PACIFIC JOURNAL OF MATHEMATICS
Vol. 127, No. 2, 1987

A GEOMETRIC FUNCTION DETERMINED BY EXTREME POINTS OF THE UNIT BALL OF A NORMED SPACE

RICHARD M. ARON AND ROBERT H. LOHMAN

A geometric function, which measures the relative distance of a vector to an extreme point of the unit ball of a normed space, is defined. This function is calculated explicitly for certain classical function and sequence spaces. Radial limits and continuity properties of this function are investigated and some applications are given.

Introduction. There are many normed spaces X, which are geometrically very different, whose closed unit balls have the following geometric property, called the λ-property, in common: each member x of the unit ball is a convex combination of an extreme point e of the unit ball and a vector y, where $\|y\| \leq 1$ and e is assigned a positive weight. If we vary e and y, looking for the "largest possible" weight in such a representation of x, we obtain a geometric function of x, called the λ-function, which measures how close x is to being an extreme point of the unit ball. In Section 1, we make these ideas more precise and calculate explicit formulas for the λ-function for the classical spaces $C_X(T)$, $l_1(X)$, $l_\infty(X)$ and $c(X)$. It is also shown when the "largest possible" weight is attained in these spaces. Section 2 investigates continuity properties of the λ-function. These include existence of radial limits (Theorem 2.2) and Lipschitz properties (Corollaries 2.8 and 2.9). In Section 3, it is shown how the uniform λ-property is related to uniformly convergent series expansions of vectors in terms of infinite convex combinations of extreme points of the unit ball (Theorem 3.1). Local boundedness of the λ-function away from zero (Theorem 3.5) is also discussed. Section 4 contains a list of questions and open problems.

0. Notation. If X is a normed space, the closed unit ball, open unit ball and unit sphere will be denoted by B_X, U_X and S_X, respectively. The symbols $l_1(X)$, $l_\infty(X)$ and $c(X)$ denote the spaces of all X-valued sequences $x = (x_n)$ which are absolutely summable, bounded and convergent, respectively. $l_1(X)$ is endowed with the norm $\|x\| = \sum_{n=1}^{\infty}\|x_n\|$, while the norm in $l_\infty(X)$ and $c(X)$ is given by $\|x\| = \sup_n\|x_n\|$. If T is a compact Hausdorff space, $C_X(T)$ denotes the space of continuous X-valued functions on T endowed with the sup norm. If $x, y \in X$, then $(x:y)$ denotes $\{\lambda x + (1-\lambda)y\colon 0 < \lambda < 1\}$, ($(x:y]$ has the obvious corresponding meaning). A point e of a convex subset A of X is an extreme point of A if $x, y \in A$ and $e \in (x:y)$ imply $e = x = y$. The set of extreme points of A is denoted by $\operatorname{ext}(A)$. The convex hull of a subset B of X is denoted by $\operatorname{co}(B)$. Recall that X is strictly convex if $\operatorname{ext}(B_X) = S_X$. A convex set A is called a polyhedron in case $\operatorname{ext}(A)$ is finite and $A = \operatorname{co}(\operatorname{ext}(A))$. We denote the set of positive integers by $\mathbf{N}$ and, if X is a normed space, $X_{\mathbf{R}}$ denotes X considered as a real vector space. The function $r\colon X\setminus\{0\} \to S_X$ is defined by $r(z) = z/\|z\|$. If z, z' are non-zero vectors, then

$$(1) \qquad \|r(z) - r(z')\| \le 2\|z - z'\| \min\{\|z\|^{-1}, \|z'\|^{-1}\}.$$

1. The λ-property and computation of the λ-function.

DEFINITION 1.1. Let X be a normed space and $x \in B_X$. If $e \in \operatorname{ext}(B_X)$, $\|y\| \le 1$, $0 < \lambda \le 1$ and $x = \lambda e + (1-\lambda)y$, we say the ordered triple (e, y, λ) is amenable to x. In this case, we define

$$(2) \qquad \lambda(x) = \sup\{\lambda\colon (e, y, \lambda) \text{ is amenable to } x\}$$

X is said to have the λ-property if each $x \in B_X$ admits an amenable triple. If X has the λ-property and, in addition, satisfies $\inf\{\lambda(x)\colon x \in B_X\} > 0$, we say X has the uniform λ-property.

There are several elementary facts which we record for future use.

PROPOSITION 1.2. *Let X be a normed space.*

(a) *If $e \in \operatorname{ext}(B_X)$, then $\lambda(e) = 1$.*

(b) *If (e, y, λ) is amenable to x and $\lambda < 1$, $\|y\| < 1$, there exist $\lambda' > \lambda$ and $y' \in S_X$ such that $y \in (y':x]$ and (e, y', λ') is amenable to x.*

(c) *If (e, y, λ) is amenable to x and $0 < \lambda' < \lambda$, there exists $y' \in (y:x)$ such that (e, y', λ') is amenable to x.*

(d) *If X has the λ-property, then $\lambda(x) \le (1 + \|x\|)/2$ for all $x \in B_X$.*

(e) *If X is a strictly convex space, then $\lambda(x) = (1 + \|x\|)/2$ for all $x \in B_X$ and $\lambda(x)$ is attained.*

(f) *If X has the λ-property and Y is a linear subspace of X such that Y has the λ-property, and $\operatorname{ext}(B_Y) \subset \operatorname{ext}(B_X)$, then $\lambda_Y(x) \le \lambda_X(x)$ for all x in Y, where λ_Y and λ_X are the λ-functions defined by (2) in B_Y and B_X.*

Proof. (a) This is clear since $(e, e, 1)$ is amenable to e.

(b) Since $\lambda < 1$, $x \neq e$ and so there is a $t > 0$ such that $y' = y + t(x - e)$ has norm one. A straightforward calculation shows that $\lambda' = (\lambda + t - t\lambda)/(1 + t - t\lambda)$ works.

(c) We may take

$$y' = \frac{\lambda - \lambda'}{\lambda(1 - \lambda')} x + \left(1 - \frac{\lambda - \lambda'}{\lambda(1 - \lambda')}\right) y.$$

(d) This follows from the fact that if (e, y, λ) is amenable to x, then $x - \lambda e = (1 - \lambda) y$ and so

$$\lambda - \|x\| \leq \|x - \lambda e\| \leq 1 - \lambda.$$

(e) If $x \in B_X$ and $x \neq 0$, then $(x/\|x\|, -x/\|x\|, (1 + \|x\|)/2)$ is amenable to x so that $\lambda(x) \geq (1 + \|x\|)/2$. An appeal to (d) completes the proof. On the other hand, it is clear that $\lambda(0) = 1/2$.

(f) This is clear from (2).

Before computing the λ-function for $C_X(T)$, we will need the following version of the Borsuk-Dugundji extension theorem.

LEMMA 1.3. *Let T be a compact metric space, let T_0 be a non-empty closed subset of T and let X be an infinite-dimensional normed space. If g: $T_0 \to S_X$ is a continuous mapping, there exists a continuous mapping g: $T \to S_X$ such that $g|_{T_0} = g$.*

Proof. This follows from Theorem 4.1 and 6.1 of [3] (a considerable strengthening of Theorem 6.1 is found in [1]).

If e is an extreme point of the unit ball of $C_X(T)$, one cannot conclude that $e(t) \in \text{ext}(B_X)$ for $t \in T$ (see [2] for a four-dimensional space X in which this fails for C_X ([0, 1])). However, all we need here is the following elementary result whose proof is given for the sake of completeness.

LEMMA 1.4. *Let T be a compact Hausdorff space and let X be a normed space. If e is an extreme point of the unit ball of $C_X(T)$, then $\|e(t)\| = 1$ for all $t \in T$.*

Proof. Suppose there exists $t_0 \in T$ such that $\|e(t_0)\| = \alpha < 1$. Let $\delta = (1 - \alpha)/4$ and set $V = \{t \in T\colon \|e(t)\| \leq \alpha + \delta\}$, $W = \{t \in T\colon \|e(t)\| \geq 1 - \delta\}$. Then $t_0 \in V$ and, since $\|e\| = 1$, $W \neq \varnothing$. By Urysohn's lemma, there is a continuous function $f\colon T \to [0,1]$ such that $f(V) = \{1\}$, $f(W) = \{0\}$. Fix $x_0 \in S_X$ and define $u, v \in C_X(T)$ by $u(t) = e(t) + \delta f(t)x_0$, $v(t) = e(t) - \delta f(t)x_0$. Then u, v are in the unit ball of $C_X(T)$, $u \neq e \neq v$ and $e = (u + v)/2$, contradicting the fact that e is an extreme point of the unit ball of $C_X(T)$.

REMARK 1.5. If $e \in C_X(T)$ and $e(t) \in \text{ext}(B_X)$ for all $t \in T$, then e is an extreme point of the unit ball of $C_X(T)$. Consequently, if X is a strictly convex normed space, the converse of Lemma 1.4 is true.

THEOREM 1.6. *Let T be a compact metric space and let X be an infinite-dimensional strictly convex normed space. Then $C_X(T)$ has the uniform λ-property. In fact, if $x \in C_X(T)$ and $\|x\| \leq 1$, then $\lambda(x) = (1 + m)/2$, where $m = \inf\{\|x(t)\|\colon t \in T\}$. Moreover, if $x(t) \neq 0$ for all $t \in T$, $\lambda(x)$ is attained.*

Proof. Suppose $x = \lambda e + (1 - \lambda)y$, where $0 \leq \lambda \leq 1$, $\|y\| = 1$ and e is an extreme point of the unit ball of $C_X(T)$. By Lemma 1.4, $\|e(t)\| = 1$ for all $t \in T$. Since $\lambda e(t) = x(t) - (1 - \lambda)y(t)$, we have

$$\lambda = \|\lambda e(t)\| \leq \|x(t)\| + (1 - \lambda)\|y(t)\| \leq \|x(t)\| + 1 - \lambda.$$

Taking the infimum over all $t \in T$ yields $\lambda \leq (1 + m)/2$. Taking the supremum over all such λ yields $\lambda(x) \leq (1 + m)/2$.

In order to obtain the reverse inequality, first note that if $m = 1$, then $\|x(t)\| = 1$ for all $t \in T$. Therefore, x is an extreme point of the unit ball of $C_X(T)$ and $\lambda(x) = 1 = (1 + m)/2$. Hence, we may assume $m < 1$. Next, suppose $x(t) \neq 0$ for all $t \in T$. In this case, we define $e, y \in C_X(T)$ by

$$e(t) = \frac{x(t)}{\|x(t)\|}, \qquad y(t) = \left[\frac{2\|x(t)\| - 1 - m}{(1 - m)\|x(t)\|}\right]x(t).$$

Then e is an extreme point of the unit ball of $C_X(T)$. Also, for each $t \in T$ we have

$$\|y(t)\| \leq \frac{|2\|x(t)\| - 1 - m|}{1 - m} \leq \frac{|\,\|x(t)\| - m| + |\,\|x(t)\| - 1|}{1 - m} \leq 1,$$

and

$$x = \left(\frac{1 + m}{2}\right)e + \left(\frac{1 - m}{2}\right)y,$$

proving $\lambda(x) \geq (1 + m)/2$.

Finally, suppose $x(t) = 0$ for some $t \in T$. Then $(1 + m)/2 = 1/2$ and we want to show $\lambda(x) \geq 1/2$. To see this, let $0 < \lambda < 1/2$ and choose $\delta > 0$ such that $4\delta < 1 - 2\lambda$. Let the closed subset T_0 of T be defined by

$$T_0 = \{ t \in T \colon \| x(t) \| \leq \delta \text{ or } \| x(t) \| \geq 2\delta \}.$$

Fix $x_0 \in X$, $\|x_0\| = 1$, and define $e\colon T_0 \to S_X$ by

$$e(t) = \begin{cases} x_0, & \text{if } \|x(t)\| \leq \delta, \\ \dfrac{x(t)}{\|x(t)\|}, & \text{if } \|x(t)\| \geq 2\delta. \end{cases}$$

Since e is continuous on T_0, Lemma 1.3 guarantees that there exists a continuous mapping $e'\colon T \to S_X$ that extends e. e' is clearly an extreme point of the unit ball of $C_X(T)$. Define $y \in C_X(T)$ by

$$y = (x - \lambda e')/(1 - \lambda).$$

To see that $\|y\| \leq 1$, observe that $\|x(t)\| \geq 2\delta$ implies

$$\| y(t) \| = \frac{\|x(t) - \lambda x(t)/\|x(t)\|\|}{1 - \lambda} = \frac{|\|x(t)\| - \lambda|}{1 - \lambda} \leq 1,$$

while $\|x(t)\| < 2\delta$ implies

$$\| y(t) \| \leq \frac{\|x(t)\| + \lambda}{1 - \lambda} < \frac{2\delta + \lambda}{1 - \lambda} < \frac{1/2}{1 - \lambda} < 1.$$

Since $x = \lambda e' + (1 - \lambda)y$ and $0 < \lambda < 1/2$ is arbitrary, we have $\lambda(x) \geq 1/2$.

Remark 1.7. If X is a finite-dimensional strictly convex space, the conclusion of Theorem 1.6 may fail. In fact, $C_X(T)$ may even fail to have the λ-property. For example, let $T = \{ z \in \mathbf{C}\colon |z| \leq 1\}$ and let $X = \mathbf{C}$. Define x in the unit ball of $C_{\mathbf{C}}(T)$ by $x(z) = z$ for all $z \in T$. Assume (e, y, λ) is amenable to x. Then $|e(z)| = 1$ for all $z \in T$ and $0 < \lambda < 1$. If $|z| = 1$, we have $z = x(z) = \lambda e(z) + (1 - \lambda)y(z)$ so that

$$|z - \lambda e(z)| \leq 1 - \lambda.$$

This means that $\lambda e(z)$ lies in both and closed disc with center 0 and radius λ and the closed disc with center z and radius $1 - \lambda$. It follows that $\lambda e(z) = \lambda z$; that is, $e(z) = z$ for all z on the unit circle. The contradiction is reached by observing that e must then be a retract of the unit disc T onto the unit circle. This example clearly generalizes to show that $C_{\mathbf{R}^n}(T)$ fails to have the λ-property if T is the closed unit ball of $\mathbf{R}^n$.

Although Remark 1.7 suggests that infinite dimensionality of X is needed in order to obtain results similar to Theorem 1.6, this is not always the case. If T is given stronger properties, infinite dimensionality of X can be relaxed. To see an example of this, we will need

LEMMA 1.8. *Let $a < b$ and let X be a normed space satisfying* $\dim X_{\mathbf{R}} \geq 2$. *If $u, v \in S_X$, there exists a continuous function $f\colon [a,b] \to S_X$ such that $f(a) = u$, $f(b) = v$. If, in addition, $\|v - u\| < 1$, then f may be chosen so that for $t, t' \in [a,b]$*

$$\|f(t) - f(t')\| \leq \frac{2|t - t'|\,\|u - v\|}{(b - a)(1 - \|u - v\|)^2}. \tag{3}$$

Proof. If $v \neq -u$, then $(1 - s)u + sv \neq 0$ for any $s \in [0,1]$. We let $f(t) = r(h(t))$, where

$$h(t) = \left(\frac{b - t}{b - a}\right)u + \left(\frac{t - a}{b - a}\right)v, \qquad a \leq t \leq b.$$

Then $f\colon [a,b] \to S_X$ is continuous, $f(a) = u$, $f(b) = v$ and by (1) f satisfies

$$\|f(t) - f(t')\| \leq 2\|h(t) - h(t')\| \min\left\{\|h(t)\|^{-1}, \|h(t')\|^{-1}\right\}.$$

If $\|v - u\| < 1$, we obtain (3) by observing that

$$\|h(t) - h(t')\| = \frac{|t - t'|}{b - a}\|u - v\| \quad \text{and}$$

$$\|h(t)\| = \left\|u + \frac{t - a}{b - a}(v - u)\right\| \geq 1 - \|u - v\|.$$

If $v = -u$, choose $w \in S_X$, $w \neq \pm u$, and let $c = (a + b)/2$. By the preceding observation, there are continuous functions $f_1\colon [a,c] \to S_X$, $f_2\colon [c,b] \to S_X$ such that $f_1(a) = u$, $f_1(c) = w = f_2(c)$, $f_2(b) = v$. In this case, combine f_1 and f_2 to obtain f.

THEOREM 1.9. *Let X be a strictly convex normed space satisfying* $\dim X_{\mathbf{R}} \geq 2$. *Then $C_X([0,1])$ has the uniform λ-property. In fact, if $x \in C_X([0,1])$, $\|x\| \leq 1$ and $m = \inf\{\|x(t)\|\colon t \in [0,1]\}$, then $\lambda(x) = (1 + m)/2$. Moreover, if $x(t) \neq 0$ for all $t \in [0,1]$, $\lambda(x)$ is attained.*

Proof. One proceeds exactly as in the proof of Theorem 1.6, noting that only the case in which $x(t) = 0$ for some $t \in T$ needs to be modified. In this case, let $0 < \lambda < 1/2$, choose $\delta > 0$ such that $4\delta < 1 - 2\lambda$ and let the closed subset T_0 of $[0,1]$ be defined by $T_0 = T_1 \cup T_2$, where $T_1 = \{t \in [0,1]\colon \|x(t)\| \le \delta\}$, $T_2 = \{t \in [0,1]\colon \|x(t)\| \ge 2\delta\}$. Fix $x_0 \in S_X$ and define $e\colon T_0 \to S_X$ as before. The set $[0,1] \setminus T_0$ is a countable disjoint union of open intervals (a, b), where $a, b \in T_0$. Extend e to $\tilde{e}\colon [0,1] \to S_X$, defining $\tilde{e}$ on each such interval (a, b) by $\tilde{e}(t) = f(t)$ for all $t \in (a, b)$, where f is chosen as in Lemma 1.8 with $u = e(a)$, $v = e(b)$. By uniform continuity of x on $[0,1]$, there exists $\eta > 0$ such that $\|x(t) - x(t')\| < \delta/2$ whenever $t, t' \in [0,1]$ and $|t - t'| < \eta$. Hence, if $t, t' \in T_0$ and $|t - t'| < \eta$, we have $t, t' \in T_1$ or $t, t' \in T_2$. Consequently, the endpoints a, b are both in the same set T_1 or T_2 for all but finitely many of the open intervals (a, b). By the uniform continuity of e on T_0, we may also assume $\|e(t) - e(t')\| < 1$ whenever $t, t' \in T_0$ and $|t - t'| < \eta$.

We now show that $\tilde{e}$ is continuous. It is clear that $\tilde{e}$ is continuous at each point of $[0,1] \setminus T_0$ and at each point interior to T_0. If t_0 is in the *boundary of T_0, then t_0 is an endpoint, say a, of one of the distinguished* intervals (a, b) mentioned above. Consequently $\tilde{e}$ is continuous from the right at t_0. On the other hand, let (t_n) be a sequence in $[0,1]$ such that $t_n \uparrow t_0$. Since e is continuous at t_0, we may assume that each t_n lies in $[0,1] \setminus T_0$. If t_0 is also a right-hand endpoint of one of the distinguished open intervals whose disjoint union is $[0,1] \setminus T_0$, $\tilde{e}$ is continuous from the left at t_0. Thus, by taking n sufficiently large, we may assume that $t_n \in (a_n, b_n)$, where (a_n, b_n) is one of the distinguished open intervals, $t_0 - \eta < a_n < b_n < t_0$ and a_n, b_n are in the same T_i. If $t_0 \in T_1$, then $a_n, b_n \in T_1$, which implies $\tilde{e}(t_n) = x_0 = e(t_0)$ for all n. If $t_0 \in T_2$, then $a_n, b_n \in T_2$ and $\|e(a_n) - e(b_n)\| < 1$ for all n. By Lemma 1.8, we may assume

$$\|e(a_n) - \tilde{e}(t_n)\| \le \frac{2\|e(a_n) - e(b_n)\|}{(1 - \|e(a_n) - e(b_n)\|)^2}.$$

Since $\|\tilde{e}(t_0) - \tilde{e}(t_n)\| \le \|e(t_0) - e(a_n)\| + \|e(a_n) - \tilde{e}(t_n)\|$, the facts that $e(a_n) \to e(t_0)$ and $e(a_n) - e(b_n) \to 0$ imply $\tilde{e}(t_n) \to e(t_0)$, establishing continuity of $\tilde{e}$. The last part of the proof of Theorem 1.6 now completes the proof.

REMARK 1.10. The conclusion of Theorem 1.9 fails if $\dim X_{\mathbf{R}} = 1$. In fact, $C_X([0,1])$ fails the λ-property, since in $C_{\mathbf{R}}([0,1])$, the only extreme points of the unit ball are the constant functions ± 1. Thus, if we define the unit vector $x \in C_{\mathbf{R}}([0,1])$ by $x(t) = 1 - 2t$, it is easy to see that there is no triple (e, y, λ) amenable to x. Also, see Remark 1.7.

THEOREM 1.11. *Let X be a strictly convex normed space. Then $l_1(X)$ has the λ-property but not the uniform λ-property. In fact, if $x = (x_n) \in l_1(X)$, $\|x\| \leq 1$ and $M = \sup\{\|x_n\|: n \in \mathbf{N}\}$, then*

$$\lambda(x) = (1 - \|x\| + 2M)/2.$$

Moreover, $\lambda(x)$ is attained.

Proof. Suppose (e, y, λ) is amenable to x. We may assume $\lambda < 1$. Write $e = (e_n)$, $y = (y_n)$ and observe that there is a positive integer m such that $e_m \in \operatorname{ext}(B_X)$ and $e_n = 0$ if $n \neq m$. Therefore,

$$1 = \sum_{n \neq m} \|y_n\| + \|y_m\| = \frac{\|x\| - \|x_m\|}{1 - \lambda} + \frac{\|x_m - \lambda e_m\|}{1 - \lambda}$$

which implies $1 - \lambda \geq \|x\| + \lambda - 2\|x_m\|$. It follows that

$$\lambda(x) \leq (1 - \|x\| + 2M)/2.$$

On the other hand, if $x = 0$, the result is clear. Hence we may assume $x \neq 0$. In addition, if x has a coordinate x_n with $\|x_n\| = 1$, then x is an extreme point of the unit ball of $l_1(X)$ and the result is clear. Consequently, we assume $\|x_n\| < 1$ for all n. Pick a positive integer N such that $\|x_N\| = M$. Let $\lambda = (1 - \|x\| + 2M)/2$, $e = (e_n)$ and $y = (y_n)$, where

$$e_n = \begin{cases} 0, & n \neq N, \\ \dfrac{x_N}{\|x_N\|}, & n = N, \end{cases}$$

$$y_n = \begin{cases} \dfrac{2}{1 + \|x\| - 2M} x_n, & n \neq N, \\ \dfrac{\|x\| - 1}{(1 + \|x\| - 2M)M} x_n, & n = N. \end{cases}$$

Then (e, y, λ) is amenable to x. This shows that $l_1(X)$ has the λ-property, establishes the formula for $\lambda(x)$ and proves that $\lambda(x)$ is attained. In order to see that $l_1(X)$ does not have the uniform λ-property, fix $x' \in S_X$. If n is any positive integer then the unit vector

$$x = \big(\underbrace{x'/n, \ldots, x'/n}_{n}, 0, 0, \ldots\big)$$

of $l_1(X)$ satisfies $\lambda(x) = 1/n$.

REMARK 1.12. Only a minor change in notation is required to show that Theorem 1.11 is valid for $(\oplus \sum_n X_n)_{l_1}$, where each X_n is a strictly convex normed space.

THEOREM 1.13. *Let X be a strictly convex normed space. Then $l_\infty(X)$ has the uniform λ-property. In fact, if $x = (x_n) \in l_\infty(X)$, $\|x\| \le 1$ and $m = \inf\{\|x_n\|: n \in \mathbf{N}\}$, then $\lambda(x) = (1+m)/2$. Moreover, $\lambda(x)$ is attained.*

Proof. First, suppose $x_{n_0} = 0$ for some index n_0. If $x = \lambda e + (1-\lambda)y$, where $0 \le \lambda \le 1$, $e = (e_n)$ is an extreme point of the unit ball of $l_\infty(X)$ and $y = (y_n)$ has norm one, then $\lambda e_{n_0} + (1-\lambda)y_{n_0} = x_{n_0} = 0$ implies $\lambda/(1-\lambda) = \|y_{n_0}\| \le 1$. Thus, $\lambda \le 1/2$ which yields $\lambda(x) \le 1/2$. On the other hand, if $n \ne n_0$, then by (e) of Proposition 1.2, we may write $x_n = \lambda_n e_n + (1-\lambda_n)y_n$, where $e_n \in \text{ext}(U_X)$, $\|y_n\| = 1$ and $\lambda_n = (1 + \|x_n\|)/2$. Since $\lambda_n \ge 1/2$, part (c) of Proposition 1.2 shows that $(e_n, z_n, 1/2)$ is amenable to x_n for some $z_n \in B_X$. Let

$$e = (e_1, \dots, e_{n_0-1}, e_{n_0-1}, e_{n_0+1}, e_{n_0+2}, \dots),$$

$$z = (z_1, \dots, z_{n_0-1}, -e_{n_0-1}, z_{n_0+1}, z_{n_0+2}, \dots).$$

Then e is an extreme point of the unit ball of $l_\infty(X)$, $\|z\| = 1$ and $x = \frac{1}{2}e + \frac{1}{2}z$. This, together with $\lambda(x) \le 1/2$, yields $\lambda(x) = 1/2$ and establishes our assertion in this case. Hence, we may assume $0 < \|x_n\| \le 1$ for all n. The assertion is also true if $m = 1$ because this implies x is an extreme point of the unit ball of $l_\infty(X)$. Thus, we also assume $m < 1$. We claim $\lambda(x) \le (1+m)/2$. To see this, choose a subsequence (x_{n_k}) of (x_n) such that $\|x_{n_k}\| \to m$ (in case $\|x_n\| = m$ for some n, the claim is proved in a manner similar to what follows). If $e = (e_n)$ is an extreme point of the unit ball of $l_\infty(X)$, $y = (y_n)$ has norm at most one, $0 \le \lambda \le 1$ and $x = \lambda e + (1-\lambda)y$, then $\lambda < 1$, since $\lambda = 1$ forces $x = e$ and $m = 1$. Then $x_{n_k} = \lambda e_{n_k} + (1-\lambda)y_{n_k}$ implies

$$\frac{\lambda}{1-\lambda} \le \frac{\|x_{n_k}\|}{1-\lambda} + \|y_{n_k}\| \le \frac{\|x_{n_k}\|}{1-\lambda} + 1.$$

Letting $k \to \infty$ yields

$$\frac{\lambda}{1-\lambda} \le \frac{m}{1-\lambda} + 1$$

or $\lambda \le (1+m)/2$, which proves the claim.

In order to see that $\lambda(x) \ge (1+m)/2$, let $\lambda = (1+m)/2$, $e = (e_n)$, $y = (y_n)$, where

$$e_n = x_n/\|x_n\|, \tag{4}$$

$$y_n = \begin{cases} x_n, & \|x_n\| = 1, \\ \dfrac{(2\|x_n\| - 1 - m)}{(1-m)\|x_n\|} x_n, & \|x_n\| < 1. \end{cases}$$

Then $\|y\| \leq 1$ because if $\|x_n\| < 1$, we have

$$\|y_n\| = \frac{|2\|x_n\| - 1 - m|}{1 - m} \leq \frac{|\,\|x_n\| - m| + |\,\|x_n\| - 1|}{1 - m} \leq \frac{1 - m}{1 - m} = 1.$$

Since (e, y, λ) is amenable to x, the proof is complete.

REMARK 1.14. Only a minor change in notation is required to show that Theorem 1.13 is valid for $(\oplus \Sigma_n X_n)_{l_\infty}$, where each X_n is a strictly convex normed space.

The next result is essentially a corollary to Theorem 1.6. Since, however, $\lambda(x)$ can be attained under more general circumstances than indicated in Theorem 1.6, we present this result as

THEOREM 1.15. *Let X be an infinite-dimensional strictly convex normed space. Then $c(X)$ has the uniform λ-property. In fact, if $x = (x_n) \in c(X)$, $\|x\| \leq 1$ and $m = \inf\{\|x_n\|: n \in \mathbf{N}\}$, then $\lambda(x) = (1 + m)/2$. Moreover, if $\lim_n x_n \neq 0$, then $\lambda(x)$ is attained.*

Proof. Since $c(X)$ is isometrically isomorphic to $C_X(T)$, where T is the one-point compactification of $\mathbf{N}$ when $\mathbf{N}$ has the discrete topology, all of the assertions above, except the last one, follow from Theorem 1.6. In order to complete the proof, write $x_\infty = \lim_n x_n$ and assume $x_\infty \neq 0$. We may assume $m < 1$; otherwise, x is an extreme point of the unit ball of $c(X)$. If $m > 0$, define $e = (e_n)$, $y = (y_n)$ as in (4). Then $e, y \in c(X)$ because $\lim_n e_n = x_\infty/\|x_\infty\|$ and

$$\lim_n y_n = \frac{2\|x_\infty\| - 1 - m}{(1 - m)\|x_\infty\|} x_\infty.$$

As in the proof of Theorem 1.13, (e, y, λ) is amenable to x, where $\lambda = (1 + m)/2$. Next, suppose $m = 0$. Fix $x_0 \in X$, $\|x_0\| = 1$, and note that the set $D = \{n: x_n = 0\}$ is finite. Define $e = (e_n)$, $y = (y_n) \in c(X)$ by

$$e_n = \begin{cases} x_0, & n \in D, \\ \dfrac{x_n}{\|x_n\|}, & n \in \mathbf{N} \setminus D, \end{cases}$$

$$y_n = \begin{cases} -x_0, & n \in D, \\ \dfrac{2\|x_n\| - 1}{\|x_n\|} x_n, & n \in \mathbf{N} \setminus D. \end{cases}$$

Then $(e, y, 1/2)$ is amenable to x, completing the proof.

By Theorems 1.11 and 1.13, the dual spaces l_1 and l_∞ have the λ-property and uniform λ-property, respectively. Since unit balls of dual spaces are rich in extreme points, one might expect (or at least hope) that dual spaces satisfy the λ-property. To see that this is not the case in general, let $X = C_{\mathbf{R}}([0,1])^*$, which, using the Riesz representation theorem, is identified with the Banach space of regular Borel measures on the Borel subsets of $[0,1]$. If m denotes Lebesgue measure on $[0,1]$, then $m \in B_X$. Assume there exists a triple (e, μ, λ) that is amenable to m. Then we can write $e = \pm\delta_t$ for some $t \in [0,1]$, where δ_t is point evaluation at t. If $A = [0,1]\setminus\{t\}$, we obtain $1 = m(A) = (1-\lambda)\mu(A)$ or $\|\mu\| \geq \mu(A) = 1/(1-\lambda) > 1$, which is a contradiction. Consequently, $C_{\mathbf{R}}([0,1])$ and $C_{\mathbf{R}}([0,1])^*$ both fail to have the λ-property.

We close this section by showing that all finite-dimensional normed spaces have the uniform λ-property.

THEOREM 1.16. *Let X be a finite-dimensional normed space. Then X has the uniform λ-property. In fact, if $x \in B_X$, then $\lambda(x) \geq 1/(1 + \dim X_{\mathbf{R}})$.*

Proof. Let $n = \dim X_{\mathbf{R}}$. Then each $x \in B_X$ can be written as $x = \sum_{k=1}^{n+1} \lambda_k e_k$, where $e_k \in \operatorname{ext}(B_X)$, $\lambda_k \geq 0$ for all k and $\sum_{k=1}^{n+1}\lambda_k = 1$ (see p. 10 of [4]). There is an index k_0 with $\lambda_{k_0} \geq 1/(n+1)$. If $\lambda_{k_0} = 1$, then $x \in \operatorname{ext}(B_X)$ and $\lambda(x) = 1$. Otherwise,

$$\left(e_{k_0}, \sum_{k \neq k_0} \frac{\lambda_k}{1-\lambda_{k_0}} e_k, \lambda_{k_0}\right)$$

is amenable to x, completing the proof.

2. Continuity properties of the λ-function. The λ-functions which were explicitly calculated for the classical normed spaces of §1 are all continuous. However, it is not difficult to construct norms in the plane for which the λ-function fails to be continuous on B_X. For example, in $X = \mathbf{R}^2$, let u_n, v_n denote those points having polar coordinates

$$\left(1, \frac{n\pi}{2(n+1)}\right) \quad \text{and} \quad \left(1, \pi - \frac{n\pi}{2(n+1)}\right),$$

respectively, for $n = 0, 1, 2, \ldots$. Take $\|\cdot\|$ to be the norm on X whose unit ball is the closed convex hull of all the points $\pm u_n$, $\pm v_n$. If $w_n = (u_n + u_{n+1})/2$ and e has polar coordinates $(1, \pi/2)$, then $e \in \operatorname{ext}(B_X)$ (so that $\lambda(e) = 1$), $w_n \to e$ and $\lambda(w_n) = 1/2$ for all n. Although the λ-function may fail to be continuous at some points of B_X, it does possess some continuity properties in important general cases. Such properties are investigated in this section.

LEMMA 2.1. *Let X be a normed space with the λ-property. If* $x \in B_X$ *and* $x \neq 0$, *then*

$$\lambda(x) \geq \frac{1+\|x\|}{2}\lambda\left(\frac{x}{\|x\|}\right).$$

Proof. The assertion is trivially true if $\|x\| = 1$, so we assume $\|x\| < 1$. Write $z = x/\|x\|$, $y = -x/\|x\|$. Then $\|z\| = \|y\| = 1$ and

$$x = \left(\frac{1+\|x\|}{2}\right)z + \left(1 - \frac{1+\|x\|}{2}\right)y.$$

Given $\varepsilon > 0$, there is a triple (e, y', λ) that is amenable to z for which $\lambda(z) - \varepsilon < \lambda$. Letting

$$\lambda' = \left(\frac{1+\|x\|}{2}\right)\lambda \quad \text{and}$$

$$y'' = \frac{(1+\|x\|)(1-\lambda)y' + (1-\|x\|)y}{2-(1+\|x\|)\lambda},$$

a routine computation shows that (e, y'', λ') is amenable to x. This shows

$$\lambda(x) \geq \left(\frac{1+\|x\|}{2}\right)\left(\lambda\left(\frac{x}{\|x\|}\right) - \varepsilon\right),$$

completing the proof.

THEOREM 2.2. *Let X be a normed space satisfying the λ-property, and let τ be a Hausdorff vector topology on X which is weaker than the norm topology. If* $\operatorname{ext}(B_X)$ *is τ-sequentially compact (respectively, τ-compact) and* B_X *is τ-sequentially closed (respectively, τ-closed), then radial limits of λ satisfy*

$$\lambda(x) = \lim_{r \to 1^-} \lambda(rx), \qquad x \in S_X.$$

Proof. First, assume $\operatorname{ext}(B_X)$ is τ-sequentially compact and B_X is τ-sequentially closed. Let $x \in S_X$ and (r_n) be a sequence of positive numbers increasing to 1. It suffices to show $\lambda(r_n x) \to \lambda(x)$. For each $n \in \mathbf{N}$, there is a triple (e_n, y_n, λ_n) amenable to $r_n x$ such that $\lambda(r_n x) - 1/n < \lambda_n$. By Lemma 2.1, $\lambda(r_n x) \geq [(1 + r_n)/2]\lambda(x)$ which implies

$\liminf \lambda(r_n x) \geq \lambda(x)$. Write $\lambda = \limsup \lambda(r_n x)$ and choose a subsequence (λ_{n_k}) of (λ_n) such that $\lambda_{n_k} \to \lambda$. By passing to a subsequence of (e_{n_k}), we may assume there exists $e \in \text{ext}(B_X)$ such that $e_{n_k} \to e$ relative to τ. Then $\lambda_{n_k} e_{n_k} \to \lambda e$ relative to τ, which implies $(1 - \lambda_{n_k}) y_{n_k} \to x - \lambda e$ relative to τ.

If $\lambda = 1$, then $x = e$ and we have

$$\lambda(x) \leq \liminf \lambda(r_n x) \leq \limsup \lambda(r_n x) = \lambda = 1 = \lambda(x),$$

implying $\lambda(r_n x) \to \lambda(x)$.

If $\lambda < 1$, then $y_{n_k} \to (x - \lambda e)/(1 - \lambda)$ relative to τ. Since B_X is τ-sequentially closed, $y = (x - \lambda e)/(1 - \lambda) \in B_X$. Then $x = \lambda e + (1 - \lambda) y$ implies $\lambda(x) \geq \lambda$. Therefore, $\lambda(r_n x) \to \lambda(x)$.

In case $\text{ext}(B_X)$ is τ-compact and B_X is τ-closed, pick λ and (λ_{n_k}) as before. Then there is a subnet $(e_{n_{k_\alpha}})$ of (e_{n_k}) and $e \in \text{ext}(B_X)$ such that $e_{n_{k_\alpha}} \to e$ relative to τ. The argument of the preceding case now applies (using subnets instead of subsequences).

COROLLARY 2.3. *Let X be a Banach space with the λ-property. Then radial limits of λ satisfy*

$$\lambda(x) = \lim_{r \to 1^-} \lambda(rx), \qquad x \in S_X$$

in the following cases

(a) *$X = Y^*$, where Y is a normed space and* $\text{ext}(B_X)$ *is weak*-sequentially compact (in particular,* $\dim X < \infty$ *and* $\text{ext}(B_X)$ *is norm closed).*

(b) *$X = Y^*$, where Y is a normed space, and* $\text{ext}(B_X)$ *is weak*-compact.*

REMARK 2.4. If $\dim X < \infty$ and $\text{ext}(B_X)$ is not norm closed, the conclusion of Corollary 2.3 may fail. To see this, let C' denote the convex hull of the union of the sets $\{(x, y, 0) \colon |x|, |y| \leq 1\}$ and $\{(x, 0, z) \colon x^2 + z^2 = 1, z \geq 0\}$ in $\mathbf{R}^3$. Let $C = (0,0,1) + C'$ and let $\|\cdot\|$ denote the norm on $\mathbf{R}^3$ whose unit ball is $B = \text{co}(C \cup -C)$. The unit vector $u = (1, 0, 1)$, which is not an extreme point of B but is a limit point of the sequence $u_m = (\cos m, 0, 1 + \sin m)$, $m \in \mathbf{N}$, from $\text{ext}(B)$, satisfies $\lambda(u) = 1/2$ and $\lim_{r \to 1^-} \lambda(ru) = 1$.

In order to consider additional results related to continuity of the λ-function, it is necessary to introduce some auxiliary functions. If $u \in S_X$, we let

$$(5) \quad \lambda(u, x) = \sup\{\lambda \colon 0 \leq \lambda \leq 1 \text{ and } x = \lambda u + (1 - \lambda) y \text{ for some } y \in B_X\}, \qquad x \in B_X,$$

$$(6)\quad \alpha(u,x) = \sup\{\alpha\colon \alpha \geq 0, \|x + \alpha(x-u)\| = 1\}, \qquad x \in B_X\setminus\{u\},$$

$$(7)\qquad y(u,x) = x + \alpha(u,x)(x-u), \qquad x \in B_X\setminus\{u\}.$$

Geometrically speaking, if $x \in B_X\setminus\{u\}$, $y(u,x)$ is the unit vector which lies on the line from u through x and is "farthest" from u. These functions have some elementary properties which are now stated and whose proofs are left to the reader.

LEMMA 2.5. *Let* $u \in S_X$.

(a) *If* $x \in B_X\setminus\{u\}$, *then* $\lambda(u,x) = \alpha(u,x)/(1+\alpha(u,x))$.

(b) *If* $x \in B_X\setminus\{u\}$, *then* $x = \lambda(u,x)u + (1-\lambda(u,x))y(u,x)$.

(c) *If* $x \in B_X\setminus\{u\}$, *then* $\lambda(u,x) = \|x - y(u,x)\|/\|u - y(u,x)\|$.

(d) *If* X *has the* λ*-property and* $x \in B_X$, *then* $\lambda(x) = \sup\{\lambda(e,x)\colon e \in \operatorname{ext}(B_X)\}$.

THEOREM 2.6. *Let* X *be a normed space.*

(a) *If* $\|u\| = 1$ *and* $\|x\|, \|z\| < 1$ *then*

$$|\alpha(u,x) - \alpha(u,z)| \leq \left[\max\left\{\frac{\alpha(u,x)(1+\alpha(u,z))}{1-\|x\|}, \frac{\alpha(u,z)(1+\alpha(u,x))}{1-\|z\|}\right\}\right]\|x - z\|.$$

(b) *If* $\|u\| = \|v\| = 1$ *and* $\|x\| < 1$, *then*

$$|\alpha(u,x) - \alpha(v,x)| \leq \frac{\alpha(u,x)\alpha(v,x)}{1-\|x\|}\|u - v\|.$$

Proof. We provide the details for (a) and note that the proof of (b) is similar. By (7),

$$y(u,z) = x + \alpha(u,z)(x-u) + (1+\alpha(u,z))(z-x).$$

It follows that

$$\begin{aligned}(8)\quad &\big|\,\|y(u,x)\| - \|x + \alpha(u,z)(x-u)\|\,\big| \\ &= \big|\,\|y(u,z)\| - \|x + \alpha(u,z)(x-u)\|\,\big| \\ &\leq \|y(u,z) - x - \alpha(u,z)(x-u)\| \leq (1+\alpha(u,z))\|x - z\|.\end{aligned}$$

We may assume X is a real normed space. Choose $f \in X^*$ such that $\|f\| = 1$ and

$$1 = f(y(u,x)) = f(x) + \alpha(u,x)f(x-u).$$

We obtain

$$(9)\qquad f(x-u)=\frac{1-f(x)}{\alpha(u,x)}\geq\frac{1-\|x\|}{\alpha(u,x)}.$$

By (8),

$$\begin{aligned}(10)\quad &(\alpha(u,z)-\alpha(u,x))f(x-u)\\ &=f(x+\alpha(u,z)(x-u))-f(x+\alpha(u,x)(x-u))\\ &=f(x+\alpha(u,z)(x-u))-\|y(u,x)\|\\ &\leq\|x+\alpha(u,z)(x-u)\|-\|y(u,x)\|\leq(1+\alpha(u,z))\|x-z\|.\end{aligned}$$

An application of (9) to (10) yields

$$(11)\qquad \alpha(u,z)-\alpha(u,x)\leq\frac{\alpha(u,x)(1+\alpha(u,z))}{1-\|x\|}\|x-z\|.$$

Interchanging the roles of x and z in (11) yields a similar inequality which, when used with (11), produces (a).

COROLLARY 2.7. *Let X be a normed space.*

(a) *If* $\|u\|=1$ *and* $\|x\|, \|z\|<1$, *then*

$$|\lambda(u,x)-\lambda(u,z)|\leq\left[\max\left\{\frac{\lambda(u,x)}{1-\|x\|},\frac{\lambda(u,z)}{1-\|z\|}\right\}\right]\|x-z\|.$$

(b) *If* $\|u\|=\|v\|=1$ *and* $\|x\|<1$, *then*

$$|\lambda(u,x)-\lambda(v,x)|\leq\frac{\lambda(u,x)\lambda(v,x)}{1-\|x\|}\|u-v\|.$$

Proof. These both follow from Theorem 2.6 and (a) of Lemma 2.5.

COROLLARY 2.8. *Let X be a normed space and let $0\leq r<1$. If* $\|u\|=\|v\|=1$ *and* $\|x\|, \|z\|\leq r$, *then*

$$|\lambda(u,x)-\lambda(v,z)|\leq\frac{1+r}{2(1-r)}[\|x-z\|+\|u-v\|].$$

Consequently, the mapping $(u,x)\to\lambda(u,x)$ on $S_X\times B_X$ is a Lipschitz mapping.

Proof. This follows from Corollary 2.7 by observing

$$|\lambda(u,x)-\lambda(v,z)|\leq|\lambda(u,x)-\lambda(u,z)|+|\lambda(u,z)-\lambda(v,z)|$$

and that if $\|w\|=1$ and $\|y\|\leq 1$, then $\lambda(w,y)\leq(1+\|y\|)/2$.

An immediate consequence of part (d) of Lemma 2.5 and Corollary 2.8 is

COROLLARY 2.9. *Let X be a normed space with the λ-property, and let $0 \leq r < 1$. If $\|x\|, \|z\| \leq r$, then*

$$|\lambda(x) - \lambda(z)| \leq \frac{1+r}{2(1-r)}\|x - z\|.$$

Consequently, the λ-function is a Lipschitz mapping on rB_X and is continuous on U_X.

Although the λ-function is continuous on U_X when X is a finite-dimensional normed space, the example of Remark 2.4 shows that points of discontinuity may exist on S_X. In that particular example, the point of discontinuity ($u = (1, 0, 1)$) is a limit of extreme points of B_X but is not an extreme point. We now show that this situation always leads to points of discontinuity of the λ-function.

THEOREM 2.10. *Let X be a finite-dimensional normed space. If $x \in \overline{\text{ext}(B_X)} \setminus \text{ext}(B_X)$, then $\lambda(x) < 1$. Consequently, the λ-function is not continuous at x.*

Proof. Assume, to the contrary, that $\lambda(x) = 1$. Since $x \notin \text{ext}(B_X)$, there is a triple (e_1, y_1, λ_1) that is amenable to x for which $e_1 \neq x \neq y_1$. Choose $\varepsilon_1 > 0$ such that $e_1, y_1 \notin U(x, \varepsilon_1)$, where $U(x, \varepsilon_1)$ denotes the open ball with center x and radius ε_1. Since $\lambda(x) = 1$, there exists a triple (e_2, y_2, λ_2) that is amenable to x for which $\lambda_2 > \max\{\lambda_1, (2 - \varepsilon_1)/2\}$. The equality $x = \lambda_2 e_2 + (1 - \lambda_2) y_2$ implies

$$\|x - e_2\| = (1 - \lambda_2)\|y_2 - e_2\| \leq 2(1 - \lambda_2) < \varepsilon_1.$$

Then x is in the relative interior of $\text{co}(e_1, y_1)$ in M_1, the one-dimensional linear manifold containing x and e_1, and $e_2 \notin M_1$.

Assume that triples (e_i, y_i, λ_i), $1 \leq i \leq n+1$, amenable to x, have been selected, where $\lambda_1 < \cdots < \lambda_{n+1} < 1$, the linear manifold M_n containing x and $e_1, \ldots, e_n$ is n dimensional, $e_{n+1} \notin M_n$ and x is in the relative interior of $\text{co}(e_1, \ldots, e_n, y_1, \ldots, y_n)$ in M_n. Then the linear manifold M_{n+1} containing x and $e_1, \ldots, e_{n+1}$ is of dimension $n + 1$ and there exists $\varepsilon_n > 0$ such that $M_n \cap U(x, \varepsilon_n) \subset \text{co}(e_1, \ldots, e_n, y_1, \ldots, y_n)$. It follows that there exists $0 < \varepsilon_{n+1} < \min_{1 \leq i \leq n}\{\|x - e_i\|, \|x - y_i\|\}$ such that $M_{n+1} \cap U(x, \varepsilon_{n+1}) \subset \text{co}(e_1, \ldots, e_{n+1}, y_1, \ldots, y_{n+1})$. Since $\lambda(x) = 1$, there exists a triple $(e_{n+2}, y_{n+2}, \lambda_{n+2})$ amenable to x such that $\lambda_{n+1} < \lambda_{n+2}$ and $\|x - e_{n+2}\| < \varepsilon_{n+1}$. If $e_{n+2} \in M_{n+1}$, then $e_{n+2} \in \operatorname{co}(e_1, \ldots, e_{n+1}, y_1, \ldots, y_{n+1})$ which implies e_{n+2} is one of the e_i's or y_i's, $1 \leq i \leq n+1$. Since this is impossible by the choice of ε_{n+1}, we obtain $e_{n+2} \notin M_{n+1}$.

By induction, we obtain an increasing sequence (M_n) of linear submanifolds of X such that M_n has dimension n for all n. Since $\dim X < \infty$, the contradiction establishes $\lambda(x) < 1$.

We do not yet know what general conditions will guarantee continuity of λ on B_X. In the simplest of cases (i.e., B_X a polyhedron), however, λ is well behaved and it is this result we now proceed to demonstrate.

LEMMA 2.11. *If B_X is a polyhedron and $e \in \operatorname{ext}(B_X)$, then the λ-function is continuous at e.*

Proof. Let $e, e_1, \ldots, e_m$ denote the distinct extreme points of B_X. There exists $\delta > 0$ such that $\|e - y\| \geq \delta$ for $y \in \operatorname{co}(e_1, \ldots, e_m)$. Suppose (x_n) is a sequence in B_X such that $x_n \to e$. For each n, write $x_n = \lambda_n e + \sum_{k=1}^m \lambda_{k_n} e_k$, where $\lambda_n, \lambda_{k_n} \geq 0$ and $\lambda_n + \sum_{k=1}^m \lambda_{k_n} = 1$. If $\lambda_n < 1$, then

$$y_n = \sum_{k=1}^m \frac{\lambda_{k_n}}{1 - \lambda_n} e_k \in \operatorname{co}(e_1, \ldots, e_m) \quad \text{and}$$

$$x_n = \lambda_n e + (1 - \lambda_n) y_n.$$

In this case, $\|x_n - e\| = (1 - \lambda_n)\|y_n - e\| \geq (1 - \lambda_n)\delta$. From this, we have $\lambda_n \to 1$ which implies $\lambda(x_n) \to 1 = \lambda(e)$.

REMARK 2.12. Let B_X be a polyhedron and let $e_0, e_1, \ldots, e_m$ denote the distinct extreme points of B_X. If $x \in B_X$, $\lambda(e_0, x)$ is attained. Thus, we may write

$$x = \lambda(e_0, x) e_0 + (1 - \lambda(e_0, x)) \sum_{k=0}^m \lambda_k e_k,$$

where $\lambda_k \geq 0$ for all k and $\sum_{k=0}^m \lambda_k = 1$. Suppose $x \notin \operatorname{ext}(B_X)$; that is, $\lambda(e_0, x) < 1$. If $\lambda_0 > 0$, choose a positive number δ small enough so that $\lambda(e_0, x) + \delta < 1$ and $\delta/(1 - \lambda(e_0, x) - \delta) < \lambda_0$. Write $\mu = \delta/(1 - \lambda(e_0, x) - \delta)$. A direct computation shows that if y is taken to be the convex combination

$$y = (\lambda_0 - \mu) e_0 + \sum_{k=1}^m \lambda_k e_k + \sum_{k=0}^m \mu \lambda_k e_k$$

of $e_0, e_1, \ldots, e_m$, then $(e_0, y, \lambda(e_0, x) + \delta)$ is amenable to x. The contradiction shows $\lambda_0 = 0$. Geometrically, this means that if one considers the line L from e_0 through x, then the unit vector u on L that is farthest from e_0 lies on a face of B_X that does not contain e_0.

THEOREM 2.13. *If X is a finite-dimensional normed space such that B_X is a polyhedron, then the λ-function is continuous on B_X.*

Proof. By Corollary 2.9 and Lemma 2.11, it only remains to show that the λ-function is continuous at each $x \in S_X \setminus \operatorname{ext}(B_X)$. Since $\operatorname{ext}(B_X)$ is finite and $\lambda(\cdot) = \max\{\lambda(e, \cdot)\colon\ e \in \operatorname{ext}(B_X)\}$, it suffices to show that each function $\lambda(e, \cdot)$ is continuous at each such x. To this end, fix $e \in \operatorname{ext}(B_X)$ and $x \in S_X \setminus \operatorname{ext}(B_X)$. Let (x_n) be a sequence in B_X such that $x_n \to x$. All of the numbers $\lambda(e, x)$, $\lambda(e, x_n)$ are attained. We first consider two special cases.

Case. I. $\|x_n\| = 1$ for all n.
We write

$$x = \lambda(e, x)e + (1 - \lambda(e, x))y(e, x),$$
$$x_n = \lambda(e, x_n)e + (1 - \lambda(e, x_n))y(e, x_n).$$

Let $(\lambda(e, x_{n_k}))$ be any convergent subsequence of $(\lambda(e, x_n))$, say $(\lambda(e, x_{n_k}))$ converges to λ. Since $x \neq e$, it follows that $\lambda < 1$ and $(y(e, x_{n_k}))$ converges to $y = (x - \lambda e)/(1 - \lambda)$. Thus, $\|y\| = 1$ and $x = \lambda e + (1 - \lambda)y$ which implies $0 \leq \lambda \leq \lambda(e, x)$. Assume that $\lambda < \lambda(e, x)$. By (a) of Lemma 2.5, $\alpha(e, x_{n_k}) \to \alpha$, where $\alpha < \alpha(e, x)$ and $\lambda = \alpha/(1 + \alpha)$. Choose $\varepsilon > 0$ such that

$$(\alpha + \varepsilon)/(1 + \alpha + \varepsilon) = (\lambda + \lambda(e, x))/2.$$

For each k, write

$$y_k = x_{n_k} + (\alpha(e, x_{n_k}) + \varepsilon)(x_{n_k} - e) = y(e, x_{n_k}) + \varepsilon(x_{n_k} - e). \tag{12}$$

If $F_1, \ldots, F_p$ denote the distinct faces of B_X, there exist $f_j \in X^*$, $\|f_j\| = 1$, such that $F_j = B_X \cap f_j^{-1}(1)$, $1 \leq j \leq p$. Moreover, by Remark 2.12 for each k, there is a face $F_{j(k)}$ that contains $y(e, x_{n_k})$ but not e. Therefore, $f_{j(k)}(e) < 1$ and $f_{j(k)}(y(e, x_{n_k})) = 1$. Since

$$x_{n_k} - e = (1 - \lambda(e, x_{n_k}))(y(e, x_{n_k}) - e),$$

we obtain from (12) that

$$\begin{aligned} f_{j(k)}(y_k) &= f_{j(k)}(y(e, x_{n_k})) + \varepsilon f_{j(k)}(x_{n_k} - e) \\ &= 1 + \varepsilon(1 - \lambda(e, x_{n_k}))(1 - f_{j(k)}(e)). \end{aligned} \tag{13}$$

By finiteness of the number of faces, we may assume, by (13) and by passing to a subsequence, that there is a common index j_0 among the $j(k)$'s satisfying

$$f_{j_0}(y_k) = 1 + \varepsilon\big(1 - \lambda(e, x_{n_k})\big)\big(1 - f_{j_0}(e)\big)$$

for all k. Consequently, $\|y_k\| \geq 1 + \varepsilon(1 - \lambda(e, x_{n_k}))(1 - f_{j_0}(e))$ for all k. But $y_k \to y + \varepsilon(x - e) = x + (\alpha + \varepsilon)(x - e)$ and $\alpha + \varepsilon < \alpha(e, x)$. This implies $\|y + \varepsilon(x - e)\| = 1$ from which we obtain $\lambda(e, x_{n_k}) \to 1$. The contradiction shows that $\lambda(e, x_{n_k}) \to \lambda(e, x)$, implying $\lambda(e, x_n) \to \lambda(e, x)$.

Case II. $\|x_n\| < 1$ for all n.

The obvious notational modification of the proof of Lemma 2.1 yields

$$\lambda(e, x_n) \geq \frac{1 + \|x_n\|}{2}\lambda\big(e, x_n/\|x_n\|\big) \quad \text{for all } n.$$

Case I then gives $\lambda(e, x_n/\|x_n\|) \to \lambda(e, x)$. This implies $\liminf \lambda(e, x_n) \geq \lambda(e, x)$. On the other hand, the same argument as in the first part of the proof of Case I shows that any cluster point λ of $(\lambda(e, x_n))$ satisfies $\lambda \leq \lambda(e, x)$. Therefore, $\limsup \lambda(e, x_n) \leq \lambda(e, x)$ completing the proof in this case.

The proof of the general case follows from Cases I and II.

3. Further properties of the λ-function.

THEOREM 3.1. *Let X be a normed space having the uniform λ-property. If $0 < \lambda < \inf\{\lambda(x): x \in B_X\}$, then for each $x \in B_X$, there is a sequence (e_k) in $\operatorname{ext}(B_X)$ such that*

$$\left\| x - \sum_{k=1}^{n} \lambda(1 - \lambda)^{k-1} e_k \right\| \leq (1 - \lambda)^n, \qquad n = 1, 2, \ldots.$$

Proof. By (c) of Proposition 1.2, there is a triple (e_1, x_1, λ) amenable to x; that is, $x = \lambda e_1 + (1 - \lambda)x_1$. Note that $\|x - \lambda e_1\| \leq 1 - \lambda$. By the same reasoning, there is a triple (e_2, x_2, λ) amenable to x_1. This yields

$$x = \lambda e_1 + \lambda(1 - \lambda)e_2 + (1 - \lambda)^2 x_2,$$

and

$$\|x - \lambda e_1 - \lambda(1 - \lambda)e_2\| \leq (1 - \lambda)^2.$$

The sequence (e_k) is obtained by repeating the preceding observations in a simple inductive argument.

REMARK 3.2. If $0 < \lambda < 1$, then $\sum_{k=1}^{\infty} \lambda(1-\lambda)^{k-1} = 1$. Thus, if X has the uniform λ-property, Theorem 3.1 shows: (a) each $x \in B_X$ admits an expansion $\sum_{k=1}^{\infty} \lambda_k e_k$ as an infinite convex combination of members of $\text{ext}(B_X)$ and (b) the sequences of partial sums of these series converge uniformly for $x \in B_X$. It is easy to check that (a) implies X has the λ-property. Moreover, the converse of Theorem 3.1 holds; that is, if (a) and (b) hold, then X has the uniform λ-property. To verify the last assertion, note that if (a) and (b) hold, then X (by (a)) has the λ-property and (by (b)) there is a positive integer N such that if $x \in B_X$, we can write $x = \sum_{k=1}^{\infty} \lambda_k e_k$, where (e_k) is a sequence in $\text{ext}(B_X)$, $\lambda_k \geq 0$ for all k, $\sum_{k=1}^{\infty} \lambda_k = 1$ and $\|x - \sum_{k=1}^{N} \lambda_k e_k\| \leq 1/2$. In particular, if $x \in S_X$, $1/2 \leq \sum_{k=1}^{N} \lambda_k$ which implies $1/2N \leq \lambda_{k_0}$, for some index k_0. If $\lambda_{k_0} = 1$, $\lambda(x) = 1$; if $\lambda_{k_0} < 1$, then

$$\left(e_{k_0}, \sum_{k \neq k_0} \frac{\lambda_k}{1 - \lambda_{k_0}} e_k, \lambda_{k_0} \right)$$

is amenable to x and so $\lambda(x) \geq 1/2N$. By Lemma 2.1, $\lambda(x) \geq 1/4N$ for all $x \in B_X$.

We do not know if the λ-property implies (a). As the following result shows, however, it does imply a similar but weaker statement.

THEOREM 3.3. *Let X be a normed space satisfying the λ-property.*

(i) *If a convex function $f\colon B_X \to \mathbf{R}$ attains its maximum value, then it attains its maximum value at a member of* $\text{ext}(B_X)$.

(ii) *If X is a Banach space, then B_X is the closed convex hull of its set of extreme points.*

Proof. (i) Suppose that f attains its maximum value at x. Pick a triple (e, y, λ) that is amenable to x. Since $x = \lambda e + (1-\lambda)y$ and f is a convex function, we have $f(x) \leq \lambda f(e) + (1-\lambda) f(y)$. The fact that $0 < \lambda \leq 1$ implies $f(e) = f(x)$.

(ii) Assume, to the contrary, that there exists $x \in B_X \setminus \overline{\text{co}(\text{ext}(B_X))}$. Then there is a continuous linear functional f on $X_{\mathbf{R}}$ and a number M such that $\|f\| = 1$ and $|f(y)| \leq M < f(x)$ for all $y \in \text{co}(\text{ext}(B_X))$. By the Bishop-Phelps theorem, there is a continuous linear functional g on $X_{\mathbf{R}}$ such that $\|g\| = 1$, $\|f - g\| < (f(x) - M)/4$ and g attains its norm on

B_X. A straightforward computation shows

$$|g(y)| \le M + \frac{f(x) - M}{4} < g(x)$$

for all $y \in \mathrm{co}(\mathrm{ext}(B_X))$. Consequently, g does not attain its maximum value on B_X at a member of $\mathrm{ext}(B_X)$. This contradiction of (i) completes the proof.

REMARK 3.4. Recall that a normed space X has the Krein-Milman property if every closed and bounded convex subset of X is the closed convex hull of its set of extreme points. Theorem 3.3 shows that if X satisfies the λ-property, then X satisfies a restricted version of the Krein-Milman property; namely, B_X is the closed convex hull of its set of extreme points. The converse, however, is false. For example, the space $C_X(T)$ of Remark 1.7 fails to have the λ-property. On the other hand, since X is the set of complex numbers, the unit ball of $C_X(T)$ is the closed convex hull of its set of extreme points (see [5]).

THEOREM 3.5. *Let X be a Banach space with the λ-property. If* $\mathrm{ext}(B_X)$ *is countable, then the λ-function is locally bounded away from* 0 *in the following sense: Given any $x_0 \in B_X$ and any open neighborhood W of x_0 in B_X, there exists a point $x_0' \in W$, a neighborhood W' of x_0' in B_X and $\lambda' > 0$ such that $\lambda(x) \ge \lambda'$ for all $x \in W'$.*

Proof. Let (e_n) be an enumeration of the members of $\mathrm{ext}(B_X)$ and let (r_m) be an enumeration of the rational numbers in $(0, 1)$. If $x \in W$, there is a positive integer n and a triple (e_n, y, λ) amenable to x. Choosing m such that $r_m < \lambda$, there is a triple (e_n, y', r_m) amenable to x. Consequently, $x \in W_{mn}$, where $W_{mn} = W \cap (r_m e_n + (1 - r_m)B_X)$. This shows that $W = \bigcup_{m,n=1}^{\infty} W_{mn}$ and, since each set W_{mn} is closed in W, the Baire category theorem guarantees the existence of indices m', n' such that $W_{m'n'}$ has non-empty interior in W. Therefore, there is a point $x_0' \in W_{m'n'}$ and $\varepsilon > 0$ such that $W_{m'n'}$ contains $W' = W \cap \{x \in B_X\colon \|x - x_0'\| < \varepsilon\}$. It follows that if $x \in W'$, we have $\lambda(x) \ge r_{m'}$, completing the proof.

If X has the λ-property and the members of $\mathrm{ext}(B_X)$ are separated, then points in B_X that are close to being extreme points of B_X possess a unique representation property. In order to make this precise, we need the following.

LEMMA 3.6. *Let X be a real normed space with the λ-property. Assume there is a number* $\delta > 0$ *such that* $\|e - e'\| \geq \delta$ *whenever* $e, e' \in \text{ext}(B_X)$ *and* $e \neq e'$. *If* $x \in B_X$ *and* (e, y, λ), (e', y', λ') *are amenable to* x, *where* $\lambda, \lambda' > 3/(3 + \delta)$, *then* $e = e'$.

Proof. We have $x = \lambda e + (1 - \lambda)y = \lambda' e' + (1 - \lambda')y'$ and so

$$\lambda(e - e') = (\lambda' - \lambda)e' - (1 - \lambda)y + (1 - \lambda')y'. \tag{14}$$

If $e \neq e'$, then (14) and the fact that $\|e - e'\| \geq \delta$ imply

$$\frac{3\delta}{3+\delta} < \lambda\|e - e'\| \leq |\lambda' - \lambda| + 2 - \lambda - \lambda'$$
$$< 3\left(1 - \frac{3}{3+\delta}\right) = \frac{3\delta}{3+\delta}.$$

The contradiction shows $e = e'$.

THEOREM 3.7. *Let X be a real normed space with the λ-property. Assume there is a number* $\delta > 0$ *such that* $\|e - e'\| \geq \delta$ *whenever* $e, e' \in \text{ext}(B_X)$ *and* $e \neq e'$. *If* $x \in B_X \setminus \text{ext}(B_X)$ *and* $\lambda(x) > 3/(3 + \delta)$, *then there exists a unique pair of vectors* $e \in \text{ext}(B_X)$, $y \in S_X$ *such that* $x = \lambda(x)e + (1 - \lambda(x))y$.

Proof. Let (e_n, y_n, λ_n) be a sequence of triples that are amenable to x and for which $\lambda_n \uparrow \lambda(x)$, $\lambda_n > 3/(3 + \delta)$ for all n and $\|y_n\| = 1$ for all n. By Lemma 3.6, all the e_n's are equal, say to e. Since $x = \lambda_n e + (1 - \lambda_n)y_n$ for all n, we have $(1 - \lambda_n)y_n \to x - \lambda(x)e$. Also, $x \neq e$ implies $\lambda(x) < 1$. Thus, if we let $z = x - \lambda(x)e$, $\|z\| = 1 - \lambda(x)$. Letting $y = z/\|z\|$ shows $x = \lambda(x)e + (1 - \lambda(x))y$, proving existence. If we also have $x = \lambda(x)e' + (1 - \lambda(x))y'$, wher $e' \in \text{ext}(B_X)$ and $\|y'\| = 1$, Lemma 3.6 implies $e = e'$. Since $1 - \lambda(x) \neq 0$, we also obtain $y = y'$.

4. Questions and open problems. The following list of questions is not meant to be exhaustive. Rather, it represents those questions which are of most interest to the authors.

1. It would be useful to calculate the λ-function for other classical spaces with the λ-property. In particular, what spaces of operators have the λ-property and what does the λ-function look like for these spaces?

2. If (X_n) is a sequence of normed spaces, each having the λ-property, when do $(\oplus\sum_{n=1}^{\infty} X_n)_{l_1}$, $(\oplus\sum_{n=1}^{\infty} X_n)_{l_\infty}$ have the λ-property and what do their λ-functions look like?

3. If X is a normed space having the λ-property, characterize the points of continuity on S_X of the λ-function. Characterize those $x \in B_X$ for which $\lambda(x)$ is attained.

4. If X is a Banach space, is the λ-function of the first Baire class on B_X?

5. If X is a normed space having the λ-property, can X be renormed so as to have the uniform λ-property?

6. If a normed space X has the λ-property, does X^* have the λ-property? In considering the converse, note that l_1 has the λ-property and c_0, one of its preduals, does not. However, c is a predual of l_1 that has the λ-property.

REFERENCES

[1] Y. Benyamini, and Y. Sternfeld, *Spheres in infinite-dimensional normed spaces are Lipschitz contractible*, Proc. Amer. Math. Soc., **88** (1983), 439–445.

[2] R. M. Blumenthal, J. Lindenstrauss and R. R. Phelps, *Extreme operators into $C(K)$*, Pacific J. Math., **15** (1965), 747–756.

[3] J. Dugundji, *An extension of Tietze's theorem*, Pacific J. Math., **1** (1951), 353–367.

[4] R. R. Phelps, *Lectures on Choquet's Theorem*, Van Nostrand Math. Studies, #7, 1966, Princeton.

[5] ______, *Extreme points in function algebras*, Duke Math. J., **32** (1965), 267–278.

Received December 18, 1985 and in revised form April 2, 1986.

KENT STATE UNIVERSITY
KENT, OH 44242