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(s)-NUCLEAR SETS AND OPERATORS

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The purpose of this paper is to demonstrate considerable similarities in the behaviour of compact and (s)-nuclear operators. More precisely, we obtain for (s)-nuclear operators results resembling previously known properties of compact operators; sometimes a word for word translation of a "compact theorem" holds for (s)-nuclear operators. However, we wish to emphasize that different methods for the proofs are now needed. For example, the often applied Ascoli-Arzela theorem does not have a (s)-nuclear counterpart (see §5).

1. Introduction. Given a bounded subset D of a Banach space E, denote by

$$\delta_n(D) = \inf\{r > 0: D \subset F_n + rB_E\}$$

its nth Kolmogorov diameter, $n \in \mathbf{N}$. Here the infimum is taken over all subspaces $F_n \subset E$ of dimension not greater than n and B_E denotes the closed unit ball of E. For an operator $T \in L(E, F)$ define $\delta_n(T) = \delta_n(TB_E)$. Now, D is (relatively) compact if and only if $(\delta_n(D))_1^\infty \in c_0$. Analogously we define the (s)-nuclear sets when we replace c_0 by the space (s) of rapidly decreasing sequences,

$$(s) = \left\{(\lambda_n)_1^\infty: \sup_n n^k|\lambda_n| < \infty \ \forall k \in \mathbf{N}\right\}.$$

In other words, D is called *(s)-nuclear* if $(\delta_n(D))_1^\infty \in (s)$. Note that we have no need for a separate notion for "relative" (s)-nuclear or non-closed (s)-nuclear sets.

A bounded operator $T \in L(E, F)$ is said to be (s)-nuclear if the set TB_E is (s)-nuclear, i.e. $(\delta_n(T))_1^\infty \in (s)$. That happens if and only if (see **[11]**) T has a representation

$$Tx = \sum_{i=1}^{\infty} \lambda_i \langle x, y_i' \rangle z_i,$$

where $\|y_i'\|$, $\|z_i\| \leq 1$ and $(\lambda_i)_1^\infty \in (s)$. This is the historical reason for using the term (s)-nuclear rather than (s)-compact.

Besides the whole class of all (s)-nuclear operators we discuss the properties of a class of sub-ideals, the $\Lambda(\alpha)$-nuclear operators. Here $\alpha = (\alpha_i)_1^\infty$, $0 < \alpha_1 \leq \alpha_2 \leq \cdots$ and

$$\text{(1)} \qquad \Lambda(\alpha) = \left\{(\lambda_n)_1^\infty: \sup_n R^{\alpha_n}|\lambda_n| < \infty \ \forall R \in \mathbf{R}_+\right\}.$$

If $\alpha_n = \log n$ and $R = e^k$, then $R^{\alpha_n} = n^k$ and so for this exponent sequence $\Lambda(\alpha) = (s)$. In general, we assume that $\Lambda(\alpha)$ is a nuclear space and equivalently that $\Lambda(\alpha) \subset (s)$ or that

$$\log n \leq M\alpha_n, \qquad n \in \mathbf{N}. \tag{2}$$

$\Lambda(\alpha)$-nuclear sets and operators are then defined in the obvious manner. For further information on $\Lambda(\alpha)$-nuclearity we refer to **[9]**, **[10]**, and **[11]**.

First we study (s)-nuclear sets of (s)-nuclear operators. For compact operators the problem was solved by Palmer **[7]**. He proved that, for instance, the following conditions are equivalent for a bounded closed subset $H \subset L(E, F)$:

(3) H is a compact set of compact operators.

(4) $H(B_E)$ and $H'(B_{F'})$ are both relatively compact.

Here $H(B_E) = \{Tx\colon\ T \in H, x \in B_E\}$ and $H' = \{T'\colon\ T \in H\}$. We shall give a similar result for (s)-nuclear operators. However, the implication (3) ⇒ (4) which is trivial in the compact case is, considered with a verbatim translation, false for (s)-nuclear operators (see Example 3.7). Hence we define the notion of uniform (s)-nuclearity; we say a set $H \subset L(E, F)$ consists of *uniformly* (s)*-nuclear* operators if the sequences of the diameters $(\delta_n(T))_{n=1}^{\infty}$, $T \in H$, form a bounded set in (s). The topology of (s) is, of course, given by the seminorms

$$p_k(\lambda) = \sup_n n^k|\lambda_n|, \quad \lambda = (\lambda_n)_1^{\infty}, \qquad k \in \mathbf{N}. \tag{5}$$

Now we have

1.1. THEOREM. *Let E and F be Banach spaces and assume $H \subset L(E, F)$ is bounded. Then the following conditions are equivalent.*

(a) *H is a (s)-nuclear set of uniformly (s)-nuclear operators.*

(b) *$H(B_E)$ and $H'(B_{F'})$ are (s)-nuclear.*

(c) *$H(B_E)$ is (s)-nuclear and H is of equal (s)-variation.*

(d) *The sets $H(x)$, $x \in B_E$, are uniformly (s)-nuclear and H is of equal (s)-variation.*

For the undefined notions in (c) and (d) we refer to §§2 and 3. The equivalence of (b) and (c) follows from characterizations of collective (s)-nuclearity, given in Theorem 2.5, which are presumably of independent interest. The corresponding results for compact operators were obtained by Palmer **[7]** and Geue **[6]**; see also **[4]**.

As in the compact case we get as corollaries a number of new proofs for (known) permanence properties. For example, Theorem 1.1 implies that $T \otimes_\varepsilon R$ and $T \otimes_\pi R$ are (s)-nuclear if and only if both T and R are (s)-nuclear.

Finally, we study how far one can generalize Theorem 1.1 to the subspaces $\Lambda(\alpha)$ of (s). It will turn out that the results of Theorem 1.1 hold for the $\Lambda(\alpha)$-nuclear operators if and only if the exponent sequence α satisfies

$$(6) \qquad \alpha_{n^2} \leq C\alpha_n, \qquad n \in \mathbf{N},$$

a condition which is known to be equivalent to $\Lambda(\alpha) \otimes \Lambda(\alpha) \approx \Lambda(\alpha)$ (see [**3**] or [**12**]).

2. (s)-nuclear sets. We start with yet another characterization of (s)-nuclearity.

2.1. DEFINITION. For a bounded set $D \subset E$, *the nth entropy number* $e_n(D)$ is defined as the infimum of all $r > 0$ such that there are points $y_1, \ldots, y_q$ with $q \leq 2^{n-1}$ and

$$D \subset \bigcup_1^q \{y_i + rB_E\}.$$

If $T \in L(E, F)$, we write $e_n(T) = e_n(TB_E)$. For more details on entropy numbers of operators see [**8**].

Recall that a set is called balanced if $\lambda D \subset D$ for $|\lambda| \leq 1$.

2.2. LEMMA. *A convex balanced subset $D \subset E$ is (s)-nuclear if and only if $(e_n(D))_{n=1}^\infty \in (s)$.*

Proof. We may assume that D is separable. Let $\{x_i\colon i \in \mathbf{N}\}$ be a dense subset of D, write e_i for the ith canonical basis vector of l^1 and define the operator $T \in L(l^1, E)$ by $Te_i = x_i$, $i \in \mathbf{N}$. Since $\delta_n(T) = \delta_n(D)$ and $e_n(T) = e_n(D)$, we must show that (s)-nuclear operators are characterized by rapidly decreasing entropy numbers. For this we apply the results of [**8**], Chapter 12, where only real Banach spaces are considered. The complex case can be treated similarly.

According to [**8**], 12.3.2 we have the inequality

$$(7) \qquad \delta_n(T) \leq n e_n(T), \qquad n \in \mathbf{N}.$$

To prove a converse we reason as in [**8**], 14.3.11. First, combining [**8**], 11.12.2 and 12.3.3, we get

$$(8) \qquad e_n(T) \leq 2m\delta_m(T) + \frac{8\|T\|}{2^{(n-1)/(m-1)}}, \qquad m > 1.$$

Then, if $(n-1)/\log n \leq 2km \leq 2k + (n-1)/\log n$ and $m^{2k}\delta_m(T) \leq C_k$,

$$(9) \quad e_n(T) \leq C_k m^{1-2k} + 8\|T\|n^{-k} \leq C_k\left(\frac{n-1}{k\log n}\right)^{1-2k} + 8\|T\|n^{-k}$$

$$\leq \left(A_k + 8\|T\|\right)n^{-k}, \qquad n > 1,$$

for some constants C_k, A_k depending only on k. □

2.3. COROLLARY. *If $D \subset E$ is bounded and $J \in L(E, F)$ is an isometry, then D is (s)-nuclear if and only if JD is (s)-nuclear in F.*

Proof. Since D is (s)-nuclear if and only if the balanced convex hull of D is (s)-nuclear and since $e_n(JD) \leq e_n(D) \leq 2e_n(JD)$, the claim follows from Lemma 2.2. □

Another proof of Corollary 2.3 is given in [**10**].

The next result is well known (see, for instance, [**8**], 11.7.4 and 11.12.)

2.4. LEMMA. *If $T \in K(E, F)$, then $\delta_n(T') \leq 2n\delta_n(T)$ and $\delta_n(T) \leq 2n\delta_n(T')$.*

In approximation theory a collection of operators $H \subset L(E, F)$ is called collectively compact if HB_E is relatively compact in F (c.f. [**1**] and the references therein). Hence it is natural to use the term *collectively (s)-nuclear* for sets of operators H such that HB_E is (s)-nuclear. As the main topic of this section we prove some equivalent conditions for collective (s)-nuclearity.

We introduce, for each bounded set $H \subset L(E, F)$, the notion of its sequence of *equi-variation measures* $v_n(H)$. For $n = 1, 2, \ldots$ the number $v_n(H)$ is defined as the infimum of those $r > 0$ for which there exists a cover $A_1, A_2, \ldots, A_{2^{n-1}}$ of B_E by at most 2^{n-1} sets such that for each i, $1 \leq i \leq 2^{n-1}$,

$$\sup\left\{\|Tx - Ty\| : T \in H,\ x, y \in A_i\right\} \leq r.$$

As is easily seen H is of equal variation in the sense of Vala [**13**] exactly when $(v_n(H))_1^\infty \in c_0$. Therefore H is said to be of *equal (s)-variation* if $(v_n(H))_1^\infty \in (s)$.

2.5. THEOREM. *Let $H \subset L(E,F)$ be bounded. Then the following conditions are equivalent.*

(a) *$H(B_E)$ is (s)-nuclear.*

(b) *H' has equal (s)-variation.*

(c) *There exists a sequence of subspaces $F_n \subset F'$ and a sequence of real numbers λ_n such that*

$$\|H'|_{F_n}\| = \lambda_n, \operatorname{codim} F_n \le n \quad \textit{and} \quad (\lambda_n)_1^\infty \in (s).$$

Proof. (a) $\Rightarrow$ (c). If $H(B_E)$ is (s)-nuclear and $\lambda_n = 2\delta_n(HB_E)$, we can find for each $n \in \mathbf{N}$ an n-dimensional subspace $G_n \subset F$ such that $H(B_E) \subset G_n + \lambda_n B_F$. Let $P_n \in L(F)$ be a projection onto G_n with norm $\|P_n\| \le n$. (cf. [**8**], B.4.9). Then the subspace $F_n = (I - P_n')F'$ has codimension n in F' and for any $T \in H$ we have

$$\begin{aligned}\|T'|_{F_n}\| &\le \|T'(I - P_n')\| \\ &= \|(I - P_n)T\| \le \lambda_n\|I - P_n\| \le (n+1)\lambda_n,\end{aligned}$$

where $((1+n)\lambda_n)_1^\infty \in (s)$.

(c) $\Rightarrow$ (b). If $\|H'|_{F_n}\| = \lambda_n$ and $F' = F_n \oplus E_n$, $\dim E_n \le n$, let P_n and Q_n be projections onto E_n and F_n, respectively. We may assume that $\|P_n\| \le n$, $\|Q_n\| \le (n+1)$ and that $P_n + Q_n = I$; then $T' = T'P_n + T'Q_n$.

Since $e_m(P_n) \le 4\|P_n\|2^{(1-m)/n} \le 4n2^{(1-m)/n}$ (see [**8**], p. 171), $B_{F'}$ can be partitioned into sets A_i, $1 \le i \le 2^{m-1}$, such that $\|P_n x - P_n y\| \le 8n2^{(1-m)/n}$ for all $x, y \in A_i$. So if $x, y \in A_i$ and $T \in H$,

$$\begin{aligned}\|T'x - T'y\| &\le \|T'Q_n(x-y)\| + \|T'\|\|P_n x - P_n y\| \\ &\le 2(n+1)\lambda_n + 8n\|H\|2^{(1-m)/n}\end{aligned}$$

where $\|H\| = \sup\{\|T\|\colon\ T \in H\} < \infty$. Thus $v_m(H') \le 4n\lambda_n + 8n\|H\|2^{(1-m)/n}$ and in the same way as we proved the implication (8) $\Rightarrow$ (9) we deduce $(v_n(H'))_1^\infty \in (s)$.

(b) $\Rightarrow$ (a). Denote by $L^\infty(H,E')$ the space of all bounded mappings from H into E' and equip $L^\infty(H,E')$ with the supremum norm. Moreover, define

$$J\colon F' \to L^\infty(H,E'), \quad (Jx')(T) = T'x'.$$

Since $\|Jx' - Jy'\|_\infty = \sup\{\|T'x' - T'y'\|\colon T \in H\}$, we have $e_n(JB_{F'}) \le v_n(H')$. As H' is assumed to have equal (s)-variation, we see that J is (s)-nuclear.

Next, let π_T: $L^\infty(H, E') \to E'$ be the evaluation at T. Then $\pi_T J x' = T'x'$ or $\pi_T \circ J = T'$ which gives $J' \circ (\pi_T)' = T''$. Thus $H''(B_{E''}) \subset J'(B_{G'})$, where $G = L^\infty(H, E')$, and so according to Lemma 2.4, H'' is collectively (s)-nuclear. But if I_E: $E \to E''$ is the canonical isometry, $I_F H(B_E) = H'' I_E(B_E)$. Therefore the (s)-nuclearity of HB_E follows from Corollary 2.3. □

2.6. Remark. In a similar fashion one proves the equivalence of the three conditions (α)-(γ) below:

(α) H' is collectively (s)-nuclear;

(β) H has equal (s)-variation;

(γ) There exists a sequence of subspaces $E_n \subset E$ and as equence of real numbers λ_n such that

$$\|H \mid_{E_n}\| = \lambda_n, \quad \operatorname{codim} E_n \le n \quad \text{and} \quad (\lambda_n)_1^\infty \in (s).$$

We leave the details to the reader.

3. (s)-nuclear operators. We are now ready for the proof of Theorem 1.1; we devide the proof into five steps.

3.1. Lemma. *Let $H \subset L(E, F)$ be bounded. If both $H(B_E)$ and $H'(B_{F'})$ are (s)-nuclear, then H is an (s)-nuclear set in $L(E, F)$.*

Proof. Since the mapping $T \to T'$ is an isometry, by Corollary 2.3 it suffices to show that H' is an (s)-nuclear subset of $L(F', E')$.

If we let $\delta_n = \delta_n(HB_E)$ and $\lambda_n = \delta_n(H'B_{F'})$, then by assumption $(\delta_n)_{n=1}^\infty, (\lambda_n)_{n=1}^\infty \in (s)$. Furthermore, there exist for each $n \in \mathbf{N}$ n-dimensional subspaces $F_n \subset F$ and $E_n \subset E'$ such that

$$H(B_E) \subset F_n + 2\delta_n B_F, \qquad H'(B_{F'}) \subset E_n + 2\lambda_n B_{E'}. \tag{10}$$

Now, choose projections $P \in L(F)$ onto F_n and $Q \in L(E')$ onto E_n with $\|P\|, \|Q\| \le n$. If $T \in H$,

$$T' = T'P' + T'(I - P') = QT'P' + (I - Q)T'P' + T'(I - P'),$$

where, by (10), $\|T'(I - P')\| = \|(I - P)T\| \le 2\delta_n(1 + n)$ and

$$\|(I - Q)T'P'\| \le \|(I - Q)T'\| n \le 2n(1 + n)\lambda_n.$$

On the other hand, since P' and Q have the rank n, one easily sees that the operator $\operatorname{Hom}(P', Q)$: $S \to QSP'$ has rank equal to n^2, i.e. the set $\{QSP'\colon S \in L(F', E')\}$ is an n^2-dimensional subspace of $L(F', E')$. Hence $\delta_{n^2}(H') \le 2(n + 1)\delta_n + 2n(n + 1)\lambda_n$. Consequently, if $k \in \mathbf{N}$ is fixed, we choose for each $p \in \mathbf{N}$ a natural number n such that $n^2 \leq p < (n+1)^2$; then

$$p^k \delta_k(H') < (n+1)^{2k} \delta_{n^2}(H') \leq 4^{k+1} n^{2k+1} \delta_n + 4^{k+1} n^{2k+2} \lambda_n \leq M_k < \infty$$

where M_k depends only on k (especially, not on p). □

3.2. LEMMA. *Let $D \subset E$ be convex, balanced and bounded. If $\delta_n = \delta_n(D)$, there are points $x_i \in D$, $1 \leq i \leq n$, such that*

$$n^{-1} D \subset \operatorname{bco}\{x_i\}_1^n + 5(n+1)\delta_n B_E.$$

(Here bco *denotes the balanced convex hull.)*

Proof. There exists an n-dimensional subspace $E_n \subset E$ such that $D \subset E_n + 2\delta_n B_E$. Let $P_n \in L(E)$ be a projection onto E_n with $\|P_n\| \leq n$. Then

$$D \subset P_n D + (I - P_n) D \subset P_n D + 2(n+1)\delta_n B_E.$$

Let F be the space spanned by $P_n D$ having as it norm the Minkowski functional μ of $P_n D$. The Auerbach lemma applied to F shows that there are vectors y_i, $1 \leq i \leq n$, with $\mu(y_i) \leq 1$ such that any $y \in P_n D$ has a representation

$$y = \sum_{i=1}^{n} \alpha_i y_i, \qquad |\alpha_i| \leq 1.$$

Furthermore, for each $\lambda \in (0,1)$ we can find vectors $x_i \in D$ with $P_n(x_i) = \lambda y_i \in P_n D$. Then

$$\begin{aligned} \|x_i - y_i\| &\leq (1-\lambda)\|y_i\| + \|x_i - \lambda y_i\| \\ &\leq (1-\lambda)\|P_n D\| + \|(I - P_n)x_i\| \leq 3(n+1)\delta_n, \end{aligned}$$

if and only if λ is chosen so that $(1-\lambda)\|P_n D\| \leq (n+1)\delta_n$. In that case

$$n^{-1} P_n(D) \subset \operatorname{bco}\{x_i\}_1^n + 3(n+1)\delta_n B_E. \qquad \square$$

3.3. LEMMA. *Let $H \subset L(E,F)$ be an (s)-nuclear set of uniformly (s)-nuclear operators. Then HB_E is (s)-nuclear.*

Proof. We may clearly assume that H is balanced and convex. Then, if $\delta_n = \delta_n(H)$, by Lemma 3.2 there are operators $T_i \in H$, $1 \leq i \leq n$, for which

$$n^{-1} H \subset \operatorname{bco}\{T_i\}_1^n + 10 n \delta_n B_{L(E,F)}.$$

Next, by the uniform (s)-nuclearity we have for the sequence $\mu_n = \sup\{\delta_n(T): T \in H\}$ that $(\mu_n)_1^\infty \in (s)$. Hence there exists for each i an n-dimensional subspace $F_n^i \subset F$ such that

$$T_i(B_E) \subset F_n^i + 2\mu_n B_F.$$

Consequently, if G is the linear span of $\{F_n^i: 1 \le i \le n\}$, then $\dim(G) \le n^2$ and

$$H(B_E) \subset G + (2n\mu_n + 10n^2\delta_n) B_F.$$

This gives $\delta_{n^2}(HB_E) \le 2n\mu_n + 10n^2\delta_n$ which shows, like in the proof of Lemma 3.1, that $(\delta_n(HB_E))_1^\infty \in (s)$. □

3.4. LEMMA. *Suppose that $H \subset L(E, F)$ has equal (s)-variation and that the sets $H(x)$, $x \in B_E$, are uniformly (s)-nuclear (that is, for $\mu_n = \sup\{\delta_n(H(x)): x \in B_E\}$ we have $(\mu_n)_1^\infty \in (s)$). Then HB_E is (s)-nuclear.*

Proof. Since H has equal (s)-variation, by Remark 2.6 there exists a sequence of subspaces E_n such that $\|H|_{E_n}\| = \lambda_n$, co-dim$(E_n) \le n$ and $(\lambda_n)_1^\infty \in (s)$. Hence, if $P_n \in L(E)$ is the projection onto the co-summand of E_n with $\|P_n\| \le n$,

$$HB_E \subset HP_nB_E + (n+1)\lambda_n B_F.$$

Moreover, as rank$(P_n) \le n$ and $\|P_nB_E\| \le n$, there are vectors $x_i \in E$, $1 \le i \le n$, with $\|x_i\| \le n^2$ such that P_nB_E is contained in the convex balanced hull of $\{x_i\}_1^n$. If $F_n^i \subset F$ is a subspace for which $\dim(F_n^i) \le n$ and

$$H(x_i) \subset F_n^i + 2n^2\mu_n B_F,$$

then we see that $HB_E \subset G + (2n^2\mu_n + 2n\lambda_n)B_F$ where $G = \operatorname{span}\{F_n^i: 1 \le i \le n\}$ with $\dim(G) \le n^2$. Thus $\delta_{n^2}(GB_E) \le 2n^2\mu_n + 2n\lambda_n$ and we get $(\delta_n(HB_E))_1^\infty \in (s)$. □

3.5. *The proof of Theorem* 1.1. To prove that (a) and (b) are equivalent assume that H is an (s)-nuclear set of uniformly (s)-nuclear operators. Since the mapping $T \to T'$ is an isometry and since $\delta_n(T') \le 2n\delta_n(T)$, H' is a (s)-nuclear set of uniformly (s)-nuclear operators. Then Lemma 3.3, applied to H and H', shows that both HB_E and $H'B_{F'}$ are (s)-nuclear. The converse follows from Lemma 3.1.

For the other conditions the equivalence of (b) and (c) follows from Theorem 2.5 and Remark 2.6, the implication (c) ⇒ (d) is trivial and finally, Lemma 3.4 gives the converse (d) ⇒ (c). □

3.6. REMARK. If HB_E is (s)-nuclear, then $H'B_{F'}$ (and thus H as a subset of $L(E, F)$) need not be (s)-nuclear. Take, for instance, a fixed vector y in a Hilbert space E with an orthonormal basis $\{f_k: k \in \mathbf{N}\}$ and define $T_n x = \langle x, f_n \rangle y$, $H = \{T_n: n \in \mathbf{N}\} \subset L(E)$. As HB_E is bounded and 1-dimensional, it is (s)-nuclear. However, $H'B_{E'}$ is not even relatively compact since it contains all the f_k's.

3.7. EXAMPLE. Let $\{A_k: k \in \mathbf{N}\}$ be a partition of natural numbers, i.e. $\mathbf{N} = \bigcup_{k=1}^{\infty} A_k$ and $A_j \cap A_k = \varnothing$ when $j \neq k$. Assume that $\#(A_k)$, the cardinality of A_k, satisfies $2e^k \leq \#(A_k) \leq 2e^{k+1}$. Now, let E be as in Remark 3.6 a Hilbert space with an orthonormal basis $\{f_k: k \in \mathbf{N}\}$. Denote by $P_k \in L(E)$ the orthogonal projection $P_k: E \to \operatorname{span}\{f_i: i \in A_k\}$.

If $T_k = e^{-k}P_k$, clearly $\|T_k\| = e^{-k}$, $k \in \mathbf{N}$. Thus, if $H = \{T_k: k \in \mathbf{N}\}$, for any k we have $\delta_k(H) \leq e^{-(k+1)}$. As the T_k's are finite dimensional operators, we see that H is a (s)-nuclear set of (s)-nuclear operators. However, if $e^k \leq n < e^k + 1$, then

$$\delta_n(HB_E) \geq \delta_n(T_k) = e^{-k} \geq n^{-1}.$$

Hence $\sup\{n^2\delta_n(HB_E): n \in \mathbf{N}\} = \infty$ and HB_E is not (s)-nuclear. Consequently, the requirement of uniform (s)-nuclearity in Theorem 1.1(a) cannot be replaced by mere (s)-nuclearity.

4. $\Lambda(\alpha)$-nuclear sets of $\Lambda(\alpha)$-nuclear operators. The proof of Theorem 1.1 was based essentially on the following three properties of the space (s).

(i) if $(\lambda_n)_1^\infty \in (s)$, then $(n\lambda_n)_1^\infty \in (s)$.

(ii) if $(\lambda_n)_1^\infty \in (s)$, $0 \leq \mu_{n^2} \leq \lambda_n$ and μ_n is decreasing, then $(\mu_n)_1^\infty \in (s)$.

(iii) $(\delta_n(T))_1^\infty \in (s)$ if and only if $(e_n(T))_1^\infty \in (s)$.

It is easy to see that if the subspace $\Lambda(\alpha)$ of (s) has the same three properties, then $\Lambda(\alpha)$-nuclear sets of $\Lambda(\alpha)$-nuclear operators admit a description as in Theorem 1.1.

Now the condition (i) is automatically satisfied if $\Lambda(\alpha) \subset (s)$, i.e. (2) holds. Furthermore, a natural assumption to guarantee (ii) is the condition (6), $\alpha_{n^2} \leq C\alpha_n$. It turns out that the same requirement gives (iii), too.

4.1. LEMMA. *Suppose* $\log n \leq M\alpha_n$ *and* $\alpha_{n^2} \leq C\alpha_n$, $n \in \mathbf{N}$. *Then for any* $T \in L(E, F)$,

$$(\delta_n(T))_1^\infty \in \Lambda(\alpha) \text{ if and only if } (e_n(T))_1^\infty \in \Lambda(\alpha).$$

Proof. Since $\delta_n(T) \le ne_n(T)$, cf. (7), the sufficiency part is trivial. To prove the necessity we first show that

$$\lim_{n\to\infty} \frac{n}{(\alpha_n)^2} = \infty. \tag{11}$$

For (11) define $q(n) \in \mathbf{N}$ by $\log q(n) = 2^n \log 2$. As $\alpha_{n^2} \le C\alpha_n$ and $q(n)^2 = q(n+1)$, $\alpha_{q(n)} \le C\alpha_{q(n-1)} \le C^2\alpha_{q(n-2)} \le \cdots \le C^n\alpha_{q(0)} = C^n\alpha_2$.

If now $q(n) \le p < q(n+1)$, then $\alpha_p \le \alpha_{q(n+1)} \le C^{n+1}\alpha_2$ which yields $\log\alpha_p \le (n+1)\log C + \log\alpha_2 \le (n+1)C_0$; here C_0 is a positive constant. However, $2^n \log 2 \le \log p$ and therefore we can estimate

$$\log\left(p/\alpha_p^2\right) = \log p - 2\log\alpha_p \ge 2^n \log 2 - (n+1)2C_0$$

Letting n (or p) tend to infinity gives (11).

Secondly, if $k \in \mathbf{N}$ is fixed and $(n-1)/\alpha_n \le km \le k + (n-1)/\alpha_n$, then it holds

$$\text{(i) } k\alpha_n \le (n-1)/(m-1) \quad \text{and} \quad \text{(ii) } \alpha_n \le r\alpha_m \tag{12}$$

if only $r \in \mathbf{N}$ is large enough. Indeed, the first inequality in (12) is obvious while for the other take a number $n_0 \in \mathbf{N}$ such that $(n-1)/(k\alpha_n)^2 \ge 2$, $n \ge n_0$. As $n \le 2(n-1) \le (n-1)^2/(k\alpha_n)^2 \le m^2$ for $n \ge \max\{n_0, 2\}$, there exists a constant C_1 (depending on k) such that $\alpha_n \le C_1\alpha_{m^2} \le C_1C\alpha_m$ for all $n \in \mathbf{N}$. If we choose $r \in \mathbf{N}$ larger than C_1C, we obtain (12).

The proof of the necessity follows from the formulae (8) and (12). If $\sup\{R^{\alpha_n}\delta_n(T)\colon n \in \mathbf{N}\} < \infty$ for each $R \in \mathbf{R}_+$, the claim is that then also $R^{\alpha_n}e_n(T) \le C_R$ for some constant C_R independent of n. We may clearly assume that R has the form $R = 2^k$, $k \in \mathbf{N}$. Moreover, if for each $n \in \mathbf{N}$ we pick m so that (12) holds we obtain from (8)

$$\begin{aligned} R^{\alpha_n}e_n(T) &\le 2^{k\alpha_n}2m\delta_m(T) + 8\|T\| \\ &\le C_{k,r}2^{k\alpha_n}2^{-kr\alpha_m} + 8\|T\| \le C_{k,r} + 8\|T\| < \infty. \qquad \square \end{aligned}$$

Analogous to the (s)-nuclear case we say that a subset $H \subset L(E,F)$ consists of *uniformly* $\Lambda(\alpha)$-*nuclear* operators if for the sequence $\mu_n = \sup\{\delta_n(T)\colon T \in H\}$ we have $(\mu_n)_1^\infty \in \Lambda(\alpha)$. Also, in a corresponding way we define the notions of equal $\Lambda(\alpha)$-variation and uniformly $\Lambda(\alpha)$-nuclear sets, cf. Chapters 2 and 3. Combining the above results we get now

4.2. THEOREM. *Let $\Lambda(\alpha) \subset (s)$ and suppose $\alpha_{n^2} \le C\alpha_n$. Then the following conditions are equivalent for any bounded subset $H \subset L(E,F)$.*

(a) *H is a $\Lambda(\alpha)$-nuclear set of uniformly $\Lambda(\alpha)$-nuclear operators.*

(b) *HB_E and $H'B_{F'}$ are $\Lambda(\alpha)$-nuclear.*

(c) *H has equal $\Lambda(\alpha)$-variation and the sets $H(x)$, $x \in B_E$, are uniformly $\Lambda(\alpha)$-nuclear.*

4.3. REMARK. As is easily seen also the counterpart of Theorem 2.5 holds for the ideal of $\Lambda(\alpha)$-nuclear operators if only $\Lambda(\alpha) \subset (s)$ and $\alpha_{n^2} \leq C\alpha_n$.

Theorem 4.2 has a converse, too. Applying a result of H. Apiola [3] we shall show that if α is any nuclear exponent sequence such that Theorem 4.2 holds for the $\Lambda(\alpha)$-nuclear operators, then necessarily $\alpha_{n^2} \leq C\alpha_n$.

4.4. THEOREM (*Apiola* [3], *Theorem* 3.2). *Let $\Lambda(\alpha) \subset (s)$ and suppose that for any pair of $\Lambda(\alpha)$-nuclear operators T, R also the product* Hom(T, R): *$S \to RST$ is a $\Lambda(\alpha)$-nuclear map between the corresponding operator spaces. Then we must have $\alpha_{n^2} \leq C\alpha_n$ for all $n \in \mathbf{N}$.*

4.5. COROLLARY. *Let $\Lambda(\alpha) \subset (s)$ and suppose that the conditions* (a), (b) *of Theorem* 4.2 *are equivalent for any bounded subset $H \subset L(E, F)$. Then $\alpha_{n^2} \leq C\alpha_n$.*

Proof. We shall show that "Hom-stability" is a consequence of Theorem 4.2. The claim will then follow from Apiola's theorem. A similar reasoning, based on the notion of equal variation, is given for compact operators in [2].

Now, suppose $T \in L(E_1, F_1)$ and $R \in L(E_2, F_2)$ are both $\Lambda(\alpha)$-nuclear. If we define

$$H = \mathrm{Hom}(T, R)B_{L(F_1,E_2)} = \{RST\colon \|S\| \leq 1,\ S \in L(F_1, E_2)\},$$

then we need to show that H is a $\Lambda(\alpha)$-nuclear set. But obviously $HB_{E_1} \subset \|T\|RB_{E_2}$ and $H'B_{F_2} \subset \|R\|T'B_{F'_1}$. Since also T' is $\Lambda(\alpha)$-nuclear (Lemma 2.4) and since the implication (b) $\Rightarrow$ (a) of Theorem 4.2 is assumed to hold H is, indeed, a $\Lambda(\alpha)$-nuclear subset of $L(E_1, F_2)$. □

Above we could have shown that, under the assumption of the equivalence of (a), (b) for $\Lambda(\alpha)$-nuclear maps, if Hom(T, R) is $\Lambda(\alpha)$-nuclear then both T and R are $\Lambda(\alpha)$-nuclear; the suitable subset H would have been $H = \{RST\colon \mathrm{rank}\ S = 1,\ \|S\| \leq 1\}$.

Stating this remark differently we see that the following known result is a consequence of Theorem 4.2.

4.6. COROLLARY. *Suppose* $\Lambda(\alpha) \subset (s)$ *and* $\alpha_{n^2} \leq C\alpha_n$. *Then the product* Hom$(T, R)$ *of the operators* T *and* R *is* $\Lambda(\alpha)$*-nuclear if and only if both* T *and* R *are* $\Lambda(\alpha)$*-nuclear.*

Since $(T \otimes_\pi R)' = \text{Hom}(T, R')$ and since $T \otimes_\varepsilon R$ can be identified with a restriction of Hom(T', R), Theorem 4.2 yields stability results also for tensor product operators.

4.7. COROLLARY. *Let* $\Lambda(\alpha) \subset (s)$ *and* $\alpha_{n^2} \leq C\alpha_n$. *Then* T, R *are* $\Lambda(\alpha)$*-nuclear if and only if* $T \otimes_\pi R$ *(or* $T \otimes_\varepsilon R$*) is* $\Lambda(\alpha)$*-nuclear.*

For the standard proof of Corollary 4.7 see **[12]** or **[2]**.

Finally we mention a result whose compact version was proved by Bonsall **[5]**.

4.8. COROLLARY. *Suppose* $\Lambda(\alpha) \subset (s)$ *and* $\alpha_{n^2} \leq C\alpha_n$. *If* $T \in L(E)$ *denote by* C_T *the centralizer of* T, $C_T = \{S \in L(E)\colon\ TS = ST\}$, *and define* $K \in L(C_T)$ *by* $K(S) = ST$.

Then, if T is $\Lambda(\alpha)$-nuclear, so is K.

Proof. Let $H = \{ST\colon\ S \in C_T,\ \|S\| \leq 1\}$. As $HB_E \subset TB_E$ and $H'B_{E'} \subset T'B_{E'}$, Theorem 4.2 shows that H is $\Lambda(\alpha)$-nuclear in C_T. □

5. Concluding remarks. One can easily see that the method of Theorem 1.1, the use of finite-dimensional projections, does not work without serious modifications for general compact operators. On the other hand, the known proofs for characterizations of compact sets of compact operators are all based on one form or another of the Ascoli-Arzela theorem. Such methods, however, fail in the (s)-nuclear case.

5.1. REMARK. The (s)-nuclear version of the standard (scalar valued) Ascoli Arzela theorem is not valid: Take

$$H = \{f \in C[0,1]\colon f(0) = 0,\quad |f(x) - f(y)| \leq |x - y|\ \forall x, y \in [0,1]\}.$$

Then it is readily seen that H has equal (s)-variation but it is not (s)-nuclear.

5.2. REMARK. The proofs of the several implications in Theorems 2.5 and 1.1 and the proof of Theoorem 6.5 in [4] also yield inequalities of the following type (Here $H \subset L(E, F)$ is any bounded subset):

$$\delta_{n^2}(H') \leq 4n\delta_n(HB_E) + 4n^2\delta_n(H'B_{F'})$$

$$e_m(H'B_{F'}) \leq An^2 v_n(H) + \frac{8\|H\|}{2^{(m-1)/(n-1)}}$$

$$v_{nm}(H) \leq 2e_n(H'B_{F'}) + \frac{2}{m} \qquad \text{for all } m, n$$

$$v_m(H) \leq Cn^2 e_n(H'B_{F'}) + \frac{8n\|H\|}{2^{(m-1)/(n-1)}} \qquad \text{for all } m, n.$$

The numerical constants or exponents of n in the above are not claimed to be sharp.

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