

REDUCIBLE COMPLEMENTS OF LIE ALGEBRA RADICALS

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This paper establishes for finite dimensional Lie algebras of characteristic p a counterpart of Levi's "radical splitting theorem" for Lie algebras of characteristic 0.

Reducible Lie algebras are Lie algebras whose solvable radicals $\text{Solv } L$ are nilpotent. For every Cartan subalgebra H , $\text{Solv } L$ has a reducible complement S invariant under $\text{ad } H_s$, that is, $L = S + \text{Solv } L$ and $S \cap \text{Solv } L$ is nilpotent. The key for choosing S invariant under $\text{ad } H_s$ is a generalization of Engel's Criteria for Nilpotence. When L is a Lie p -algebra, S can be chosen such that every torus of L of maximal rank is in the same toral switching class as some torus of S .

1. Introduction. This paper is concerned with the structure of Lie algebras L over a field k of characteristic $p = 0$ or $p > 0$. Notational conventions follow Jacobsen [2], Seligman [3], Winter [5]. Throughout the paper, all Lie algebras and vector spaces are assumed to be finite dimensional.

In the paper, we prove a variation of Levi's theorem, which is concerned with the existence of complements, in the following sense, of the solvable radical $\text{Solv } L$ of L . We also indicate briefly in general terms, how our results are used to generate, inductively, structural information about a given Lie algebra.

1.1. DEFINITION. A *complement* of the solvable radical $\text{Solv } L$ of a Lie algebra L is a subalgebra S of L such that $L = S + \text{Solv } L$.

Note that $S/S \cap \text{Solv } L$ is isomorphic to $L/\text{Solv } L$ for any complement S of $\text{Solv } L$ in L . Since $L/\text{Solv } L$ is semisimple, we may state this as follows.

1.2. PROPOSITION. $S/\text{Solv } S$ is isomorphic to $L/\text{Solv } L$ for any complement S of $\text{Solv } L$ in L . □

The "best possible" theorem on existence of complements holds only when the ground field k has characteristic 0, and counterexamples abound for characteristic $p > 0$.

1.3. THEOREM (*Levi* [2]). *Let L be a Lie algebra over a field of characteristic 0. Then the solvable radical of L has a semisimple complement S .* \square

In this paper, we prove the existence of *reducible* $\text{ad } H_s$ -stable complements of $\text{Solv } L$ in L , given a Cartan subalgebra H . We briefly discuss reducible Lie algebras and the use of such reducible complements in §§2 and 3. Section 4 is devoted to a generalization of Engel's theorem needed for the proof, in §5, of the existence of reducible complements stable under $\text{ad } H_s$. Questions of unicity, conjugacy and Cartan switching are discussed in §6, where it is shown that a Lie p -algebra L with triangulable radical $\text{Solv } L$ has a reducible complement S stable under $\text{ad } H_s$ such that every torus T of L of maximal dimension can be switched to a torus T' which is contained in S . Here, $\text{ad } H_s = \text{span}\{(\text{ad } x)p^i \mid x \in H, p^i \geq \dim L\}$. (Recall from Winter [11] that each element x of a Lie p -algebra decomposes as $x = x_s + x_n$ uniquely where x_s and x_n are defined in terms of the p th power operation and $\text{ad } x = \text{ad } x_s + \text{ad } x_n$ is decomposition of $\text{ad } x$ into its semisimple and nilpotent parts.)

Reducible Lie algebras are characterized in §2 in terms of their ad -modules, which we now define. The class of such modules for a given Lie algebra L contains all subquotients of the adjoint (left regular) module L of L .

1.4. DEFINITION. An *ad-representation* for a Lie algebra L is a representation ρ for L such that $\rho(x)$ is nilpotent for all $x \in L$ for which $\text{ad } x$ is nilpotent. An *ad-module* for L is a module for L whose associated representation is an ad -representation.

2. Reducible Lie algebras. When the condition in Theorem 1.3 that S be semisimple is weakened to the condition that S be reducible in the following sense, a revised theorem holds for all Lie algebras L relative to a given Cartan subalgebra H .

2.1. DEFINITION. A Lie algebra S is *reducible* if its solvable radical $\text{Solv } S$ is nilpotent.

Letting $\text{Nil } S$ denote the (unique) maximal nilpotent ideal of S , the *nilpotent radical* of S , it follows that S is reducible if and only if $\text{Solv } S = \text{Nil } S$ if and only if $S/\text{Nil } S$ is semisimple. This implies that the irreducible ad -modules V for a reducible Lie algebra S coincide with those for $S/\text{Solv } S$, since $(\text{Solv } S) \cap V = \text{Solv}\{0\}$. Taking V to be the adjoint

module of a Lie algebra S and passing from V to the sum $\bar{V} \sum_{i=1}^n \bar{V}_i$ of its composition factors \bar{V}_i , the condition $(\text{Solv } S) \bar{V}_i = \{0\}$ ($1 \leq i \leq n$) implies that $\text{Nil } S = \text{Solv } S$ and S must be reducible. We state these observations as follows.

2.2. PROPOSITION. *A Lie algebra S is reducible if and only if $(\text{Solv } S) V = \{0\}$ for every irreducible ad-module for S .* □

The characterization of reducible Lie algebras stated in Proposition 2.2 is the basis for applications of reducible Lie algebras such as those discussed in §3 below.

The following characterization of reducible complements is used in §5.

2.3. PROPOSITION. *S is a reducible complement of the solvable radical of L if and only if:*

1. $L = S + \text{Solv } L$;
2. $S \cap \text{Solv } L$ is nilpotent. □

3. Application of the existence of reductive complements. We now briefly indicate in general terms how Theorem 5.1 is used to generate, inductively, structural information about a Lie algebra L . This method is used in Winter [9, 10] to get specific information on root strings in the root system of L .

One starts with a subalgebra B of L for which one already has information about the structure of the semisimple algebra $B/\text{Solv } B$ and its irreducible representations. To use this information to obtain information about L , one takes a reducible complement S of $\text{Solv } B$ in B and a composition series $L_1 > L_2 > \dots > L_n$ of L as S -module. One then has:

1. $B/\text{Solv } B$ and $S/\text{Nil } S$ are isomorphic;
2. The composition factors L_i/L_{i+1} are irreducible modules for $S/\text{Nil } S$ when $(\text{Nil } S)L_i \subset L_{i+1}$.

But then such composition factors L_i/L_{i+1} can be regarded as irreducible modules for $B/\text{Solv } B$, by way of an isomorphism from $B/\text{Solv } B$ to $S/\text{Nil } S$. This enables one to transfer known structural information about the semisimple Lie algebra $B/\text{Solv } B$ and its irreducible representations to $S/\text{Solv } S$ and the composition factors L_i/L_{i+1} which are ad-modules for S .

An important feature of the procedure outlined above is that structural information about a semisimple Lie algebra $B/\text{Solv } B = \bar{B}$ which is expressed relative to a specified Cartan subalgebra \bar{H} is transferred to a

reducible complement S provided that S be stable under $\text{ad } H_s$. Such a choice of S is always possible, by Theorem 5.1.

4. A generalization of Engel's Theorem. Engel's theorem states that a Lie algebra L is nilpotent if $\text{ad } L$ consists of linear transformations. A linear version of it states that a Lie subalgebra L of the algebra $\text{Hom}_k V$ of linear transformations of a vector space V is nilpotent if it consists of linear transformations. The purpose of this section is to generalize these two theorems as a prelude to §6. The generalization is based on the theorem of Jacobson [1] on weakly closed sets of nilpotent linear transformations, and on methods of ascent and descent.

We begin by proving a linear version of parts 1 and 2 of the following Engel's theorem for graded Lie algebras over a group A .

4.1. THEOREM (Winter [8]). *Let $L = \sum_{a \in A} L_a$ be an A -graded Lie algebra. Suppose that $\text{ad } L_1$ consists of nilpotent linear transformations. Then:*

1. *if A is torsion free, L is nilpotent;*
2. *if $p > 0$, A is a p -group and $(\text{ad } L)^p \subset \text{ad } L$, L is nilpotent;*
3. *if A is cyclic, L is solvable.* □

In the above theorem, an A -graded Lie algebra is a Lie algebra L which is the direct sum $L = \sum_{a \in A} \oplus L_a$ of subspaces L_a (some of which may be $\{0\}$) such that $[L_a, L_b] \subset L_{ab}$ for all $a, b \in A$. In the following theorem, an A -graded linear Lie algebra is an A -graded Lie algebra $L = \sum_{a \in A} L_a$ which is a Lie subalgebra of $\text{Hom}_k V$ where V is a finite dimensional graded vector space $V = \sum_{a \in A} \oplus V_a$ such that $L_a V_b \subset V_{ab}$ for all $a, b \in A$.

4.2. THEOREM. *Let $L = \sum_{a \in A} L_a$ be a graded linear Lie algebra, $V = \sum_{a \in A} V_a$ the underlying graded vector space. Suppose that L_1 consists of nilpotent transformations of V . Then:*

1. *if either A is torsionfree, or $p > 0$, A is a (finite or infinite) p -group and $L^p \subset L$, then L consists of nilpotent linear transformations and L is nilpotent:*
2. *if A is cyclic, L is solvable;*
3. *if A is cyclic and $V_0 = \{0\}$, L consists of nilpotent linear transformations and L is nilpotent.*

Proof. By the theorem of Jacobson [2] on weakly closed sets of nilpotent linear transformations, it suffices to show that x is nilpotent for all $x \in L_a$, $a \in A$. For $a = 1$, this is true by hypothesis. Let $a \neq 1$ and

note that $x^n V_b \subset V_a n_b$ for all $b \in A$ and all positive integers n . For A torsionfree or $V_0 = \{0\}$, this implies that x is nilpotent. Suppose next that $p > 0$, and A is a p -group. Since L is finite dimensional, we can assume that A is finite without any loss of generality. Let $|a| = p^e$ and $L^p \subset L$. Then $x^{p^e} V_b \subset V_a^{p^e} b = V_b$ for all $b \in A$. Writing $x^{p^e} \in L^{p^e} \subset L$ as $x^{p^e} = y \in L$, and writing y as $y = \sum_{c \in A} y_c \in \sum_{c \in A} L_c$, we conclude for $v_b \in V_b$ that V_b contains $y v_b = \sum_{c \in A} y_c v_b$. Thus, $y_c v_b = 0$ for $c \neq 1$, $b \in A$. It follows that $y = y_1 \in L_1$ and y is nilpotent. Since $y = x^{p^e}$, x is nilpotent, as asserted. Finally, suppose that A is cyclic. Since L_0 consists of nilpotent linear transformations, so does $\text{ad } L_0$, so that L is solvable by Theorem 4.1. \square

We now formulate the theorem needed for §5 in terms of the p -closure of L . This is defined as the span \bar{L} of $\{x^{p^e} | x \in L, e = 0, 1, 2, \dots\}$ ($p^e = 1$ if $p = 0$), so that $\bar{L} = L$ for $p = 0$. The growth in passing from L to \bar{L} is related to the strength of Theorem 4.3 for a given L , which is greatest when $L = \bar{L}$. The *departure* \bar{L}/L of \bar{L} from L is constrained by the following relationships between L and \bar{L} , observed in Winter [6, 7], which show that \bar{L}/L is an abelian Lie algebra isomorphic to $\hat{H}/(\hat{H} \cap H)$:

1. $\bar{L}^{(1)} \subset L$ and $\bar{L} = \hat{H} + L$ for every Cartan subalgebra \hat{H} of \bar{L} ;
2. \bar{L} has a Cartan subalgebra \hat{H} such that $\hat{H} \cap L \subset H$. Those relationships are not needed for the proof of Theorem 4.3, but they and the construction in Winter [7] of \hat{H} from H are relevant to understanding when it can be applied. For the proof, one does need to know that $[\bar{N}, \bar{L}] \subset [N, L]$ for any subalgebra N of $\text{Hom}_k V$ such that $[N, L] \subset L$. To see this, observe, firstly, that $[\bar{N}, L] \subset L$ since the normalizer of L is closed under p th powers. Then note that $x \in \bar{N}$, $y \in L$ implies $[x, y^{p^e}] \in L$, which is clear upon rewriting $[x, y^{p^e}]$ as $[\dots[x, y], y, \dots, y]$ (y occurs p^e times) with $[x, y] \in [\bar{N}, L] \subset L$ and $y \in L$.

4.3. THEOREM. *Let L be a Lie subalgebra of $\text{Hom}_k V$ and let N be a nilpotent Lie subalgebra of $\text{Hom}_k V$ such that $[N, L] \subset L$. Suppose that $\bar{L}_0(\text{ad } N)$ consists of nilpotent linear transformations. Then L consists of nilpotent linear transformations and L is nilpotent.*

Proof. We take K to be the algebraic closure of k and ascend from k, V, N, L, \bar{L} to $K, V_K = K \otimes_k V, N_K = K \otimes_k N, L_K = K \otimes_k L, \bar{L}_K = K \otimes_k \bar{L}$. Since $\bar{L}_0(\text{ad } N)$ is the solution space of a system of linear equations (defined by p^e th powers $(\text{ad } x)^{p^e}$ with $p^e \geq \dim \bar{L}$ of $\text{ad } x \in \text{ad } N$, for $p > 0$, and by semisimple parts of $\text{ad } x \in \text{ad } N$ for $p = 0$),

$\bar{L}_{K0}(\text{ad } N_K)$ is the K -span $K\bar{L}_0(\text{ad } N)$ of $\bar{L}_0(\text{ad } N)$. Thus, the zero subalgebra of the graded Lie algebra $\bar{L}_K = \sum_{a \in G} \bar{L}_{Ka}(\text{ad } N_K) = \sum_{a \in G} \bar{L}_{Ka}$ is $\bar{L}_{K0} = K\bar{L}_0(\text{ad } N)$. It follows from Engel's theorem that $\bar{L}_0(\text{ad } N)$ and, therefore, $K\bar{L}_0(\text{ad } N) = \bar{L}_{K0}$ are triangulable algebras of nilpotent linear transformations of V_K . Now \bar{L}_K is a graded linear Lie algebra with underlying graded vector space $V_K = \sum_{a \in G} V_{Ka}(N_K)$. Since G is a finitely generated additive group of functions from N_K to K , G is torsionfree for $p = 0$ or an elementary abelian p -group for $p > 0$. For $p > 0$, the condition $L^p \subset L$ implies that $(L_K)^p \subset L_K$, by Theorem 11, p. 190 of Jacobson [1]. We may, therefore, conclude that \bar{L}_K consists of nilpotent transformations of V_K and \bar{L}_K is nilpotent, by Theorem 4.2. Thus, L is a nilpotent Lie algebra of nilpotent linear transformations of V , as asserted. \square

The above counterpart of Engel's theorem for linear Lie algebras has, as immediate consequence, the following counterpart of Engel's theorem for abstract Lie algebras.

4.4. THEOREM. *Let L be a Lie algebra and let N be a nilpotent Lie algebra of derivations of L . Suppose that $(\text{ad } \bar{L})_0(\text{ad } N)$ consists of nilpotent linear transformations. Then L is nilpotent.* \square

5. Existence of reducible complements relative to a given Cartan subalgebra. In Theorem 5.1 below, we show that Lie algebra L with Cartan subalgebra H has a reducible complement S stable under $\text{ad } H_s$, that is, $\text{ad } H_s(S) \subset S$ and:

1. $L = S + \text{Solv } L$;
2. $S \cap \text{Solv } L$ is nilpotent.

We then show for $L/\text{Solv } L$ simple that the proper ideals of any minimal $\text{ad } H_s$ -stable reducible complement S of $\text{Solv } L$ in L are contained in $\text{Nil } S$.

5.1. THEOREM. *Let L be a Lie algebra over a field k , and let H be a Cartan subalgebra of L . Then the solvable radical of L has a reducible $\text{ad } H_s$ -stable complement S . If L is restricted, S can be taken to be restricted.*

Proof. If $\text{Solv } L$ is nilpotent, it suffices to let $S = L$.

Suppose next that $R = \text{Solv } L$, is not nilpotent. By Theorem 4.3 $(\text{ad } \bar{R})_0(\text{ad } \text{ad } H)$ has an element X_1 which is not nilpotent on R . Since we need the action of X_1 on L later in the proof, $\text{ad } x$ (x in R or L) is

regarded as left translation on L , rather than on R . Let $R_1 = R_0(X_1) = \{x \in R \mid X_1^{\dim L}(x) = 0\}$ and note that $R \neq R_1$. We continue to choose X_2, X_3, \dots and $\overline{R_2}, R_3, \dots$ according to the rules.

1. $X_{i+1} \in (\overline{\text{ad } R_i})_0(\text{ad } H_s)$ where X_{i+1} is not nilpotent on R_i ;
2. $R_{i+1} = R_{i0}(X_{i+1})$.

Note that R_i is stable under $\text{ad } H$ since $[X_i^{p^e}, \text{ad } H] = 0$ for $p^e \geq \dim L$. Since $R_1 > R_2 > \dots > R_i$, by condition (1), there is a terminal R_i which is reached when the condition “ X is nilpotent on R_i for all $X \in (\overline{\text{ad } R_i})_0(\text{ad } H)$ ” is met. But this is the condition of Theorem 4.3 for nilpotency of $\text{ad } R_i$, so that R_i is nilpotent. Since $X_1 \in (\overline{\text{ad } R_0})_0(\text{ad } H)$, we have $H \subset L_1 = L_0(X_1) = \{x \in L \mid X_1^{p^e}(x) = 0\}$ ($p^e \geq \dim L$). Moreover, $X_1 \in \overline{\text{ad } R}$ and $\text{ad } R(L) \subset R$ imply that $X_1(L) \subset R$, so that $L_1 + R = L_0(X_1) + R = L_0(X_1) + L_*(X_1) = L$. Since $X_2 \in (\overline{\text{ad } R_1})_0(\text{ad } H)$, we have $H \subset L_2 = L_{10}(X_2)$. Moreover, we have $X_2(L) \subset R$ and $L_2 + R = L_{10}(X_2) + R = L_{10}(X_2) + L_{1*}(X_2) + R = L_1 + R = L$. Continuing, we have $H \subset L_i$ and $L = L_i + R$. Finally, we have $L_i \cap R = L_0(X_1) \cap L_0(X_2) \cap \dots \cap L_0(X_i) \cap R = R_0(X_1) \cap R_0(X_2) \cap \dots \cap R_0(X_i) = R_i$, so that $L_i \cap R$ is nilpotent. The following conditions are now established, which completes the proof of the existence of S since we may take $s = L_i$:

1. $L = L_i + \text{Solv } L, H \subset L_i$;
2. $L_i \cap \text{Solv } L$ is nilpotent.

Clearly, if L is restricted, $S = L_i$ is also. □

By the above theorem, a Lie algebra L with Cartan subalgebra H has a reducible complement stable under $\text{ad } H_s$; so L has a minimal $\text{ad } H_s$ -stable complement S . The ideals I of S correspond to ideals $I + \text{Solv } L$ of L , since $L = S + \text{Solv } L$. The ideals $S^i, S^{(i)}$ are $\text{ad } H$ -stable ideals of S . If $L/\text{Solv } L$ is simple (and nonzero), then $L = S^2 + \text{Solv } L$ and S^2 is a reducible complement. But then $S = S^2$ by the minimality. More generally, we have the following consequence of Theorem 5.2.

5.2. THEOREM. *Let L be a Lie algebra with Cartan subalgebra H and let S be a minimal $\text{ad } H_s$ -stable reducible complement of $\text{Rad } L$ in L . Suppose that $L/\text{Solv } L$ is simple. Then S is a local Lie algebra in the sense that every proper $\text{ad } H_s$ -stable ideal of S is contained in $\text{Nil } S$.* □

The condition that S be a local Lie algebra is generally much stronger than the condition that $S/\text{Nil } S$ be simple. In fact, this is illustrated by such easily constructed algebras as $S = T \oplus U$ (direct) with U nonzero and nilpotent, $T/\text{Nil } T$ simple (and nonzero), $\text{Nil } T$ nonzero, which are not local since T is a proper ideal not contained in $\text{Nil } S = \text{Nil } T \oplus U$.

Proof of Theorem 5.2. Let I be a proper ideal of S . Since S is an ideal of $H + S$, I^∞ is an ideal of $H + S$ by Jacobson [2], Exercise 9, p. 29. Since I^∞ is $\text{ad } H$ -stable, the ideal $I^\infty + \text{Solv } L$ of L is proper, by minimality of S . By simplicity of $L/\text{Solv } L$, it follows that $I^\infty \subset \text{Solv } L$, so that $I^\infty \subset S \cap \text{Solv } L = \text{Nil } S$. But then the nilpotent ideal $I + \text{Nil } S/\text{Nil } S$ of the simple Lie algebra $S/\text{Nil } S$ must be 0 and $I \subset \text{Nil } S$. \square

6. Uniqueness, conjugacy and switching of Cartan subalgebras. For Lie algebras L over an algebraically closed field of characteristic 0, the rich supply of automorphisms $\exp \text{ad } x = \sum_{n=0}^{\infty} (\text{ad } x)^n/n!$ ($\text{ad } x$ nilpotent) insure that the group $\text{Aut } L$ of automorphisms of L act transitively on the set of Cartan subalgebras, and on the set of semisimple complements (Levi factors), of L .

For Lie algebras L over an algebraically closed field of characteristic $p > 0$, the objects of discussion for this section, such $\exp \text{ad } x$ can be defined by truncating to $\exp \text{ad } x = \sum_{n=0}^{p-1} (\text{ad } x)^n/n!$. However, the truncated $\exp \text{ad } x$ are not always automorphisms, and Cartan subalgebras are not always conjugate under $\text{Aut } L$.

The effect of this is that nonconjugate Cartan subalgebras must be taken into account. The emerging classification theory for simple Lie algebras of Block-Wilson [1], Wilson [4] involves starting from a given Cartan subalgebra H , and then using the exponential Cartan switching operators e^x of Winter [11] to change H to a Cartan subalgebra $H' = e^{x_1} \cdots e^{x_n} H$ in an “optimal” conjugacy class. Theorem 6.1 shows that, under certain circumstances, there is no loss of generality in doing this within a given reducible complement S containing H .

There is no known general procedure for “switching” nonnilpotent subalgebras, such as reducible complements of S , thereby generating switching classes of such S . However, Theorem 6.1 can be viewed as a method for comparing reducible complements S, S' of a Lie p -algebra L which respectively contain tori T, T' of l of maximal dimension.

In the theorem, a Lie p -algebra L is p -triangular if its restricted representations are triangulable or, equivalently, if $x^{p^e} = 0$ for $p^e \geq \dim L$ for all $x \in L^{(1)}$.

6.1. THEOREM. *Let L be a Lie p -algebra with maximal torus H over an algebraically closed field of characteristic $p > 0$ such that $\text{Solv } L$ is p -triangular. Then L has a reducible complement S of $\text{Solv } L$ containing H such that every torus T of L of maximal dimension can be switched to a torus $T' = e^{x_1} \cdots e^{x_n} T$ contained in S .*

Proof. By the special properties of Lie p -algebras and their Cartan subalgebras discussed in Winter [7], $S = L_0(\text{ad}(H \cap \text{Solv}L))$ has intersection $S \cap \text{Solv}L = (\text{Solv}L)_0(\text{ad}(H \cap \text{Solv})) \supset \hat{H}$, a maximal torus of $\text{Solv}L$. Since $\text{Solv}L$ is p -triangular, it follows that $\text{Solv}L = \hat{H} + (\text{Solv}L)^{(1)}$ where $(\text{Solv}L)^{(1)}$ is a p -nil ideal of L . Now $L = S + (\text{Solv}L)^{(1)}$ with $S \cap \text{Solv}L$ nilpotent, so that S is a reducible complement. The decomposition $L = S + (\text{Solv}L)^{(1)}$ of L with $(\text{Solv}L)^{(1)}$ a p -nil ideal of L enables us to invoke Theorem 4.3 of Winter [11] in concluding that S contains some $e^{x_1} \cdots e^{x_n}T$. \square

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