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QUOTIENTS OF NEST ALGEBRAS WITH TRIVIAL COMMUTATOR

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The main result of this paper is to show that every operator T commuting with a nest algebra modulo a two-sided ideal $\mathscr{J}$ of $\mathscr{L}(H)$ is of the form $T = \lambda I + J$ for some $\lambda \in C$, $J \in \mathscr{J}$.

Introduction. The structure of commutators of non-selfadjoint operator algebras has received considerable interest in recent years [**4**, **5**, **6**, **8**, **9**, **13**, **16** and their references] ([**7**] contains a good survey of known results). However, results for perturbed algebras in general and finite perturbations in particular are not available except for the special case of the ideal $\mathscr{K}$ of all compact operators. To put the results proven here into perspective, we mention two well known and particularly useful special cases. For any subalgebra $\mathscr{A}$ of $\mathscr{L}(H)$ and any subset $\mathscr{M}$ of $\mathscr{L}(H)$, denote by $C(\mathscr{A}, \mathscr{M})$ the collection $\{T \in \mathscr{L}(H)\colon\ AT - TA \in \mathscr{M}$ for every $A \in \mathscr{A}\}$. We now state:

I. (Calkin [**3**].) Given any ideal $\mathscr{J}$ (two-sided) of $\mathscr{L}(H)$,

$$C(\mathscr{L}(H), \mathscr{J}) = CI + \mathscr{J}.$$

Using the results of Johnson and Parrott [**11**] on $C(\mathscr{B}, \mathscr{K})$ for $\mathscr{B}$, a type I von Neumann algebra, Christensen and Peligrad were able to show the following.

II. (Christensen and Peligrad [**5**].) For any nest algebra $\mathscr{A}$,

$$C(\mathscr{A}, \mathscr{K}) = CI + \mathscr{K}.$$

It should be mentioned that II was shown to have an extension to the von Neumann-Schatten p-classes in [**7**].

The central result of this paper is to show that I and II above are "endpoints" of a very general theorem concerning commutators. This result can be stated as:

III. For any nest algebra $\mathscr{A}$ and any ideal $\mathscr{J}$ of $\mathscr{L}(H)$,

$$C(\mathscr{A}, \mathscr{J}) = CI + \mathscr{J}.$$

Combining III with the main result of **[4]**, we obtain:

IV. Any derivative of a nest algebra into an ideal (two-sided) $\mathscr{J}$ of $\mathscr{L}(H)$ is implemented by an operator from $\mathscr{J}$.

I would like to thank C. Apostol for his helpful conversation.

For the purpose of this paper, $\mathscr{A}$ will denote the nest algebra of all operators acting on a fixed separable Hilbert space H leaving invariant a (complete) totally ordered nest of subspaces N. Denote by $\mathscr{E}$ the corresponding totally ordered nest of orthogonal projections onto the members of $\mathscr{N}$. If $\mathscr{E} = \{E_n\}_{n \in Z}$, let Δ_i be the orthogonal projection $E_i - E_{i-1}$. $\mathscr{J}$ will denote an arbitrary but non-zero two-sided ideal of $\mathscr{L}(H)$. It is well known **[10]** that $\mathscr{F} \subseteq \mathscr{J} \subseteq \mathscr{K}$, where $\mathscr{F}$ denotes the ideal of all finite rank operators. (Note that all the results below are obviously true for $\mathscr{J} = (0)$.)

Essential use will be made of the identification between such an ideal $\mathscr{J}$ and its corresponding ideal set $\tilde{\mathscr{J}}$ of s-numbers in $c_0(N)$ satisfying

(i) $\{\lambda_i\}$, $\{\mu_i\}$ in $\tilde{\mathscr{J}}$ implies $\{\lambda_i + \mu_i\}$ in $\tilde{\mathscr{J}}$.

(ii) $\{\lambda_i\} \in \tilde{\mathscr{J}}$ and $0 \leq \mu_i \leq \lambda_i$ for every $i \in N$ implies $\{\mu_i\} \in \tilde{\mathscr{J}}$.

(iii) For any permutation π: $\mathbf{N} \to \mathbf{N}$, $\{\lambda_i\}$ in $\tilde{\mathscr{J}}$ implies that $\{\lambda_{\pi(i)}$ is in $\tilde{\mathscr{J}}$.

This identification is given by s: $T \to \sigma((T^*T)^{1/2})$. We will use the standard notation $s_j(T)$ for the jth eigenvalue of $(T^*T)^{1/2}$. Given T in $\mathscr{L}(H)$, denote by δ_T the map from $\mathscr{A}$ to $\mathscr{L}(H)$ given by $\delta_T(A) = AT - TA$. Let $x \otimes y$ be the rank one operator $(x \otimes y)z = \langle z, x\rangle y$. By c.l.s. $\{S\}$ will be meant the closed linear span in the norm topology of the set S.

Commutants of nest algebras modulo $\mathscr{J}$. In order to prove III, we initially divide the problem into three cases:

(i) There exists a projection E into $\mathscr{E}$ with infinite range and kernel.

(ii) There exists an increasing sequence $\{E_n\}_{n=0}^{\infty}$ of finite dimensional projections in $\mathscr{E}$, with $E = \sup E_n$ having finite dimensional kernel.

(iii) There exists a decreasing sequence $\{E_n\}_{n=0}^{\infty}$ of finite co-dimensional projections in $\mathscr{E}$, with $E = \inf E_n$ having finite range.

Case (i). As in **[5]** we note that there will exist a partial isometry V in $\mathscr{A}$ with $VV^* = E$ and $V^*V = I - E$. Thus both $E\mathscr{L}(H)EV$ and $V(I-E)\mathscr{L}(H)(I-E)$ are subsets of $\mathscr{A}$. Let δ_K be a (bounded) derivation from $\mathscr{A}$ into $\mathscr{J}$. For any X in $\mathscr{L}(H)$, $\delta_K(EXEV) = \delta_K(EXE)V + EXE\delta_K(V)$, it will immediately follow that $\delta_k(EXE)E$ is in $\mathscr{J}$. Define the ideal $\mathscr{J}_1$ of $\mathscr{L}(EH)$ to be

$$\mathscr{J}_1 = \{ETE: T \in \mathscr{J}\}.$$

Consider the action of δ_{EKE} on $\mathscr{L}(EH)$. For any X in $\mathscr{L}(H)$,

$$\delta_{EKE}(EXE) = E(XEK - KEX)E = E\delta_K(EXE)E.$$

Thus δ_{EKE} derives $\mathscr{L}(EH)$ into $\mathscr{J}_1$. An application of I above will show that $EKE = \lambda E + T_1$ for some T_1 in $\mathscr{J}_1$. An exactly similar argument will show that $(I - E)K(I - E)$ is of the form $\mu(I - E) + T_2$, where $T_2 = (I - E)T_2(I - E)$ for some $T_2 \in \mathscr{J}$. In addition, $EK(I - E) = E\delta_K(E)(I - E) = ET_3(I - E)$ with $T_3 \in \mathscr{J}$. Similarly, $(I - E)KE = (I - E)\delta_K(I - E)E = (I - E)ET_4E$ with $T_4 \in \mathscr{J}$. Therefore, K can be written as:

$$K = \begin{bmatrix} \lambda & \\ & \mu \end{bmatrix} + \begin{bmatrix} T_1 & T_4 \\ T_3 & T_2 \end{bmatrix},$$

where the second term T is in $\mathscr{J}$. All that remains is to show $\lambda = \mu$. Note, however, that since $V \in \mathscr{A}$, we have

$$(\lambda E + \mu(I - E) + T)V - V(\lambda E + \mu(I - E) + T) \in \mathscr{J}.$$

It immediately follows that $(\lambda - \mu)E \in \mathscr{J}$, showing $\lambda = \mu$.

Case (ii). In order to prove case (ii), it will be necessary to further subdivide case (ii) into (ii) (a) $\mathscr{J} \neq \mathscr{F}$ and (ii) (b) $\mathscr{J} = \mathscr{F}$. Before beginning the proof of either, we note that it may as well be assumed that $\mathscr{E}$ is the classical nest of one-dimensional jumps on $l^2(N)$. That is, with respect to the usual basis $\{e_j\}_{n=1}^{\infty}$, E_n is given as the projection onto the closed linear span of $\{e_j\}_{j=1}^{n}$.

Case (ii)a. Let δ_K. $\mathrm{Alg}\{E_n\} \to \mathscr{J}$. It follows from II that we can assume K is compact. Fix a $c_0(N)$ sequence $\{\varepsilon_i\}$ in $\tilde{\mathscr{J}}$ satisfying $\varepsilon_1 > \varepsilon_2 > \cdots > 0$. Define a partial isometry A in $\mathscr{A}$ by $A^*e_i = e_{n_i}$, where $n_i > n_{i-1}$ and $\|\Delta_{n_i}AK\| < 2^{-i}\varepsilon_i$. That this is possible follows from the compactness of K and the observation that $(I - E_n)\downarrow 0$ strongly. It can now be seen that AK is the operator with the property that $\Delta_n AK = \Delta_{n_i}K$. We claim that $s(AK)$ is dominated by $\{\varepsilon_i\}$, and thus $AK \in \mathscr{J}$ by (iii). That this holds is an application of [**1**]. Indeed we have

$$s_{n+1}(AK) \leq \|(I - E_n)AK\| \leq \sum_{j=n+1}^{\infty} \|\Delta_j AK\| < \varepsilon_{n+1}$$

since, in particular, rank $E_n AK \leq n$.

Thus, necessarily KA is also in $\mathcal{J}$. Moreover,

$$s(KA) = s(A^*K^*) = \sigma\left[(KAA^*K^*)^{1/2}\right] = \sigma\left[(KK^*)^{1/2}\right] = s(K),$$

showing K is also in $\mathcal{J}$.

Case (ii)b. It is not too difficult to show that this result follows from case (ii)a using the fact that $\bigcap\{\mathcal{J}: \mathcal{J} \supsetneqq \mathcal{F}\} = \mathcal{F}$. However, the following proof is of independent interest in that it provides a concrete example of an operator A such that $\{\delta_T(A) \notin \mathcal{F}$ for a given $T \notin CI + \mathcal{F}$. Since $\delta_X(A) \subseteq \mathcal{F}$ if and only if $\delta_{X^*}(A^*) \subseteq \mathcal{F}$, it may as well be assumed that $\mathcal{A}$ is the algebra of all (bounded) lower triangular matrices with respect to the basis $\{e_n\}$. Let δ_T: $\mathcal{A} \to \mathcal{F}$. Suppose, contrary to the assertion of III, that $T \notin CI + \mathcal{F}$. We shall construct sequences $\{x_n\}$, $\{y_n\}$ of unit vectors together with associated projections $E_{m(n)}$ and $E_{j(n)}$ satisfying

(i) $\langle x_j, x_k \rangle = \langle Tx_j, x_k \rangle = 0$ for $j \neq k$.

(ii) $x_n = E_{m(n)} x_n$ and $y_n = (E_{j(n)} - E_{m(n)}) y_n$.

(iii) $\{Ty_k - \langle Tx_k, x \rangle y_k\}_{k=1}^{n}$ are linearly independent vectors for each $n \in N$. The construction is an inductive one.

$k = 1$. Let $x_1 = e_1$. If for every e_j, $j > 1$, $Te_j = \langle Te_1, e_1 \rangle e_j$, it will immediately follow that $T = \langle Te_1, e_1 \rangle I + K$ for K, a rank two operator, contrary to our assumption. Take $y_1 = e_k$, where k is the first integer with $Te_k \neq \langle Te_1, e_1 \rangle e_k$. It is easily seen that (x_1, y_1) satisfies (i), (ii) and (iii) above.

$k = n$ *implies* $k = n + 1$. Suppose that $\{x_i\}_{i=1}^{n}$ and $\{y_i\}_{i=1}^{n}$ have been chosen to satisfy (i) through (iii). Let H_n be c.l.s. $\{x_1, \ldots, x_n, Tx_1^*, \ldots, Tx_n^*\}$ and note that $E_{2n+1}(H_n) \subsetneqq E_{2n+1}(H)$. From this we deduce the existence of a unit vector $x_{n+1} = E_{2n+1} x_{n+1}$ satisfying (i) for $j, k \leq n + 1$. Take $E_{m(n+1)} = E_{2n+1}$.

Define $\tilde{H}_n$ to be c.l.s. $\{y_1, \ldots, y_n, Ty_1^*, \ldots, Ty_n^*\}$ and $\lambda = \langle Tx_{n+1}, x_{n+1} \rangle$. Suppose that, for every $I > E \geq E_{m(n+1)}$ and $y \in (E - E_{m(n+1)})\tilde{H}_n$, $Ty - \lambda y$ is in $\tilde{H}_n$. It would immediately follow that $(T - \lambda)(I - E_{m(n+1)}) \in \mathcal{F}$. That is, $T = \lambda I + F$ for some F in $\mathcal{F}$, contrary to our assumption. Thus, for some $j(n + 1) > m(n + 1)$, we have both $y_{n+1} \in (E_{j(n+1)} - E_{m(n+1)})H$ and $Ty_{n+1} - \lambda y_{n+1} \notin \tilde{H}_n$.

Let A be the operator

$$A = \sum_{n=1}^{\infty} x_n \otimes y_n.$$

Now each $x_n \otimes y_n$ is in $\mathscr{A}$ and $\mathscr{A}$ is strongly closed; therefore, $A \in \mathscr{A}$. Consider the vector $w_k = (TA - AT)x_k = Ty_k - \langle Tx_k, x_k \rangle y_k$. From (iii) it follows that, for each n, $\{w_k\}_{k=1}^n$ are linearly independent vectors in the range of $\delta_T(A)$.

Case (iii). If X derives $\mathscr{A}$ into $\mathscr{J}$, then X^* derives $\mathscr{A}^*$ into $\mathscr{J}$. Since $\mathscr{A}^* = \mathrm{Alg}\{I - E_n\}$, where $\{I - E_n\}$ satisfies the hypotheses of case (ii), we obtain case (iii).

In order to prove IV, we simply combine III with the main result of [4], which says that any derivation of a nest algebra into $\mathscr{L}(H)$ is inner.

COROLLARY. *It easily follows that for any generalized commutator pair AB, with $AT - TB$ in $\mathscr{J}$ for all T in $\mathscr{A}$ implies A, B are both in $CI + \mathscr{J}$.*

REMARK. There has been considerable recent interest in automorphisms of perturbed algebras [14], determining under which circumstances an automorphism of $\mathscr{A} + \mathscr{J}$ is inner. For nests indexed by $\mathbf{N}$ and $\mathscr{J} = \mathscr{K}$, it is shown in [14] that every automorphism is inner. In the general situation there will exist outer automorphisms (for example, the bilateral shift acting on the classical nest of one-dimensional jumps indexed by $\mathbf{Z}$). Indeed, it is shown in [16] and [6] that these have a rather rich structure being isomorphic to the group of all dimension preserving order isomorphisms of the underlying nest. However, a key to all these results is the fact [2] that $\mathscr{A} + \mathscr{K}$ is precisely all operators T in $\mathscr{L}(H)$ such that $E \to (I - E)TE$ is continuous from $\mathscr{E}$ (strong operator topology) to $\mathscr{K}$ (norm topology). In the situation of arbitrary (two sided) ideals, this does not hold even for tractable classes such as symmetrically normed ideals [12].

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