

Recently Montejano [3] has generalized Miller's technique and stated the approximation theorem in terms of liftings. The same could be done here.

There are many open questions remaining in the study of topological embeddings in codimension two. In particular, it is not known whether D^k in Theorem 1 can be replaced by some other k-manifold (such as S^k or a k-manifold with only handles of index ≤ 1, etc.).

Finally, I would like to thank Mike Starbird for explaining [2] to me in an unusually clear and beautiful way. I also wish to thank Luis Montejano for listening patiently to the proofs of the piping lemmas and for offering many helpful suggestions on how those proofs should be written. Thanks also to the referee for pointing out several inconsistencies in the first write-up of the proof of Lemma 3.

1. Preliminaries. We begin with some definitions and notation, most of which are found in [2] and [7]. The reader is warned, however, that there are minor differences between the terminology used here and that in [2]. We use [7] as a general reference for PL topology.

For each positive integer k, let R^k denote k-dimensional Euclidean space and let d be the usual distance function for R^k. If $X \subset R^k$ and $\varepsilon > 0$, then $N_\varepsilon(X) = \{y \in R^k \mid d(y, X) < \varepsilon\}$. Let $I = [0,1]$. The k-cell D^k is defined by $D^k = \{(x_1, \ldots, x_k) \in R^k \mid x_i \in I \text{ for each } i\}$. For $1 \leq j \leq k$ and $0 \leq a \leq b \leq 1$, we define $D_j^k[a, b]$ to be $\{(x_1, \ldots, x_k) \in D^k \mid a \leq x_j \leq b\}$ and $D_j^k[a]$ to be $D_j^k[a, a]$.

In the rest of this paper, assume that k is a fixed positive integer and that a topological embedding $h: D^k \to R^{k+2}$ has been given.

For each j, $1 \leq j \leq k$, we define two homotopies. First θ_t^j: $D^k \to D^k$, $0 \leq t \leq 1$, is the deformation retraction defined by

$$\theta_t^j(x_1, \ldots, x_k) = \begin{cases} (x_1, \ldots, x_k) \text{ if } x_j \geq t, \quad \text{and} \\ (x_1, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_k) \quad \text{if } x_j \leq t. \end{cases}$$

Notice that θ_t^j deforms D^k to its jth face, $D_j^k[1]$. We use the notations $\theta_t^j(x)$ and $\theta^j(x, t)$ interchangeably.

As in [2], the fact that $h(D^k)$ is an ANR can be used to find a neighborhood N of $h(D^k)$ and a retraction $r\colon N \to h(D^k)$. We may assume that $N \subset N_1(h(D^k))$. Define $r_t(x)$ to be the homotopy which moves x at constant speed along the straight line from x to $r(x)$ during the time interval $0 \le t \le d(x, r(x))$ and remains fixed for other values of t. Now define a second homotopy, $\psi_t^j\colon N \to R^{k+2}$, $0 \le t \le 1$, by

$$\psi_t^j(x) = \begin{cases} r_t(x) & \text{if } 0 \le t \le d(x, r(x)) \\ h\theta^j\left(h^{-1}r(x), \dfrac{t - d(x, r(x))}{1 - d(x, r(x))}\right) & \text{if } d(x, r(x)) \le t \le 1. \end{cases}$$

The important feature of ψ_t^j is that ψ_t^j extends $h\theta_t^j h^{-1}$ to all of N.

Suppose X and Y are polyhedra and that X simplically collapses to Y via a collapse ξ. (Write $\xi\colon X \searrow Y$.) We think of ξ as specifying not only which elementary simplicial collapses are to be done, but also the timing of those collapses during the time interval $0 \le t \le 1$. The collapse ξ then induces a strong deformation retraction ξ_t of X to Y in a natural way (see [2], p. 408). Given $\xi\colon X \searrow Y$ and a subpolyhedron Z of X, we will consider the following subpolyhedron of X: $\text{Trail}_\xi(Z) = \xi(Z \times I)$.

Let $f\colon K \to R^n$ be a piecewise linear map. The *singular set* of f is defined by $S(f) = \overline{\{y \in K \mid f^{-1}f(y) \neq \{y\}\}}$. Observe that if $\xi\colon K \searrow L$ is a collapse, then $f(K) \searrow f(L) \cup f(\text{Trail}_\xi S(f))$. The *support* of a map f is defined by $\text{supp}(f) = \overline{\{x \mid f(x) \neq x\}}$.

2. The first piping lemma. In this section we prove an ε-controlled version of Zeeman's Piping Lemma. Piping is a technique in which part of the singular set of a homotopy is pushed off the edge of the track of the homotopy. The important consequence is that a hole is punched in each top dimensional simplex of the singular set, allowing us to collapse out things below that simplex. Figure 0 illustrates the usual Zeeman Piping procedure.

The usual procedure is not good enough for our purposes here because the pipe must, in general, be very long and thus the modification moves some points too far. We will describe a more elaborate procedure which produces the same holes in the singular set but does so without moving any point very far. The reader can refer ahead to Figures 4 and 5 for pictures of the kind of modification we will make. Since the horizontal distance in Figure 4 can be made arbitrarily small, no point is moved very far by the push pictured there.

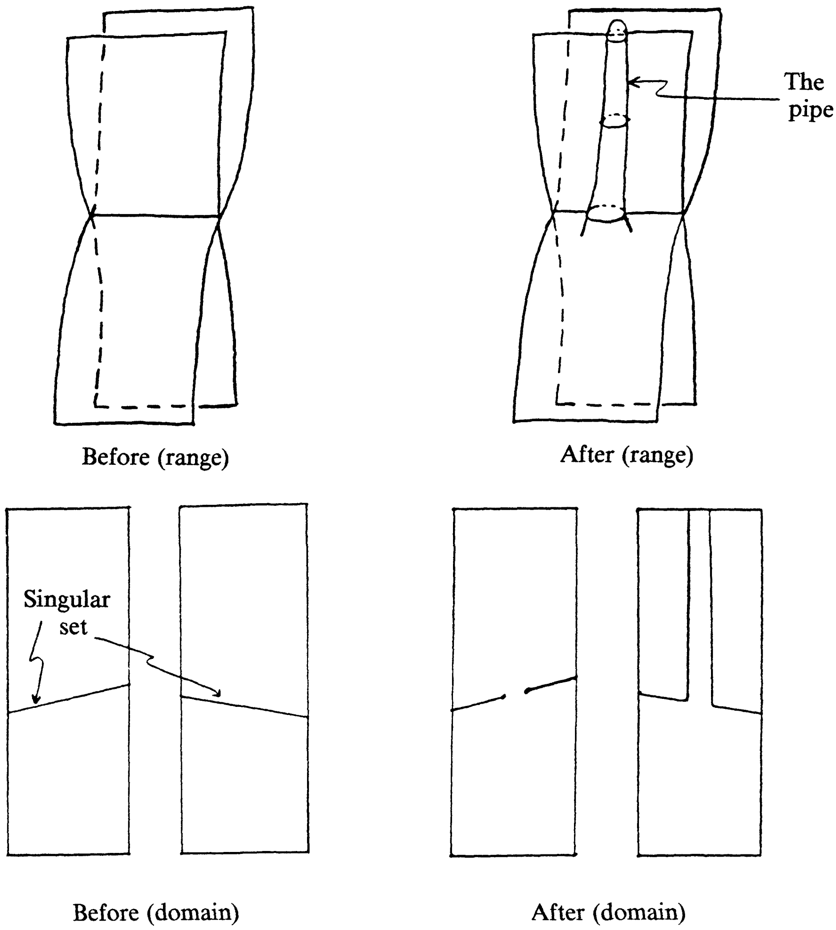


FIGURE 0

The reader is warned that the proof of Lemma 1 should be understood thoroughly before the proofs of the subsequent lemmas are attempted. All the details of the epsilonics are spelled out in the proof of Lemma 1 but are left out of the proofs of the later lemmas whenever they are essentially the same as those in earlier proofs.

We begin with two more definitions.

As stated above, the positive integer k and the topological embedding $h\colon D^k \to R^{k+2}$ are fixed. For each $x \in h(D^k)$ and $1 \le j \le k$, we define the *fiber through x in the jth direction* to be $F_j(x) = \{y \in h(D^k) \mid h^{-1}(x)$ and $h^{-1}(y)$ differ in at most their jth coordinates$\}$.

Suppose Y is a polyhedron in N, Z and C are subpolyhedra of Y and $\varepsilon > 0$. We say that a collapse $\xi\colon Y \searrow Z \cup C$ is a *(j, ε, Z)-collapse* if

(a) $C \subset N_\varepsilon(h(D_j^k[1]))$,

(b) $\xi_t(Y) \subset N_\varepsilon(h(D_j^k[t, 1])) \cup Z$ for all $t \in I$,

(c) supp $\xi_t \subset N_\varepsilon(h(D_j^k[0, t]))$ for all $t \in I$, and

(d) for each $y \in Y$ there exists one $x \in h(D^k)$ such that $\xi_t(y) \in N_\varepsilon(F_j(x))$ for all $t \in I$.

LEMMA 1. *For every $\varepsilon > 0$ and $1 \leq j \leq k$ there exists a $\delta > 0$ such that if X is a compact polyhedron in $N_\delta(h(D_j^k[0]))$ with* $\dim X = k - 1$, *then there exists a PL map $f\colon X \times I \to N_\varepsilon(h(D^k))$ and a subpolyhedron J_0 of $X \times I$ such that*

(1) *$f(x, 0) = x$ for each $x \in X$,*

(2) *$f(x, t) \in N_\varepsilon(h(D_j^k[t])) \cap N_\varepsilon(F_j(r(x)))$ for each $(x, t) \in X \times I$,*

(3) $\dim J_0 \leq k - 2$, *and*

(4) *there is a $(j, \varepsilon, f(J_0))$-collapse $\xi\colon f(X \times I) \searrow f(X \times \{1\}) \cup f(J_0)$.*

Proof. The lemma is trivial for $k \leq 1$, so we assume that $k \geq 2$. Begin by choosing a number $\alpha > 0$ such that if $t_1, t_2 \in I$ and $|t_1 - t_2| < \alpha$, then $h(D_j^k[t_1]) \subset N_{\varepsilon/3}(h(D_j^k[t_2]))$. Let $0 = t_0 < t_1 < t_2 < \cdots < t_n = 1$ be a partition of $[0, 1]$ with $|t_i - t_{i-1}| < \alpha$ for each i. Choose $\beta > 0$ such that if A is any subset of $N \times I$ of diameter $< \beta$, then $\operatorname{diam} \psi^j(A) < \varepsilon/3$.

The most important conclusion of the lemma is (3). To illustrate that point, we quickly choose a number δ and use it to get conclusions (1), (2), and (4) as stated. However, in this first attempt we are off by one dimension from achieving conclusion (3). The remainder of the proof then consists of showing how to choose δ more carefully and how to modify the easy first attempt to lower the dimension of J_0 by 1 and get conclusion (3) as well.

Choose $\delta > 0$ so small that $\psi_t^j(x) \in N_{\varepsilon/3}(h(D_j^k[t]))$ and $d(x, r(x)) < \varepsilon/3$ for every $x \in N_\delta(h(D_j^k[0]))$ and for every $t \in I$. Suppose $X \subset N_\delta(h(D_j^k[0]))$ is a compact polyhedron of dimension $k - 1$. Let $f\colon X \times I \to N_\varepsilon(h(D^k))$ be a PL, general position $(\varepsilon/3)$-approximation to $\psi^j \mid X \times I$, keeping $X \times \{0\}$ fixed. Conclusions (1) and (2) of the Lemma are then obvious.

Triangulate $X \times I$ with a cylindrical triangulation which has mesh less than β and includes each of the sets $X \times \{t_i\}$, $i = 0, 1, \ldots, n$ and $S(f)$ as subcomplexes. Let $\mu\colon X \times I \searrow X \times \{1\}$ be a cylindrical collapse of that triangulation, timed so that all the simplices in $X \times [t_{i-1}, t_i]$ are collapsed out during the time interval $t_{i-1} \leq t \leq t_i$. Let $J_0 = \operatorname{Trail}_\mu(S(f))$. Define $\xi\colon f(X \times I) \searrow f(X \times \{1\}) \cup f(J_0)$ to be the collapse induced by μ.

We claim that ξ is a $(j, \varepsilon, f(J_0))$-collapse. It is necessary to check conditions (a)–(d).

(a) $C = f(X \times \{1\})$ is contained in $N_\varepsilon(h(D_j^k[1]))$ by the choice of δ.

(b) Let $t \in [0,1]$. Then $t \in [t_{i-1}, t_i]$ for some i. Now $\mu_t(X \times I) \subset X \times [t_{i-1}, 1]$, so

$$\begin{aligned}\xi_t(f(X \times I)) &\subset f(X \times [t_{i-1}, 1]) \cup f(J_0)\\ &\subset N_{\varepsilon/3}(\psi^j(X \times [t_{i-1}, 1])) \cup f(J_0)\\ &\subset N_{2\varepsilon/3}\big(h\big(D_j^k[t_{i-1}, 1]\big)\big) \cup f(J_0)\\ &\subset N_\varepsilon\big(h\big(D_j^k[t, 1]\big)\big) \cup f(J_0).\end{aligned}$$

(c) Let $t \in [0,1]$. Again $t \in [t_{i-1}, t_i]$ for some i. Since $\operatorname{supp} \mu_t \subset X \times [0, t_i]$, we must have

$$\begin{aligned}\operatorname{supp}(\xi_t) &\subset f(X \times [0, t_i])\\ &\subset N_{\varepsilon/3}(\psi^j(X \times [0, t_i]))\\ &\subset N_{2\varepsilon/3}\big(h\big(D_j^k[0, t_i]\big)\big)\\ &\subset N_\varepsilon\big(h\big(D_j^k[0, t]\big)\big).\end{aligned}$$

(d) Let $y \in f(X \times I)$. Then $y = f(x, s)$ for some $(x, s) \in X \times I$. There is a simplex σ of X such that $x \in \sigma$. Because μ is a cylindrical collapse, $\xi_t(y) \in f(\sigma \times I)$ for every t. Thus

$$\begin{aligned}\xi_t(y) &\in f(\sigma \times I)\\ &\subset N_{\varepsilon/3}(\psi^j(\sigma \times I))\\ &\subset N_{2\varepsilon/3}(\psi^j(\{x\} \times I)) \quad \text{(by the choice of } \beta)\\ &\subset N_\varepsilon\big(F_j(r(x))\big).\end{aligned}$$

Hence f, J_0 satisfy all the conclusions of the Lemma except (3). The best we can hope for is that, for J_0 as defined above,

$$\begin{aligned}\dim J_0 &\le \dim S(f) + 1\\ &\le 2[(k-1)+1] - (k+2) + 1 = k - 1.\end{aligned}$$

As is usual in a piping argument, we want to homotope f around in such a way that a hole is punched in each top-dimensional simplex of $S(f)$, allowing us to collapse out what is below that simplex without running into $S(f)$. That will require a more complicated choice of δ.

The choice of δ.

We choose δ inductively.

First choose δ_0 so that $0 < \delta_0 \leq \varepsilon/3$ and $N_{\delta_0}(h(D_j^k[0, t_{i-1}])) \cap N_{\delta_0}(h(D_j^k[t_i, 1])) = \varnothing$ for each $i = 1, \ldots, n$.

Next choose δ_1, $0 < \delta_1 \leq \delta_0$, such that if $x_1, x_2 \in h(D^k)$ and $N_{\delta_1}(F_j(x_1)) \cap N_{\delta_1}(F_j(x_2)) \neq \varnothing$, then $N_{\delta_1}(F_j(x_1)) \subset N_{\delta_0}(F_j(x_2))$. (The existence of δ_1 follows from the uniform continuity of h and h^{-1}.)

Now for the induction. Suppose $\delta_0, \delta_1, \ldots, \delta_i$ have been chosen, $1 \leq i \leq n-1$. Choose numbers $s_i^i, s_{i+1}^i, \ldots, s_{n-1}^i$ such that $t_i < s_i^i < t_{i+1} < s_{i+1}^i < \cdots < t_{n-1} < s_{n-1}^i < t_n = 1$ and such that $h(D_j^k[t_q, s_q^i]) \subset N_{\delta_i}(h(D_j^k[t_q]))$ for $q = i, i+1, \ldots, n-1$. Let $\eta_i \leq \delta_i$ be a positive number so small that the sets $N_{\eta_i}(h(D_j^k([0, t_i]))$ and $\{N_{\eta_i}(h(D_j^k[s_q^i, t_{q+1}]))\}_{q=i}^{n-1}$ are pairwise disjoint. We also make η_i small enough so that $N_{\eta_i}(h(D_j^k[t_q, s_q^i])) \subset N_{\delta_i}(h(D_j^k[t_q]))$ for $q = i, i+1, \ldots, n-1$. By [5, Corollary 2.2], there exists a positive number γ_i with the following property: If K is any compact, 1-dimensional polyhedron in $N_{\gamma_i}(h(D_j^k[s_q^i, t_{q+1}]))$ for some $q \geq i$, then there is a PL ambient isotopy H_t such that

(i) $H_0 = \text{id}$,

(ii) $\operatorname{supp} H_t \subset N_{\eta_i}(h(D_j^k[s_q^i, t_{q+1}])) - N_{\gamma_i}(h(D_j^k[0, s_q^i]))$,

(iii) $H_1(K) \subset N_{\eta_i}(h(D_j^k[s_q^i]))$, and

(iv) $H_t(N_\gamma(F_j(x))) \subset N_{\eta_i}(F_j(x))$ for every $t \in I$ and $x \in h(D^k)$.

Moreover, there exists a PL map g: $K \times I \to N_{\eta_i}(h(D_j^k[s_q^i, t_{q+1}]))$ such that H_t can be chosen to have its support in an arbitrarily small neighborhood of $g(K \times I)$.

Now choose δ_{i+1}, $0 < \delta_{i+1} \leq \gamma_i$ such that if $x_1, x_2 \in y(D^k)$ and $N_{\delta_{i+1}}(F_j(x_1)) \cap N_{\delta_{i+1}}(F_j(x_2)) \neq \varnothing$, then $N_{\delta_{i+1}}(F_j(x_1)) \subset N_{\gamma_i}(F_j(x_2))$.

After δ_0 through δ_n have been chosen inductively, we finally choose $\delta > 0$ such that $\psi_t^j(x) \in N_{\delta_n}(h(D_j^k[t])) \cap N_{\delta_n}(F_j(r(x)))$ for each $x \in N_\delta(h(D_j^k[0]))$ and for every $t \in I$.

The first approximation of f.

Suppose X is as in the statement of the lemma. Let f_n: $X \times I \to N$ be a PL, general position map which satisfies $f_n(x, 0) = x$ for each $x \in X$ and which so closely approximates $\psi^j \mid X \times I$ that $f_n(x, t) \in N_{\delta_n}(h(D_j^k[t])) \cap N_{\delta_n}(F_j(r(x)))$ for each $(x, t) \in X \times I$. Triangulate $X \times I$ with a cylindrical triangulation of mesh $< \beta$ which includes each of the sets $X \times \{t_i\}$ and $S(f_n)$ as subcomplexes. Let μ: $X \times I \searrow X \times \{1\}$ be a cylindrical collapse, timed as before. This time, let $S = S(f_n)$ and let $J_0 = \operatorname{Trail}_\mu(S^{(k-3)})$ where $S^{(k-3)}$ denotes the $(k-3)$-dimensional skeleton of S. Notice that $\dim J_0 \leq (k-3) + 1 = k-2$. We hope to homotope f_n, keeping the image of $J_0 \cup (X \times \{0, 1\})$ fixed, to a map f_0 with the property that $f_0(X \times I) \searrow f_0(X \times \{1\}) \cup f_0(J_0)$.

Let $\pi: X \times I \to X$ denote projection onto the first coordinate. We may assume that if σ is a top-dimensional simplex in S, then $\pi\sigma \cap \pi(\overline{S - \sigma}) \subset \pi(\partial\sigma)$ [7, Chapter VII, Sublemma 1]. It may also be assumed that each top-dimensional simplex σ of S is horizontal in $X \times I$; i.e., $\pi \mid \sigma$ is a homeomorphism. (Those two assumptions will require slight modifications of the map f_n. The modifications are explaned in [**1**, Theorem 2.5].)

The $(k - 2)$-dimensional simplices of S will come in pairs, $\{\sigma_1, \sigma_2\}$, with the property that each simplex in a pair is embedded and the two have the same image. Thus $f_n(\sigma_1) = f_n(\sigma_2)$, but $f_n(\text{int}\,\sigma_i) \cap f_n(\text{int}\,\sigma) = \phi$ for every $(k - 2)$-simplex σ in S with $\sigma \neq \sigma_1$ and $\sigma \neq \sigma_2$.

The two basic constructions.

Let $\{\sigma_1, \sigma_2\}$ be a matched pair of $(k - 2)$-simplices in S. The choice of δ_0 guarantees that either there is an i such that $\sigma_1 \cup \sigma_2 \subset X \times [t_{i-1}, t_i]$ or an i such that $\sigma_1 \subset X \times [t_{i-1}, t_i]$ and $\sigma_2 \subset X \times [t_i, t_{i+1}]$ (relabel if necessary). There are two different modifications we might make, depending on which of the two cases occurs.

Case 1. $\sigma_1 \cup \sigma_2 \subset X \times [t_{i-1}, t_i]$. Following the proof of Zėeman [7, Lemma 48] exactly, pipe the image of σ_1 off the t_{i-1} end of the image of $X \times [t_{i-1}, t_i]$. As long as $i > 1$, the result is a new singular set which is the same as the old one except that σ_2 is replaced by a $(k - 2)$-cell σ_2' which curves up over the t_i-level as indicated in Figure 1.

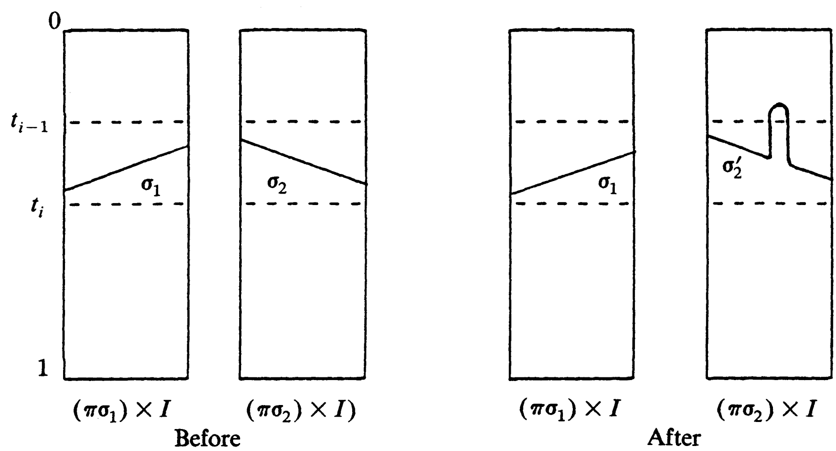


FIGURE 1

If $i = 1$, we would actually push the image of the barycenter of σ_1 completely off the end of $f_n(X \times I)$. The picture would then be as indicated in Figure 2; we would have succeeded in punching the desired holes in σ_1 and σ_2.

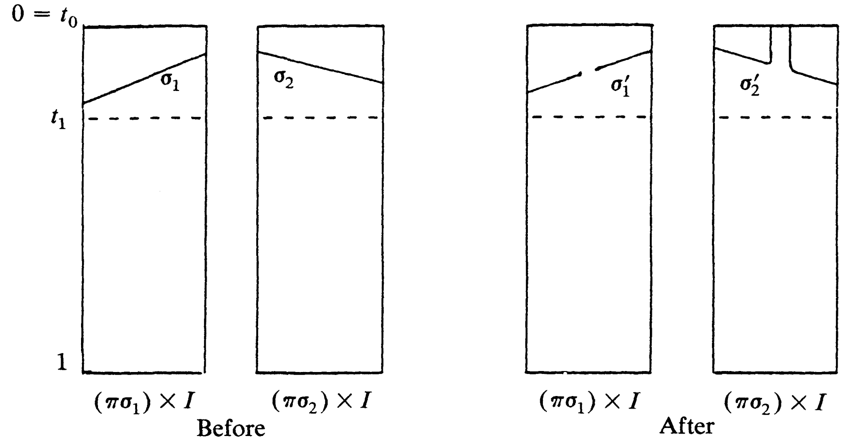


FIGURE 2

We will refer to the modification done in Case 1 as a "Type I Piping." A Type I piping is simply a Zeeman piping done to a small slice, $X \times [t_{i-1}, t_i]$, of $X \times I$.

Case 2. $\sigma_1 \subset X \times [t_{i-1}, t_i]$, $\sigma_2 \subset X \times [t_i, t_{i+1}]$. Let A be the vertical arc in $X \times I$ joining the barycenter of σ_2 to the level $X \times \{t_i\}$. Notice that both ends of $f_n(A)$ lie in $N_{\delta_n}(h(D_j^k[t_{i-1}, t_i]))$. The objective is to push all of $f_n(A)$ near $h(D_j^k[t_{i-1}, t_i])$ keeping the ends of $f_n(A)$ fixed. Suppose for the moment that that can be done and that f_n' is the map f_n followed by such a push. A new map is defined which is composed of a vertical push straight down along A in $X \times I$ followed by f_n'. The new map has exactly the same image set as f_n'; the only difference is in how that image is parametrized. Figure 3 shows what happens to the singular set. We will call this kind of modification a "Type II Piping."

The details of how we push $f_n(A)$ to $f_n'(A)$ while still maintaining ε-control over where individual points go are exactly the same as those in the proofs of [**4**, Lemma 3] and [**5**, Lemma 3.2], but we give a condensed version here as well.

Let K be a compact subplyhedron of $f_n(A)$ such that

$$f_n(A) - N_{\delta_{i+1}}\left(h\left(D_j^k\left[t_i, s_i^i\right]\right)\right) \subset K \subset N_{\delta_{i+1}}\left(h\left(D_j^k\left[s_i^i, t_{i+1}\right]\right)\right).$$

By the choice of γ_i, there exists a PL map $g\colon K \times I \to N_{\eta_i}(h(D_j^k[s_i^i, t_{i+1}]))$ such that $f_n(A)$ can be pushed into $N_{\eta_i}(h(D_j^k[t_{i-1}, t_i]))$ with an ambient isotopy having support in an arbitrarily small neighborhood of $g(K \times I)$. What we have to watch out for is that the isotopy may pull some part of $f_n(X \times [t_{i+1}, t_{i+2}])$ out of a neighborhood of $h(D_j^k[t_{i+1}, t_{i+2}])$. Put g in

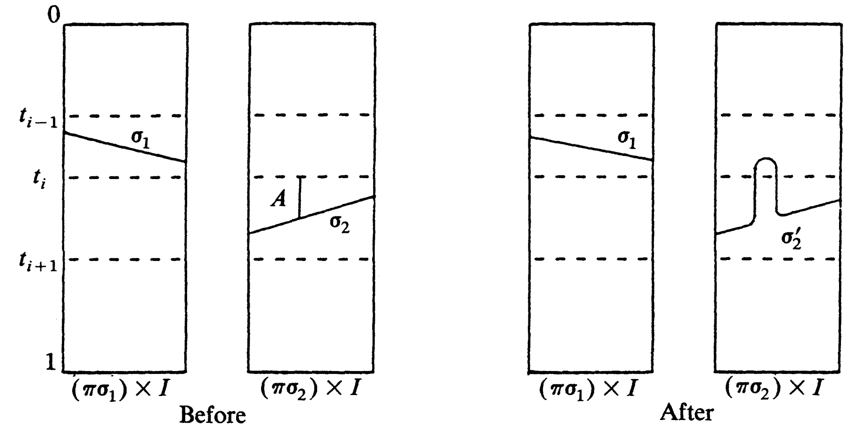


FIGURE 3

general position and consider $g(K \times I) \cap f_n(X \times [t_{i+1}, t_{i+2}])$. That intersection consists of a finite number of points. Let A_1 be the union of a finite number of vertical arcs in $X \times I$ joining those points to $X \times \{t_{i+1}\}$. Associated with A_1 there is a compact 1-dimensional polyhedron $K_1 \subset f_n(A_1)$ and a PL map g_1: $K_1 \times I \to N_{\eta_n}(h(D_j^k[s_{i+1}^i, t_{i+2}]))$ such that $f_n(A_1)$ can be pulled near $h(D_j^k[t_i, t_{i+1}])$ along $g_1(K_1 \times I)$. Next consider $g_1(K_1 \times I) \subset f_n(X \times [t_{i+2}, t_{i+3}])$—again a finite number of points. There will be a collection A_2 of vertical arcs associated with those points, a subset $K_2 \subset f_n(A_2)$ and a PL map g_2 of $K_2 \times I$, etc. The procedure is contained inductively down to $f_n(X \times [t_{n-1}, t_n])$.

There will be arcs A_{n-i-1}, a subset $K_{n-i-1} \subset f_n(A_{n-i-1})$ and a PL map g_{n-i-1}: $K_{n-i-1} \times I \to N_{\eta_i}(h(D_j^k[t_{n-1}, t_n]))$. Push $f_n(A_{n-i-1})$ across $g_{n-i-1}(K_{n-i-1} \times I)$ into a neighborhood of $h(D_j^k[t_{n-2}, t_{n-1}])$. Then push the t_{n-1}-level of $X \times I$ straight down near A_{n-i-1} so that $A_{n-i-1} \subset X \times [t_{n-2}, t_{n-1}]$. Next push $f_n(A_{n-i-2})$ across $g_{n-i-2}(K_{n-i-2} \times I)$ into a neighborhood of $h(D_j^k[t_{n-3}, t_{n-2}])$ and then push the t_{n-2}-level of $X \times I$ down near A_{n-i-2} until $A_{n-i-2} \subset X \times [t_{n-3}, t_{n-2}]$. This is continued back up the levels until eventually $A \subset X \times [t_{i-1}, t_i]$ and the Type II piping is complete.

Because the various ambient pushes have support on disjoint sets (by the choice of η_i) and because the fibers of $f_n(X \times I)$ are left setwise fixed by the vertical pushes in $X \times I$, the fibers $\{f_n(\{x\} \times I)\}$ are still where they should be. In addition, each slice $f_n(X \times [t_{q-1}, t_q])$ is still near $h(D_j^k[t_{q-1}, t_q])$, so we have the desired control over where individual points go. The details of those epsilonics are worked out in the proof of the claim below.

Construction of one pipe.

Suppose again that σ_1, σ_2 are two $(k-2)$-simplices of S which are identified under f_n. Assume for the moment also that $\sigma_1 \cup \sigma_2 \subset X \times [t_{i-1}, t_i]$, $i > 1$. One Type I piping results in a new singular set which includes σ_1 but has a new $(k-2)$-cell σ_2^1 in place of σ_2. The boundary of σ_2' is in $X \times [t_{i-1}, t_i]$, but the center lies in $X \times [t_{i-2}, t_{i-1}]$. If we restrict our attention to small simplices σ_1 and σ_2 interior to σ_1 and σ_2', we have a pair which fits Case 2, above. We can do a Type II piping to that pair. In the centers of the resulting $(k-2)$-cells we could find another pair of cells, both of which are contained in $X \times [t_{i-2}, t_{i-1}]$. Go ahead and do a Type I piping to them, then a Type II, etc. Eventually we will find ourselves doing a Type I piping in $X \times [t_0, t_1]$. Stop after doing that move; the pipe is now constructed.

Figure 4 indicates the overall result of all of those moves in the range.

Figure 5 shows the overall result in the domain, $X \times I$.

If we have started in the situation $\sigma_1 \subset X \times [t_{i-1}, t_i]$, $\sigma_2 \subset X \times [t_i, t_{i+1}]$, we would just have had one extra Type II piping to do, but could still have achieved the same end result.

Since all the modifications involved in the moves described above are done in neighborhoods of 2-dimensional sets and $f_n(S \cup J_0 \cup X \times \{0,1\})$ has codimension 3, we can do all that and leave $f_n \mid (S - \sigma_1 \cup \sigma_2) \cup J_0 \cup X \times \{0,1\}$ unchanged.

Construction of all the pipes simultaneously.

Rather than working on one pipe at a time, we actually construct them all at once as follows: First do all the Type I pipings in $X \times [t_{n-1}, t_n]$, then all the Type II pipings in $X \times [t_{n-2}, t_n]$, next all the Type I's in

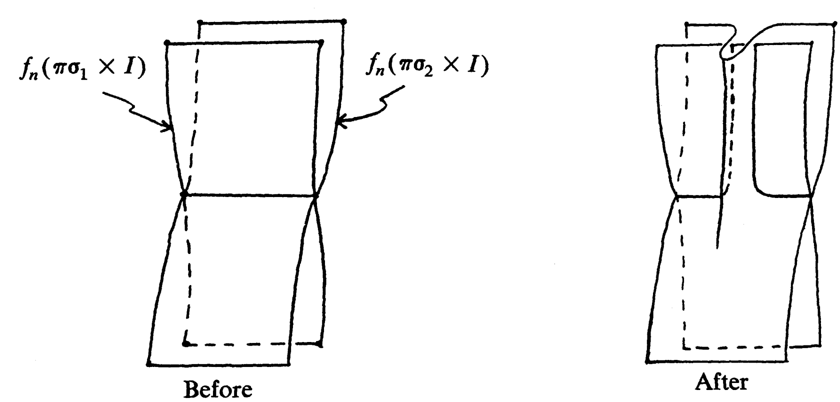


FIGURE 4

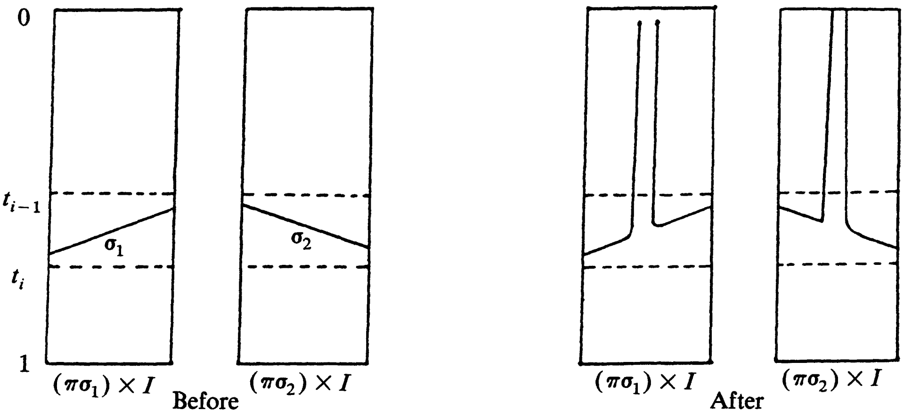


FIGURE 5

$X \times [t_{n-2}, t_{n-1}]$ and so on, alternating Type I and Type II pipings and ending with all the Type I pipings in $X \times [t_0, t_1]$.

Let f_n' denote the map which results from doing all the Type I pipings in $X \times [t_{n-1}, t_n]$ to f_n and let f_{n-1} denote the map which results from doing all the Type II pipings in $X \times [t_{n-2}, t_n]$ to f_n'. In general, f_i' will denote the map which we have after all the Type I pipings in $X \times [t_{i-1}, t_i]$ have been done and f_{i-1} will denote the map which results from doing all the Type II pipings in $X \times [t_{i-2}, t_i]$ to f_i'. In each case f_i is a PL map of $X \times I$ into R^{k+2}.

Claim. We can do the construction so carefully that each f_i satisfies $f_i(\{x\} \times I) \subset N_{\delta_i}(F_j(r(x)))$ for every $x \in X$ and $f_i(X \times [t_{q-1}, t_q]) \subset N_{\delta_i}(h(D_j^k[t_{q-1}, t_q]))$ for every $q = 1, \ldots, n$.

The proof of the claim is by downward induction on i. The case $i = n$ is obvious from the definition of f_n. So suppose the claim is true for some f_i. We get from f_i to f_{i-1} in two steps: we first do all the Type I pipings in $X \times [t_{i-1}, t_i]$ to get f_i' and then the Type II pipings in $X \times [t_{i-2}, t_i]$ to get f_{i-1}.

Now f_i' agrees with f_i everywhere except near a finite number of points in $X \times (t_{i-1}, t_i)$. Each of those points is piped off the image of a vertical arc in $X \times [t_{i-1}, t_i]$. Thus we can easily make sure that f_i' still satisfies

$$f_i'\left(X \times \left[t_{q-1}, t_q\right]\right) \subset N_{\delta_i}\left(h\left(D_j^k\left[t_{q-1}, t_q\right]\right)\right) \subset N_{\gamma_{i-1}}\left(h\left(D_j^k\left[t_{q-1}, t_q\right]\right)\right).$$

Further, the choice of δ_i ensures that $f_i'(\{x\} \times I) \subset N_{\gamma_{i-1}}(F_j(r(x)))$. Hence we see that f_i' satisfies the conclusions of the claim with δ_i replaced by γ_{i-1}.

The change from f_i' to f_{i-1} involves a finite number of Type II pipings in $X \times [t_{i-2}, t_i]$. What actually happens is that the image set is moved by an ambient isotopy, then reparametrized, then moved by an ambient isotopy, then reparametrized again, and so forth. The numbers s_q^{i-1} and η_{i-1} have been chosen in such a way that no point is moved by more than one of those ambient isotopies. (They are supported on disjoint sets.) But the reparametizations leave the images of the fibers $\{x\} \times I$ setwise fixed. Thus property (iv) of the isotopy H_t ensures that

$$f_{i-1}(\{x\} \times I) \subset N_{\eta_{i-1}}(F_j(r(x))) \subset N_{\delta_{i-1}}(F_j(r(x)))$$

for every $x \in X$.

Also, the complicated construction of the Type II piping was specifically designed to avoid moving any part of the image of $X \times [t_{q-1}, t_q]$ out of $N_{\delta_{i-1}}(h(D_j^k[t_{q-1}, t_q]))$. If a point was ever in danger of being pulled out of its correct neighborhood by one of the isotopies, we first moved it out of the way. Therefore $f_{i-1}(X \times [t_{q-1}, t_q]) \subset N_{\delta_{i-1}}(h(D_j^k[t_{q-1}, t_q]))$ for each q. This completes the proof of the claim.

The constructions above have now produced a map f_1. We finally obtain the map f that we are looking for by doing all the Type I pipings in $X \times [t_0, t_1]$ to f_1. The choice of δ_1 then guarantees that f satisfies $f(\{x\} \times I) \subset N_{\delta_0}(F_j(r(x))) \subset N_{\varepsilon/3}(F_j(r(x)))$ for each $x \in X$ and $f(X \times [t_{q-1}, t_q]) \subset N_{\delta_0}(h(D_j^k[t_{q-1}, t_q])) \subset N_{\varepsilon/3}(h(D_j^k[t_{q-1}, t_q]))$ for each q.

We now complete the proof of the Lemma by checking that f satisfies all of the conclusion.

(1) We never change $f_i \mid X \times \{0\}$, so $f(x, 0) = f_n(x, 0) = x$ for every $x \in X$.

(2) As noted above, $f(x, t) \in N_{\varepsilon/3}(F_j(r(x)))$ for every $(x, t) \in X \times I$. The choice of α together with the fact that $f(x, t) \in N_{\varepsilon/3}(h(D_j^k[t_{q-1}, t_q]))$ whenever $t_{q-1} \leq t \leq t_q$ shows that $f(x, t) \in N_\varepsilon(h(D_j^k[t]))$.

(3) Since all the modifications to f_n were done in the complement of $f_n(J_0)$, we still have $f_n(J_0) = f(J_0) \subset f(X \times I)$ and $\dim J_0 \leq k - 2$.

(4) Consider $S_0 = S(f)$. Then $S_0^{(k-3)} \supset S^{(k-3)}$. The difference is that each $(k-2)$ simplex $\sigma \subset S$ has been replaced by an annulus $\Sigma \approx (\partial\sigma) \times I$ with $\Sigma \subset (\pi\sigma) \times I$ (refer to Figure 5). Notice that for each i and each σ,

$$\begin{aligned}(\pi\sigma) \times [t_{i-1}, t_i] &\searrow (\pi\sigma) \times \{t_i\} \cup (\pi\partial\sigma) \times [t_{i-1}, t_i] \\ &\quad \cup (\Sigma \cap X \times [t_{i-1}, t_i]) \\ &\searrow (\pi\sigma) \times \{t_i\} \cup (\pi\partial\sigma) \times [t_{i-1}, t_i].\end{aligned}$$

Now $f \mid \Sigma$ is an embedding and each such Σ is matched with another $\Sigma' \subset S_0$ just like it such that $f(\Sigma) = f(\Sigma')$. Also, the only place where Σ attaches to $S^{(k-3)}$ is in $\partial\sigma$. Therefore there is a collapse ξ: $f(X \times I) \searrow f(X \times \{1\}) \cup f(J_0)$. Further, we can time ξ so that any simplex in $X \times [t_{i-1}, t_i]$ which is collapsed by ξ is collapsed during the time interval $t_{i-1} \leq t \leq t_i$. If B is any simplex of X then $\xi_t(f(B \times I)) \subset f(B \times I)$ for all t. We claim that ξ is a $(j, \varepsilon, f(J_0))$-collapse. The argument is the same as that given at the beginning of the proof of Lemma 1. □

3. Building $(j, \varepsilon, \varnothing)$-collapses. In this section we will continue the construction begun in §2. By attaching certain polyhedra to $f(X \times I)$ along $f(J_0)$, we will engulf the $(k-1)$-dimensional polyhedron X in a k-dimensional polyhedron for which there is a $(j, \varepsilon, \varnothing)$-collapse. We first state a definition. It should be noted that this definition is one of those which differs slightly from the corresponding one in [**2**].

DEFINITION. Let A be a polyhedron and $\varepsilon > 0$. A homotopy g: $A \times I \to R^{k+2}$ is a (j, ε)-*homotopy* if

(a) $g_t(A) \subset N_\varepsilon(h(D_j^k[t, 1]))$ for each t,

(b) $g_t \mid g_0^{-1}(N_\varepsilon(h(D_j^k[t,1]))) = g_0 \mid g_0^{-1}(N_\varepsilon(h(D_j^k[t,1])))$ for each t, and

(c) $g_t(x) \in N_\varepsilon(F_j(r(g_0(x))))$ for every $x \in A$ and $t \in I$.

To begin the construction, let X, f, and J_0 be as in Lemma 1. Define Y to be the abstract union

$$Y = f(X \times I) \cup_{f(J_0)} f(J_0) \times I$$

where each $x \in f(J_0) \subset f(X \times I)$ is identified with $(x, 0) \in f(J_0) \times I$. Let g: $Y \to R^{k+2}$ be the map defined by

$$g(y) = \begin{cases} y & \text{if } y \in f(X \times I), \\ \psi^j(z, s) & \text{if } y = (z, s) \in f(J_0) \times I. \end{cases}$$

LEMMA 2. *For every $\varepsilon' > 0$ there is a $\delta' > 0$ such that if $X \subset N_{\delta'}(h(D_j^k[0]))$ is a compact $(k-1)$-dimensional polyhedron and Y is constructed as in the paragraph above, then there is a collapse λ: $Y \searrow C$ with $g \circ \lambda_t$ a (j, ε')-homotopy.*

Proof. This is just a formal statement of a result which is proved on p. 410 of [**2**] and which is also stated as Lemma 4.1 of [**3**]. The collapse consists of two parts. Use ξ (the collapse defined in Lemma 1) to collapse $f(X \times I)$ to $f(J_0) \cup f(X \times \{1\})$ and at the same time, as bits of $f(J_0)$ are exposed, start to collapse them down the product structure of $f(J_0) \times I$—always keeping the second part of the collapse a little behind ξ.

More precisely, λ is constructed as follows. Choose a small positive number β. First collapse $f(X \times I)$ to $f(X \times [\beta, 1]) \cup f(J_0 \cap X \times [0, \beta])$ using ξ. Then collapse $f(J_0) \times I$ to

$$(f(J_0 \cap X \times [\beta, 1]) \times I) \cup (f(J_0 \cap X \times [0, \beta]) \times [\beta, 1])$$

by collapsing straight down the product structure. Next continue the collapse of $f(X \times I)$ to $f(X \times [2\beta, 1]) \cup f(J_0)$ and follow that by a further collapse down the product structure of $f(J_0) \times I$ to

$$(f(J_0 \cap X \times [2\beta, 1]) \times I) \cup (f(J_0 \cap X \times [0, 2\beta]) \times [2\beta, 1]).$$

Continue to alternate back and forth in that manner.

If δ' is small enough, $\psi^j \mid f(J_0)$ will have the property that $\psi^j(y, t) \in N_{\varepsilon'}(h(D_j^k[t, 1]))$ for each $(y, t) \in f(J_0) \times I$. Therefore $g \circ \lambda_t$ will be a (j, ε')-homotopy.

Our next job will be to work on the singular set of g and do some piping to modify it. Before doing so, however, we must change the way in which $g(Y)$ is parametrized. The problem is that for $(z, t) \in f(J_0) \times I$, we cannot be certain that $g(z, t) \in N_{\varepsilon'}(h(D_j^k[t]))$, but only that $g(z, t) \in N_{\varepsilon'}(h(D_j^k[t, 1]))$.

Suppose β in the proof above is of the form $\beta = 1/m$ where m is an integer. Define $F \subset f(J_0) \times I$ by

$$F = \bigcup_{i=1}^{m} (f(J_0 \cap X \times [0, i/m])) \times [(i-1)/m, i/m].$$

Notice that the image of any vertical segment from $f(J_0) \times I - F$ has small diameter and that if $(z, t) \in F$ then $g(z, t) \in N_{\varepsilon'}(h(D_j^k[t]))$. Let F_0 be a small neighborhood of F in $f(J_0) \times I$ with the property that there is a PL homeomorphism H: $F_0 \to f(J_0) \times I$. The homeomorphism H should be chosen to have two further properties: H preserves first coordinates, and supp(H) is contained in a small neighborhood of the frontier of F_0 in $f(J_0) \times I$.

Now define $Y^* = f(X \times I) \cup F_0$ where each $x \in f(J_0)$ is identified with $H^{-1}(x, 0) \in F_0$ and define g^*: $Y^* \to R^{k+2}$ by

$$g^*(x) = \begin{cases} x & \text{if } x \in f(X \times I), \text{ and} \\ g(H(x)) & \text{if } x \in F_0. \end{cases}$$

If H is chosen carefully enough, g^* will have the property that for every $(z, t) \in F_0$, $g^*(z, t) \in N_{\varepsilon'}(F_j(r(z))) \cap N_{\varepsilon'}(h(D_j^k[t]))$.

Put g^* in general position, keeping $f(X \times I)$ fixed, and let $S = S(g^*)$. Since $g^* \mid f(X \times I)$ is an embedding,

$$\begin{aligned}\dim S &\le \dim f(X \times I) + \dim F_0 - (k+2) \\ &\le k + (k-1) - (k+2) = k - 3.\end{aligned}$$

Define J_1 to be $\mathrm{Trail}_\lambda(S^{(k-4)})$. Then $\dim J_1 \le k - 3$. As in the proof of Lemma 1, we wish to use a piping argument to punch a hole in each top-dimensional simplex of S so that there is a $(j, \varepsilon, g^*(J_1))$-collapse of $g^*(Y^*)$. That is the first step in an inductive argument which will eventually produce a $(j, \varepsilon, \varnothing)$-collapse.

LEMMA 3. (*Second Piping Lemma.*) *For every* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that if* $X \subset N_\delta(h(D_j^k[0]))$ *is a compact* $(k-1)$*-dimensional polyhedron, then* f, Y^*, g^*, λ, *and* J_1 *can be constructed in such a way that* f *is homotopic* rel $X \times \{0,1\} \cup f^{-1}(J_1)$ *to a PL map* $\bar{f}$: $X \times I \to R^{k+2}$ *and* $g^* \mid F_0$ *is homotopic* rel $H^{-1}(J_0 \times \{0\}) \cup J_1$ *to a PL map* $\bar{g}$: $F_0 \to R^{k+2}$ *such that there is a* $(j, \varepsilon, \bar{g}(J_1))$*-collapse* θ: $\bar{f}(X \times I) \cup \bar{g}(F_0) \searrow \bar{g}(J_1) \cup C_1$.

Proof. Let $\alpha, \beta > 0$ and $0 = t_0 < t_1 < \cdots < t_n = 1$ be exactly as at the beginning of the proof of Lemma 1. Choose δ to be the δ' of Lemma 2 corresponding to

$$\varepsilon' = \min_{i=1,\ldots,n} \left\{ \mathrm{dist}\left(h\left(D_j^k[0, t_{i-1}]\right), h\left(D_j^k[t_i, 1]\right)\right)\right\}.$$

Now suppose $X \subset N_\delta(h(D_j^k[0]))$ and that f, Y^*, g^*, λ, S, and J_1 are all defined as above. Put g^* in general position.

As usual, the fact that g^* is in general position implies that the top-dimensional simplices of S come in pairs, each of which is identified under g^*. Let $\{\sigma_1, \sigma_2\}$ be a matched pair of $(k-3)$-simplices in S with $\sigma_1 \subset f(X \times I)$ and $\sigma_2 \subset F_0$. The situation is not symmetric as it was in the proof of Lemma 1, so there are now three cases to consider.

Case 1. $\sigma_1 \subset f(X \times [t_{i-1}, t_i])$ and $\sigma_2 \subset f(J_0) \times [t_{i-1}, t_i]$ for some i.

Case 2. $\sigma_1 \subset f(X \times [t_i, t_{i+1}])$ and $\sigma_2 \subset f(J_0) \times [t_{i-1}, t_i]$ for some i.

Case 3. $\sigma_1 \subset f(X \times [t_{i-1}, t_i])$ and $\sigma_2 \subset f(J_0) \times [t_i, t_{i+1}]$ for some i.
One of these three cases must occur by the choice of ε'.

Construction of one pipe.

Suppose $\{\sigma_1, \sigma_2\}$ is a pair which fits either Case 1 or Case 2. (Pairs which fit Case 3 can present a special difficulty. Later in the proof we will use a small trick to change each Case 3 pair into a Case 1 pair and thus avoid that problem.) We will pipe the singularity over the edge of $g^*(Y^*)$ in two steps. Step 1 will push it off the fin (but into $f(X \times I)$) and is just like the construction of one pipe in Lemma 1. Step 2 is then used to push things off $f(X \times I)$ and involves some new complications.

If Case 1 is the case which occurs initially, do a Type I piping by pushing $g^*(\sigma_1)$ along the image of a vertical arc in F_0 above the barycenter of σ_2. The result is a new pair which fits Case 2.

If Case 2 is the one which occurs, we do a Type II piping. This time we push $g^*(\sigma_2)$ along the image of an arc above the barycenter of σ_1. By general position there will be no points of $S(f)$ above the barycenter of σ_1 and so we can do this without obstruction. The result of this piping is a new pair which fits Case 1 again.

By alternating back and forth between Type I and Type II moves, we will eventually reach the top edge of F_0. Since F_0 consists of only part of $f(J_0) \times I$, we are likely to push the singularity off the edge of the fin before we reach the 0-level. This completes Step 1.

REMARK. Notice that we were forced by the lack of symmetry to do things in a very particular way. We make certain that we do the Type I pipings in such a way that Case 1 changes to Case 2. If we were to do the Type I pipings in the other obvious way, then a pair which fits Case 3 would result and we would be stuck. The reason is simply that an arc above σ_2 might not reach all the way to the t_{i-1}-level of $f(J_0) \times I$.

Assume that Step 1 has been completed. The result is a hole punched in that part of the intersection between F_0 and $f(X \times I)$ represented by $\{\sigma_1, \sigma_2\}$. Unfortunately the top of the fin is attached to $f(X \times I)$, so we pay for this improvement in the form of a new self-intersection in $f(X \times I)$. Figure 6 shows a picture of what has happened so far in the domain.

As stated above, the way in which we plan to deal with the new singularities is by continuing to pipe, but now doing so along two sheets of $f(X \times I)$. So we focus our attention on exactly what the preimages of the singularities look like all the way back in $X \times I$.

Let a be the point of $f(J_0)$ directly above the barycenter of σ_2 and let σ_1^* denote the preimage of the adjusted σ_1 under f. Notice that σ_1^* will consist of an annulus ($S^{k-4} \times [0,1]$) together with a $(k-2)$-complex

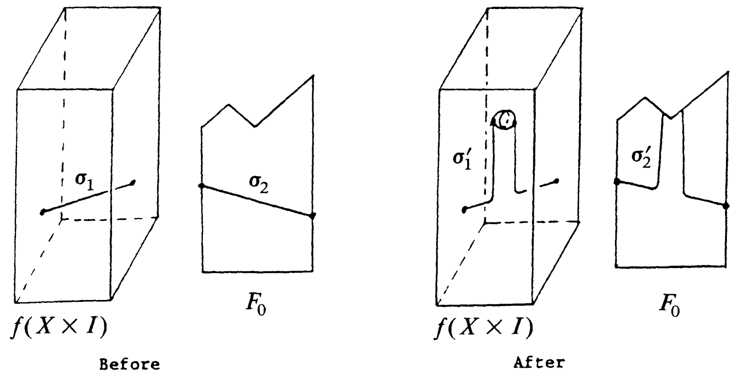


FIGURE 6

attached to that annulus. If $\pi(f^{-1}(a))$ is in the interior of a $(k-1)$-simplex and if we do Step 1 carefully, then that $(k-2)$-complex will be exactly a $(k-2)$-sphere attached to the free boundary component of the annulus as pictured in Figure 7. In general, the $(k-2)$-complex will be the union of a finite number of $(k-2)$-cells, one for each $(k-1)$-simplex of X which contains $\pi(f^{-1}(a))$. These cells are joined together along their common boundary. Let C denote their union. Now $f(C) \cap f(J_0)$ will consist exactly of a $(k-4)$-sphere, A, which contains a in its interior relative to $f(J_0)$. We can write C as the join of $f^{-1}(A)$ and a 1-complex K^1. Let B^2 denote the cone on K^1. We can assume that B^2 is contained in a small neighborhood of σ_1^* as indicated in Figure 7 and that $B^2 \cap \sigma_1^* = K^1$.

Step 2 now consists of pushing B^2 straight along $f(X \times I)$ towards $f(X \times \{0\})$. This is done by means of the same back-and-forth alternation between Type I and Type II pipings as always, but there are some new complications.

Whereas earlier we pushed only the barycenter of a simplex, we now push the entire image of B up, keeping $f(A)$ fixed. We use Type I pipings to push $f(B)$ up along $f(J_0)$. This works just like the Type I pipings in the proof of Lemma 1. But the Type II pipings are different. We take an arc α above B, push its image up near the t_i-level, and then reparametrize in such a way that not only is the arc incorporated into the slice $X \times [t_{i-1}, t_i]$, but so is all of B. Figure 8 shows the result in $X \times I$.

There is now another obstacle to be faced. As we push $f(B)$ up along $f(J_0)$, everything is simply a product until we reach the top of $f(J_0)$. There we will see part of $f(S(f))$. Attached to this part of $f(S(f))$ may be $(k-2)$-dimensional annuli of $f(S(f))$ as shown in Figure 7. We must avoid pushing $f(K^1)$ into those annuli because that would create still

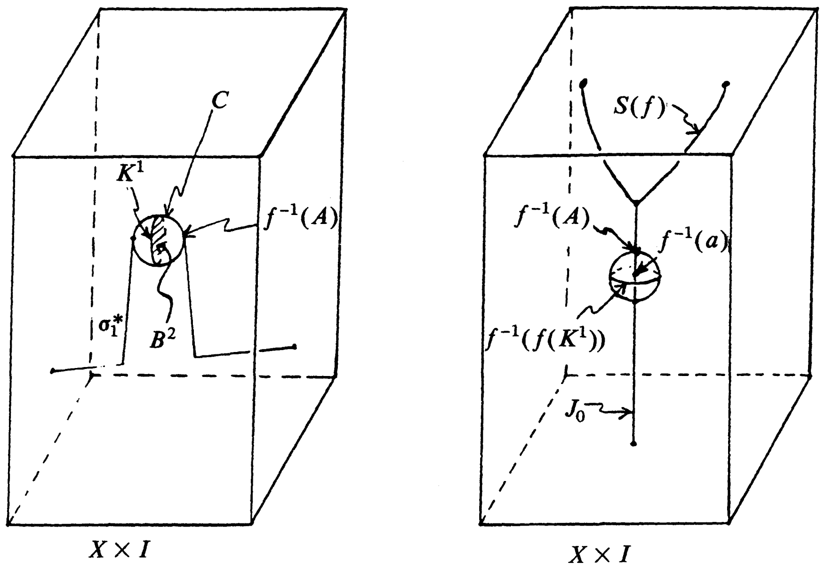


FIGURE 7

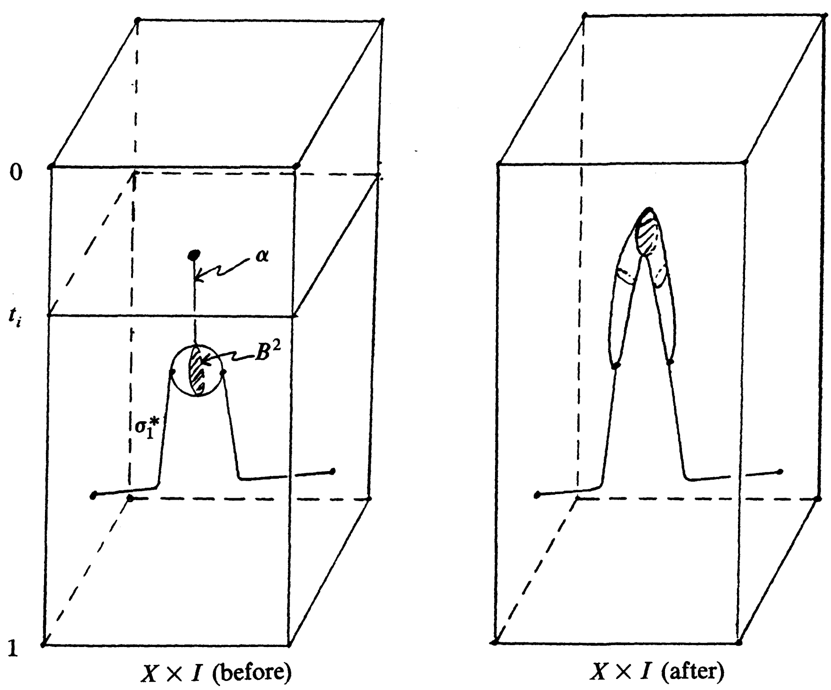


FIGURE 8

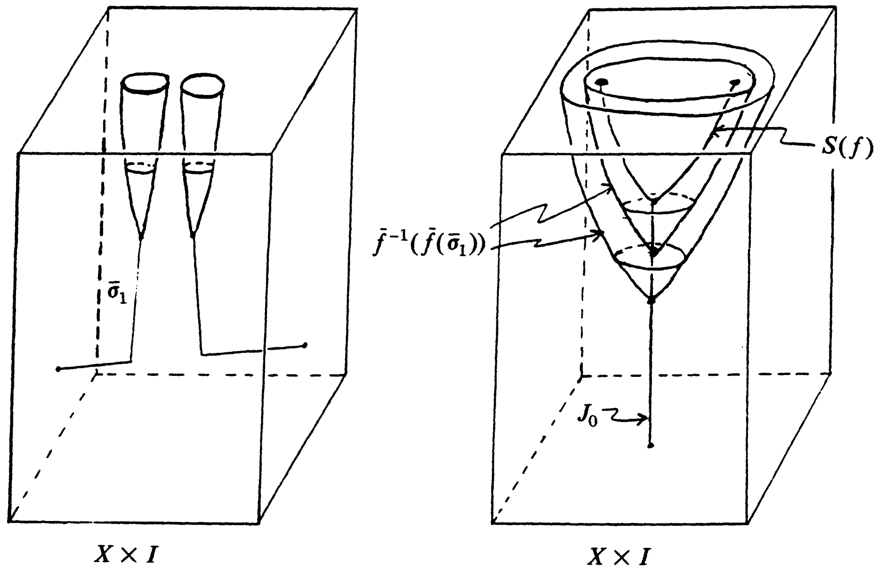


FIGURE 9

more self-intersections in $f(X \times I)$. (Other sheets of $f(X \times I)$ are attached there.) But $(\pi(K^1) \times I) \cap S(f)$ will consist of a finite set of points, one for each $(k-2)$-annulus of $f(S(f))$ attached to the $(k-3)$-simplex above a. To avoid pushing K^1 into $f(S(f))$, simply push that point out radially and then continue up through the hole in $f(S(f))$ which was created in the proof of Lemma 1.

Figure 9 shows the results of these changes in the domain, $X \times I$. Notice that σ_1^* has been replaced by a complex $\bar{\sigma}_1$ which collapses and that part of $\bar{\sigma}_1$ is identified, under $\bar{g}f$, with the sets labeled $\bar{f}^{-1}(\bar{f}(\bar{\sigma}_1))$ in the right-hand panel. It is not possible to draw a dimensionally accurate picture of what occurs in the range, but Figure 10 gives the general idea.

There is one last problem to be faced. In Step 2 we push $f(B^2)$ near to parts of $f(S(f))$. But there are other sheets of $f(X \times I)$ (not shown in the figures above) which cross there. Thus we have probably created new intersections with those sheets of $f(X \times I)$. The way in which we intend to avoid this problem is simple: the singularities we are worrying about in Step 2 are ones which were created (by us) in the course of doing Step 1 and so we will just go back and do Steps 1 and 2 in such a way that the problem does not arise.

Consider what happened in Step 1 when σ_1 was pushed off the edge of the fin. Instead of pushing σ_1 all the way through $f(X \times I)$, we could just push it into $f(X \times I)$ and stop there. Then, at the end of Step 1, the

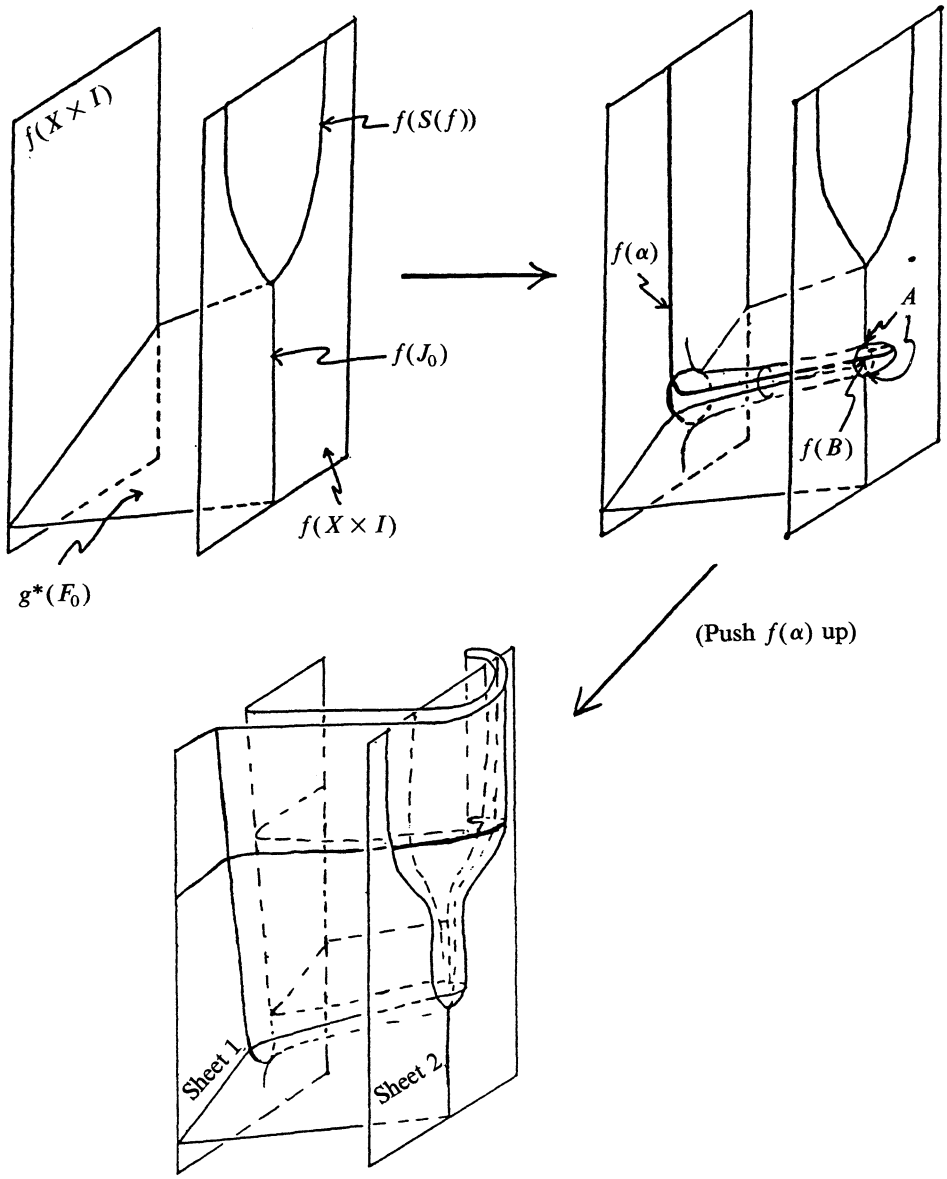


FIGURE 10

fin would look the same as before but the two sheets of $f(X \times I)$ would have a k-dimensional set in common. This k-dimensional set is the join of B and $f^{-1}(A)$. If we now proceed to do Step 2, the construction will be the same except that we will end up with the two sheets still intersecting in a k-dimensional set. We push B^2 off the top of $f(X \times I)$, so now the intersection of the two sheets is homeomorphic with the join of B^2 and

$f^{-1}(A)$ minus a neighborhood of B^2, which collapses via a collapse which induces a (j, ε)-homotopy.

This means that when we are finished with Step 2, the singular set of $\bar{f}$ will be k-dimensional. The picture will be a little different from that presented first. In Figure 10, the part of Sheet 1 which pokes through Sheet 2 in the second and third pictures is projected to the left so that it lies flat in Sheet 2.

Construction of all the pipes simultaneously.

So far we have explained how an individual pipe is constructed. In order to complete the proof of the Lemma, we must deal with two remaining points: first, how to avoid pairs which fit Case 3, and second, how to do all this and maintain the desired ε-control so that the collapse θ exists.

Here is a trick which will exchange each pair which fits Case 3 for one which fits Case 1. Given the partition $0 = t_0 < t_1 < \cdots < t_n = 1$, choose a refinement $0 = t'_0 < t'_0 < \cdots < t'_m = 1$ of much smaller mesh. Now suppose that δ was chosen to be small compared with the mesh of the new partition. Then each pair of $(k-3)$-simplices in S would fit one of the three cases listed above, but this time relative to the finer partition. Let $\{\sigma_1, \sigma_2\}$ be a pair such that $\sigma_1 \subset f(X \times [t'_{i-1}, t'_i])$ and $\sigma_2 \subset f(J_0) \times [t'_i, t'_{i+1}]$. Simply reparametrize the map f so that the barycenter of σ_1 lies in $f(X \times [t'_1, t'_{i+1}])$. This changes the pointwise control on g^* by very little and does not cause trouble since we do this one time only. Now each pair fits either Case 1 or Case 2 relative to the first partition.

The collapse θ is constructed from the composition of four different collapses.

$$\bar{f}(X \times I) \searrow f(X \times \{1\}) \cup f(J_0) \cup J_1 \cup \bar{f}(S(\bar{f})) \cup [\bar{f}(X \times I) \cap \bar{g}(F_0)]$$
$$\searrow f(X \times \{1\}) \cup f(J_0) \cup J_1 \cup [\bar{f}(X \times I) \cap \bar{g}(F_0)].$$

But

$$J_1 \cup f(J_0) \cup \bar{g}(F_0) \searrow \bar{f}(S(\bar{g})) \cup J_1 \cup g(J_0 \times \{1\}) \searrow J_1 \cup g(J_0 \times \{1\}).$$

The way in which θ is constructed from those four collapses is just like the construction of λ in the proof of Lemma 2. Achieving the desired ε-control of course requires a much more careful choice of δ. The details are just like those in the proof of Lemma 1 and so we omit them.

This completes the proof of Lemma 3. □

We now come to the large inductive construction which is the culmination of Sections 2 and 3.

LEMMA 4. *For every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that if* X *is a compact,* $(k-1)$*-dimensional polyhedron in* $N_\delta(h(D_j^k[0]))$*, then* $X \subset Y \subset N_\varepsilon(h(D^k))$ *where* Y *is a* k*-dimensional polyhedron for which there is a* $(j, \varepsilon, \varnothing)$*-collapse* $\xi\colon Y \searrow C$.

Proof. In Lemma 1 we showed that δ could be chosen so that X is contained in a polyhedron $f(X \times I)$ for which there is a (j, ε, J_0)-collapse with $\dim J_0 \leq k-2$. Then in Lemma 3 we showed that δ could be further refined and $f(X \times I)$ modified and enlarged to a polyhedron for which there is a (j, ε, J_1)-collapse with $\dim J_1 \leq k-3$. These are obviously the first two steps of an inductive proof of Lemma 4; we keep reducing the dimension of the image of the collapse until eventually it is empty. Rather than get bogged down in the notation of a full inductive proof of the lemma, we will concentrate instead on just the third step and then the inductive construction will be clear.

The construction described below is built on that of Lemma 3. We claim that for any given ε there is an $\varepsilon' > 0$ such that if the construction of Lemma 3 is done to within a tolerance ε', then we can do the construction below so carefully that when we are done we have a (j, ε, J_2)-collapse with $\dim J_2 \leq k-4$. Proving the claim requires an understanding of the construction below and then the details are just like those in the choice of δ in Lemma 1, so we omit them.

Choose δ' to be the δ' of Lemma 3 corresponding to the ε' of the claim above. Suppose $X \subset N_{\delta'}(h(D_j^k[0]))$ is a compact $(k-1)$-dimensional polyhedron. Let $f_0\colon X \times I \to R^{k+2}$ be the map given by Lemma 1. There exists $J_0 \subset f_0(X \times I)$ and a (j, ε', J_0)-collapse of $f_0(X \times I)$ to $J_0 \cup f_0(X \times \{1\})$. Now let F_0 be the fin constructed in Lemma 3, let $Y_1 = X \times I \cup F_0$ and let $f_1\colon Y_1 \to R^{k+2}$ be the PL map of Lemma 3. There is a $(k-3)$-dimensional polyhedron $J_1 \subset f_1(Y_1)$ and a (j, ε', J_1)-collapse $\theta\colon f_1(Y_1) \searrow J_1 \cup C_1$.

Define Y_2' to be $f_1(Y_1) \cup (J_1 \times I)$ where $x \in J_1 \subset f_1(Y_1)$ is identified with $(x, 0) \in J_1 \times I$. Use ψ^j to extend the identity on $f_1(Y_1)$ to a map f_1': $Y_2' \to R^{k+2}$. As in the proof of Lemma 3, there exists a subpolyhedorn F_1 of $J_1 \times I$ and a PL homeomorphism $H\colon F_1 \to J_1 \times I$ which preserves first coordinates and has the following property: if $(z, t) \in F_1$, then $f_2'(H(z, t)) \in N_{\varepsilon'}(h(D_j^k[t])) \cap N_{\varepsilon'}(F_j(r(z)))$. Define Y_2 to be $f_1(Y_1) \cup F_1$ where $x \in J_1 \subset f_1(Y_1)$ is identified with $H^{-1}(x, 0) \in F_1$. Define $f_2\colon Y_2 \to R^{k+2}$ to be the identity on $f_1(Y_1)$ and $\psi^j \circ H^{-1}$ on F_1. Then $f_1(z, t) \in N_{\varepsilon'}(h(D_j[t])) \cap N_{\varepsilon'}(F_j(r(z)))$ for every $(z, t) \in F_1$. (We need this kind of pointwise control to make our controlled piping arguments work.) By the proof of Lemma 2, there is a collapse λ: $Y_2 \searrow C_2$ such that $f_2 \circ \lambda_t$ is a (j, ε)-homotopy.

Shift f_2 into general position. Since $f_2 \mid f_1(Y_1)$ is an embedding,

$$\begin{aligned}\dim S(f_2) &\leq \dim f_1(Y_1) + \dim F_1 - (k+2)\\ &\leq k + (k-2) - (k+2) = k-4.\end{aligned}$$

Define S to be the $(k-5)$-dimensional skeleton of $S(f_2)$ and J_2 to be $f_2(\mathrm{Trail}_\lambda(S))$. As usual, the proof is completed by homotoping f_2, rel J_2, so that $f_2(Y_2)$ collapses to $J_2 \cup f_2(C_2)$.

The $(k-4)$-simplices of $S(f_2)$ come in pairs $\{\sigma_1, \sigma_2\}$ such that $f_2(\sigma_1) = f_2(\sigma_2)$, $\sigma_1 \subset f_0(X \times I)$, and $\sigma_2 \subset F_1$. By the same trick as in the proof of Lemma 3, we may assume that there exists an i such that one of the following two cases occurs:

Case 1. $\sigma_1 \subset f_0(X \times [t_{i-1}, t_i])$ and $\sigma_2 \subset J_1 \times [t_{i-1}, t_i]$,

Case 2. $\sigma_2 \subset f_0(X \times [t_i, t_{i+1}])$ and $\sigma_2 \subset J_1 \times [t_{i-1}, t_i]$.

For each such pair, we will now construct a pipe. The pipe is constructed in three stages. We first push $f_2(\sigma_1)$ off the new fin $f_2(F_1)$, then off the original fin $f_1(F_0)$, and finally off $f_0(X \times I)$. There is only one new aspect to this construction. In the proof of Lemma 3, at the second stage of the construction of one pipe, it was necessary to spread things out to avoid part of $S(f_0)$ above J_0. We do now want that spreading out to compound on us, so we now argue that, for each pipe, we will have to do that at most once.

Consider $\sigma_2 \subset F_1$. Above the barycenter b of σ_2 there is a point $a \in J_1$. Now looking in Y_1, we find a point $c \in f_1(S(f_1))$ above a. This point c is necessarily a double point of f_1 by general position. (The set of triple points has codimension two in $S(f_1)$.) The double point c will either be a place where F_0 crosses itself or a place where F_0 crosses $f_0(X \times I)$. If the latter is the case, we bend the path we pipe along just a little (as shown in Figure 11) so that the pipe goes directly from F_1 to $f_0(X \times I)$. That means that the construction of this particular pipe skips the second stage of the construction and is exactly the same as the construction of one pipe in Lemma 3. Otherwise we push $f_2(\sigma_1)$ off $f_2(F_1)$ and into $f_1(F_0)$, then off $f_1(F_0)$ into $f_0(X \times I)$ and finally off $f_0(X \times I)$. In F_0, the picture will be as in Figure 12(a) and so no spreading out will be needed in the second stage because there is no part of $S(f_1)$ above c. Figure 12(b) is what we have avoided.

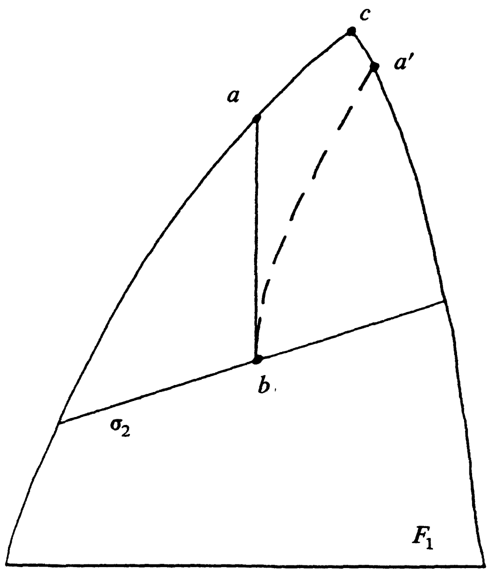
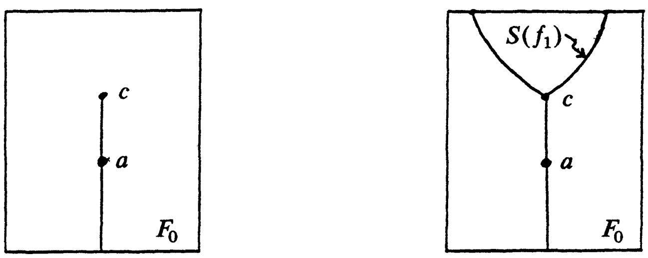


FIGURE 11



(a) (b)

FIGURE 12

This completes the proof of Lemma 4. □

REMARK. Even in later steps of the inductive proof of Lemma 4, it will only be necessary to do the "spreading out" once. This is because the double points we are concerned with at later stages will always arise as points where the new fin hits $f_0(X \times I)$. We always push directly into $f_0(X \times I)$.

4. Enlarging $(j, \varepsilon, \varnothing)$-collapses. In this section we show that a $(j, \varepsilon, \varnothing)$-collapse like that built in the preceding section can be enlarged to contain certain specified $(k-1)$-dimensional polyhedra. We first give a name to such a collapse.

DEFINITION. Suppose $\varepsilon > 0$ and X is a compact, $(k-1)$-dimensional polyhedron in $N_\varepsilon(h(D_j^k[0]))$. We say that a collapse σ: $Y \searrow C$ is a *standard* $(j, \varepsilon, \varnothing)$*-collapse for* X if Y is a k-dimensional polyhedron constructed according to the proof of Lemma 4 and ξ is the collapse described there.

The most important features of a standard $(j, \varepsilon, \varnothing)$-collapse are that Y consists of the track of a (j, ε)-homotopy of X with tracks of homotopies of lower dimensional polyhedra attached and that ξ follows those tracks. The next lemma corresponds to Miller's Lemma 9 [2].

LEMMA 5. *For every* $\varepsilon > 0$ *and for every* $\alpha > 0$, *there exists* $\delta > 0$ *with the following property. If* $X \subset N_\delta(h(D_j^k[0]))$ *is a compact* $(k-1)$*-dimensional polyhedron and* σ: $Y \searrow C$ *is a standard* $(j, \delta, \varnothing)$*-collapse for* X *and if* $Z \subset N_\delta(h(D_j^k[\alpha]))$ *is also a compact polyhedron of dimension* $\leq k-1$, *then there is a* $(j, \varepsilon, \varnothing)$*-collapse* ξ_1: $Y_1 \searrow C_1$ *such that* $Y \cup Z \subset Y_1$. *Furthermore,* $Y_1 \cap N_\delta(h(D_j^k[0])) = Y \cap N_\delta(h(D_j^k[0]))$.

Proof. Let $r = \dim Z$. By induction it suffices to consider the special case in which $Z \cap Y \supset Z^{(r-1)}$. Again we will just describe the construction of Y_1 and ξ_1 and leave the details of the choice of δ to the reader.

Let X, Y and ξ: $Y \searrow C$ be as in the statement of the Lemma. Then there exists a map f: $X \times I \to R^{k+2}$ which represents the first stage of the construction of Y. Use Lemma 4 to put Z in a standard $(j, \varepsilon, \varnothing)$-collapse Z^* with θ: $Z^* \searrow C^*$ denoting the actual collapse. Being a standard collapse for Z, Z^* contains $f^*(Z \times [\alpha, 1])$ for some (j, ε)-homotopy f^* of Z. When doing the construction of Lemma 4, choose $f^*(Z^{(r-1)} \times [\alpha, 1])$ to equal $\mathrm{Trail}_\xi(Z^{(r-1)})$ so that Y and Z^* match up nicely. Define Y^* to be the abstract union of Y and Z^*, sewn together along $f^*(Z^{(r-1)} \times [\alpha, 1])$. Then there is a $(j, \varepsilon, \varnothing)$-collapse μ: $Y^* \searrow C^*$ by Lemma 2. Define g: $Y^* \to R^{k+2}$ to be the inclusion on each of the two pieces of Y^*.

Now put g in general position, keeping Y fixed. Our plan is simply to attach fins to $g(Y^*)$ along $\mathrm{Trail}_\mu(S(g))$ and build up a $(j, \varepsilon, \varnothing)$-collapse from $g(Y^*)$. Consider $S(g)$.

$$\begin{aligned}\dim S(g) &\leq \dim Y + \dim Z^* - (k+2)\\ &\leq k + k - (k+2) = k - 2.\end{aligned}$$

As usual, the top-dimensional simplices of $S(g)$ comes in pairs $\{\sigma_1, \sigma_2\}$ where $\sigma_1 \subset f(X \times I) \subset Y$, $\sigma_2 \subset f^*(Z \times [\alpha, 1]) \subset Z^*$, and $g(\sigma_1) = g(\sigma_2)$. For each pair, construct a pipe (exactly as in the proof of Lemma 1) off the end of Z^*. Notice that, since Z^* lies near $h(D_j^k[\alpha, 1])$, we can do that without moving any points of Y near $h(D_j^k[0])$; this is now we achieve the last conclusion of the Lemma.

Let $L_0 = g(\text{Trail}_\mu(S(g)^{(k-3)}))$. After all the pipes have been built, there will be a (j, ε, L_0)-collapse of $g(Y^*)$. As in the proof of Lemma 3, we can attach a fin F_0 to $g(Y^*)$ along L_0. Put F_0 in general position with respect to $g(Y^*)$. Then

$$\dim[F_0 \cap g(Y^*)] \leq (k-1) + k - (k+2) = k - 3.$$

Once again we construct a pipe for each top-dimensional simplex of $F_0 \cap g(Y^*)$. But we must be careful how we do so because if we push a simplex of $g(Y^*)$ off F_0 and into $f(X \times I)$, we will be forced to continue piping all the way along Y to $f(X \times \{0\})$ and will not be able to claim that points near $h(D_j^k[0])$ are left fixed. Just as in the proof of Lemma 4, we have some flexibility about which side of F_0 we push off. Each double point of g arises from a place where $f^*(Z \times [\alpha, 1])$ intersects $f(X \times I)$. Just make sure that when the pipe is pushed off J_0 it is pushed off into $f^*(Z \times [\alpha, 1])$. After these pipes have been built there will be (j, ε, J_1)-collapse of $g(Y^*) \cup F_0$, where $\dim J_1 \leq k - 3$.

The next step is to attach a fin F_1 to $g(Y^*) \cup F_0$ along J_1 and then to pipe top-dimensional simplices of $F_1 \cap (F_0 \cup g(Y^*))$. This procedure is continued inductively as in Lemma 4. At later stages of the construction, building one pipe will involve pushing something off a whole sequence of fins. But it is always the case that a fin is attached where a fin which was new at the previous stage intersects $g(Y^*)$, so we can always avoid piping off the edge of the new fin into $f(X \times I)$. □

When we apply Lemma 5, we will actually need to do so in a PL submanifold of R^{k+2} which lies near $h(D^k)$, rather than in R^{k+2} itself. The key ingredient which made all the proofs go through was not so much the particular structure of R^{k+2}, but rather the existence of the (j, ε)-homotopies we needed. (Recall that we constructed them by using the fact that $h(D^k)$ is an ANR to squeeze things down to $h(D^k)$ and then sliding along the product structure of D^k.) We could just as well have proved the following Lemma. The only thing which prevented us from doing so immediately was the additional notation which would have been required. The relationship betweens Lemmas 5 and 6 is exactly the same as the relationship between Lemma 9 and Corollary 10 of [**2**].

LEMMA 6. *Fix $j < k$. Let $\{\Sigma_\delta \mid \delta > 0\}$ denote a set of collections of closed, s-dimensional PL submanifolds of R^{k+2} with each element of Σ_δ contained in $N_\delta(h(D^k))$. Suppose $\{\Sigma_\delta\}$ has the property that for every*

$\varepsilon' > 0$ there exists $\delta' > 0$ such that if $S_1 \in \Sigma_{\delta'}$ and Z is a compact subpolyhedron of S_1 of dimension $\leq s - 3$, then there is $S_2 \in \Sigma_{\varepsilon'}$ such that S_2 contains a (j, ε')-homotopy of Z and $S_2 \cap N_{\delta'}(h(D^k_{j+1}[0])) = S_1 \cap N_{\delta'}(h(D^k_{j+1}[0]))$.

Then for every $\varepsilon > 0$ and every $\alpha > 0$ there exists $\delta > 0$ such that if $S \in \Sigma_\delta$, Y is a standard $(j, \delta, \varnothing)$-collapse of dimension $\leq s - 2$ in S, and $X \subset S \cap N_\delta(h(D^k_j[\alpha]))$ is a compact polyhedron with dim $X \leq s - 3$, *then there exists $S^* \in \Sigma_\varepsilon$ with these properties.*

1. *$S^* \cap N_\delta(h(D^k_{j+1}[0])) = S \cap N_\delta(h(D^k_{j+1}[0]))$.*
2. *S^* contains a standard $(j, \varepsilon, \varnothing)$-collapse Y^* which in turn contains $Y \cup X$.*
3. $\dim(Y^* - Y) \leq \dim X + 1$.
4. *$Y^* \cap N_\delta(h(D^k_j[0])) = Y \cap N_\delta(h(D^k_j[0]))$.*

5. (j, ε)-collapses. In this section we relate the $(j, \varepsilon, \varnothing)$-collapses we have been working with to the (j, ε)-collapses of Miller. We show that every $(j, \delta, \varnothing)$-collapse is also a (j, ε)-collapse in the terminology of [**2**]. Once that has been established, we will really be finished with the proof for Theorem 1, since our Lemma 6 is a codimension two version of Miller's Corollary 10 and Miller's Lemmas 11, 12, and 13 are essentially dimension free—depending only on his Corollary 10. However, for the sake of completeness and for the sake of the reader who is not familiar with [**2**], we will include a final section in which we sketch the remainder of the proof of Theorem 1.

DEFINITION [**2**, pp. 407 and 408]. Let Y and C be subpolyhedra of N. A collapse σ: $Y \searrow C$ is called a *(j, ε)-collapse* if $d(\xi_t(x), \psi^j_t(x)) < \varepsilon$ for every $y \in Y$ and $t \in I$.

LEMMA 7. *For every $\varepsilon > 0$ and $1 \leq j \leq k$ there exists $\delta > 0$ such that each $(j, \delta, \varnothing)$-collapse is also a (j, ε)-collapse.*

Proof. Let $\varepsilon > 0$ be given. We give the details of the choice of δ. First choose $\gamma > 0$ such that $N_\gamma(F_j(y)) \cap N_\gamma(h(D^k_j[t])) \subset N_{\varepsilon/2}(F_j(y) \cap h(D^k_j[t]))$ for every $(y, t) \in h(D^k) \times I$. We next choose a sequence of five δ's.

Choose $\delta_1 > 0$ such that if $N_{\delta_1}(F_j(y_1)) \cap N_{\delta_1}(F_j(y_2)) \neq \phi$ then $N_{\delta_1}(F_j(y_1)) \subset N_\gamma(F_j(y_2))$ for every y_1, $y_2 \in h(D^k)$.

Choose $\delta_2 > 0$ such that $N_{\delta_2}(h(D^k_j[0, t])) \cap N_{\delta_2}(h(D^k_j[t, 1])) \subset N_\gamma(h(D^k_j[t]))$ for every $t \in I$.

Choose $\delta_3 > 0$ such that $\psi_t^j \mid N_{\delta_3}(h(D_j^k[t,1]))$ is within ε of the identity for every t.

Choose $\delta_4 > 0$ such that $\psi_t^j(N_{\delta_4}(h(D_j^k[0,t]))) \subset N_\gamma(h(D_j^k[t]))$ for every $t \in I$.

Choose $\delta_5 > 0$ such that $d(x, r(x)) < \delta_1$ for every $x \in N_{\delta_5}(h(D^k))$.

Define $\delta = \min\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5\}$.

Suppose σ: $Y \searrow C$ is a $(j, \delta, \varnothing)$-collapse. We must show that $d(\xi_t(x), \psi_t^j(x)) < \varepsilon$ for every $(x, t) \in Y \times I$.

Fix $(x, t) \in Y \times I$. Notice that $Y \subset N_\delta(h(D^k))$, so either $x \in N_\delta(h(D_j^k[0,t]))$ or $x \in N_\delta(h(D_j^k[t,1])) - N_\delta(h(D_j^k[0,t]))$.

Case 1. $x \in N_\delta(h(D_j^k[0,t]))$. Then $\psi_t^j(x) \in N_\gamma(h(D_j^k[t]))$ by the choice of δ_4 and $\psi_t^j(x) \in N_{\delta_1}(F_j(r(x)))$ by the choice of δ_5. Therefore $\psi_t^j(x) \in N_{\varepsilon/2}(F_j(r(x)) \cap h(D_j^k[t]))$ by the choice of γ.

On the other hand, parts (b) and (c) of the definition of (j, ε, Z)-collapse together with the choice of δ_2 imply that $\xi_t(x) \in N_\gamma(h(D_j^k[t]))$. Part (d) of the definition of (j, ε, Z)-collapse plus the choice of δ_1 give $\xi_t(x) \in N_\gamma(F_j(r(x)))$. The choice of γ again gives $\xi_t(x) \in N_{\varepsilon/2}(h(D_j^k[t]) \cap F_j(r(x)))$. Since $h(D_j^k[t]) \cap F_j(r(x))$ is just one point, we have that $d(\xi_t(x), \psi_t^j(x)) < \varepsilon$ by the triangle inequality.

Case 2. $x \in N_\delta(h(D_j^k[t,1])) - N_\delta(h(D_j^k[0,t]))$. Then $d(x, \psi_t^j(x)) < \varepsilon$ by the choice of δ_3 and $\xi_t(x) = x$ by part (c) of the definition of (j, ε, Z)-collapse. □

6. Proof of Theorem 1. In this section we give a brief outline of the remainder of the proof of Theorem 1. All the ideas are also found on pp. 413–416 of [**2**]. If T is a triangulation, we use T' to denote the first barycentric subdivision of T. If $j < k$, we will think of D^j as being a subset of D^k by identifying $(x_1, \dots, x_j) \in D^j$ with $(x_1, \dots, x_j, 0, \dots, 0) \in D^k$. We begin with a definition.

DEFINITION. Suppose $\varepsilon > 0$ and $1 \leq j \leq k$. A sequence $C^k, \dots, C^j$; $B^k, \dots, B^j$ of subpolyhedra of R^{k+2} is called a (j, ε)-*sequence* if

(1) each C^r is an (r, ε)-collapse in $N_\varepsilon(H(D^r))$ with $\dim C^r \leq r$, and

(2) there is a triangulation T_k of R^{k+2} and there are triangulations T_r of ∂B^{r+1}, $1 \leq r < k$, all of mesh $< \varepsilon$ such that C^r is a subcomplex of T_r and $B^r = N(C^r, T_r)$.

Note. B^r is an $(r+2)$-dimensional PL manifold-with-boundary.

LEMMA 8. *For every* $\varepsilon > 0$, $\alpha > 0$, *and* $1 \leq j \leq k$ *there exists* $\delta > 0$ *with the following property. If* $C^k, \ldots, C^j$; $B^k, \ldots, B^j$ *is a* (j, δ)*-sequence and* $X \subset \partial B^{j+1} \cap N_\delta(h(D_j^k[\alpha]))$ *is a polyhedron of dimension* $\leq j - 1$, *then there is a* (j, ε)*-sequence* $C_X^k, \ldots, C_X^j$; $B_X^k, \ldots, B_X^j$ *such that*

(1) $C^j \cup X \subset C_X^j$,

(2) $B_X^r \cap N_\delta(h(D^{r-1})) = B^r \cap N_\delta(h(D^{r-1}))$ *for each* r, *and*

(3) *if* $\dim X < j - 1$, *then* C_X^j *contains a* $(j - 1, \varepsilon)$*-homotopy of* X.

Proof. The proof is by downward induction on j. The case $j = k$, $\dim X = j - 1$ is just Lemma 5. Otherwise, if $j = k$ and $\dim X < j - 1$, apply Lemma 5 to the track of a $(j - 1, \delta)$-homotopy of X.

So assume that $j < k$ and that the Lemma is true for $j + 1$. Under those assumptions we will prove the Sublemma below. Lemma 8 then follows from the Sublemma together with Lemma 6. The Sublemma supplies the (j, δ)-homotopies we need to apply Lemma 6 with $\Sigma_\delta = \{\partial B^{j+1} \mid \text{there exists a } (j + 1, \delta)\text{-sequence } C^k, \ldots, C^{j+1}; B^k, \ldots, B^{j+1}\}$.

SUBLEMMA. *For every* $\varepsilon' > 0$ *there exists* $\delta' > 0$ *such that if* $C^k, \ldots, C^j$; $B^k, \ldots, B^j$ *is a* (j, δ')*-sequence and* $Z \subset \partial B^{j+1} \cap N_{\delta'}(h(D^j))$ *is a compact polyhedron of dimension* $\leq j - 1$, *then there exists a* (j, ε')*-sequence* $C_Z^k, \ldots, C_Z^{j+1}, C^j$; $B_Z^k, \ldots, B_Z^{j+1}, B^j$ *such that* ∂B_Z^{j+1} *contains a* (j, ε')*-homotopy of* Z.

Proof. Use the fact that B^{j+1} is a small regular neighborhood of C^{j+1} to homotope Z to C^{j+1} and then use the $(j + 1, \delta')$-collapse of C^{j+1} to push Z a little further to $N_{\delta'}(h(D_{j+1}^k[\gamma]))$, where γ is a small positive number. Let $\beta(Z)$ denote the image of Z there. Apply Lemma 8 to the $(j + 1, \delta')$-sequence $C^k, \ldots, C^{j+1}$; $B^k, \ldots, B^{j+1}$ with $X = \beta(Z)$. That gives a $(j + 1, \varepsilon')$-sequence $C_Z^k, \ldots, C_Z^{j+1}$; $B_Z^k, \ldots, B_Z^{j+1}$ such that C_Z^{j+1} contains a (j, ε')-homotopy of $\beta(Z)$. Furthermore, B_Z^{j+1} is the same as B^{j+1} near $h(D^j)$, so we still have $B^j \subset \partial B_Z^{j+1}$. Thus we can append C^j and B^j to form the (j, ε')-sequence $C_Z^k, \ldots, C_Z^{j+1}, C^j$; $B_Z^k, \ldots, B_Z^{j+1}, B^j$.

To finish the proof we must verify that ∂B_Z^{j+1} contains a (j, ε')-homotopy of Z. We already know that C_Z^{j+1} contains such a homotopy near $h(D_{j+1}^k[\gamma])$. To get it out into ∂B_Z^{j+1}, we collapse C_Z^{j+1} a little past the track of the homotopy. Once it lies in $B_Z^{j+1} - \xi(C_Z^{j+1})$, we use the product structure given by the fact that B_Z^{j+1}j is a regular neighborhood of $\xi(C_Z^{j+1})$ to push the track of the homotopy out into ∂B_Z^{j+1}. □

LEMMA 9. *For every* $\varepsilon > 0$ *and every* $j \leq k$, *there exists a* (j, ε)*-sequence* $C^k, \ldots, C^j$; $B^k, \ldots, B^j$ *such that there is a PL map* $g^{j-1}\colon D^{j-1} \to C^j$ *with* $d(h(x), g^{j-1}(x)) < \varepsilon$ *for every* $x \in D^{j-1}$.

Proof. The proof is by downward induction on j. If $j = k$, we apply Lemma 4. Let $X = g^{k-1}(D^{k-1})$ for some PL map which is a close approximation to $h \mid D^{k-1}$. By Lemmas 4 and 7 there is a (k, ε)-collapse C^k which contains X. Take B^k to be a small regular neighborhood of C^k.

Suppose that the Lemma is true for $j + 1$. Choose δ to be the δ of Lemma 8 and let $C^k, \ldots, C^{j+1}$; $B^k, \ldots, B^{j+1}$ be a $(j + 1, \delta)$-sequence as in the statement of the Lemma. Consider $g^j \colon D^j \to C^{j+1}$. By the same construction as in the proof of Lemma 8, there is a map $p\colon g^j(D^j) \to \partial B^{j+1}$ which is close to the identity and projects $g^j(D^j)$ into ∂B^{j+1}. Define g^{j-1} to be $p \circ g^j \mid D^{j-1}$. Now apply Lemma 8 to the (j, δ)-sequence $C^k, \ldots, C^{j+1}, \phi$; $B^k, \ldots, B^{j+1}, \phi$ with $X = g^{j-1}(D^{j-1})$. The sequence $C^k_X, \ldots, C^j_X$; $B^k_X, \ldots, B^j_X$ satisfies the conclusion of Lemma 9. □

Proof of Theorem 1. By Lemma 9 there exists a $(1, \varepsilon)$-sequence for every $\varepsilon' > 0$. Consider $g^0\colon D^0 \to C^1$. Since $g^0(D^0)$ is just one point, we can find a point $G^0(D^0)$ in ∂B^1 such that $d(G^0(D^0), h(D^0)) < 2\varepsilon'$. Extend G^0 to a PL embedding $G^1\colon D^1 \to B^1$ using the product structure on ∂B^1. Then use the $(1, \varepsilon')$-collapse of C^1 to stretch G^1 out to a PL embedding which approximates $h \mid D^1$. Now $B^1 \subset \partial B^2$, so we can use the product structure on ∂B^2 to extend G^1 to a PL embedding $G^2\colon D^2 \to B^2$ and then use the $(2, \varepsilon')$-collapse of C^2 to stretch G^2 out to a PL embedding which approximates $h \mid D^2$. Then G^2 is extended to $G^3\colon D^3 \to B^3$ and so on until we arrive at a PL embedding $G^k = g$ which approximates h on all of D^k. □

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