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REPRESENTING CLASSES IN THE BRAUER GROUP OF QUADRATIC NUMBER RINGS AS SMASH PRODUCTS

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Azumaya algebras may be constructed as smash products of Galois objects with respect to a dual pair of Hopf algebras. In this paper we explore how useful this construction is for representing the non-trivial class in the Brauer group of a real quadratic number field.

The classical representation of a class in the Brauer group of a field K as a crossed product with respect to a finite Galois extension of K is often not available for the Brauer group of a commutative ring R [**1**], if one restricts to Galois extensions of R with group G in the sense of Chase, Harrison, Rosenberg [**3**]. A well-known example is $R = Z[\sqrt{2}]$; then the Brauer group of R has two elements, and the non-trivial class of $\mathrm{Br}(R)$ is not split by a Galois extension, much less representable as a crossed product, because R has no non-trivial Galois extensions. However, if one considers extensions S of R which are Galois objects with respect to a finite Hopf algebra H, then at least for number rings, Galois H-objects can easily be found which split all classes in the Brauer group. This fact suggests that the crossed product construction involving Galois objects of Hopf algebras may provide access to Azumaya algebras and Brauer classes whose representation heretofore has been obscure.

The object of this paper is to examine this idea, in particular for crossed products arising from Galois H-objects with normal basis where H is a free rank 2 Hopf algebra. Such crossed products have an attractive presentation as the smash product of two Galois objects with normal basis. We find a collection of such smash products over rings of integers of real quadratic fields. Included is a table presenting all such smash products over rings of integers of $Q(\sqrt{m})$, $m < 100$.

We also show that even with the availability of these more general crossed products, examples may be found of classes in the Brauer group which cannot be represented by these crossed products.

1. Free rank 2 Hopf algebras and their Galois objects with normal basis. Let R be a commutative ring, and suppose H is a Hopf algebra, commutative and cocommutative, which is free of rank 2 over R. Since the augmentation map $\varepsilon: H \to R$ is a split epimorphism, $\ker \varepsilon$, the augmentation ideal, is free of rank one over R, $\ker \varepsilon = Rx$. Then, as Tate and Oort **[20]** and, subsequently, Kreimer **[15]** and Nakajima **[16]** have shown, $H \cong R[x]/(x^2 - ax)$ for some a in R, with

$$\Delta(x) = x \otimes 1 + 1 \otimes x - b(x \otimes x),$$

and $ab = 2$. Similarly, $H^* = \operatorname{Hom}_R(H, R)$ is of the same form, and, as Hurley **[14]**, following Tate-Oort **[20]** shows,

$$H^* \cong R[f]/(f^2 - bf), \tag{1.1}$$

b as above, where $\langle f, x \rangle = -1$.

Notation. Set $H_a = R[x]/(x^2 - ax)$, with

$$\Delta(x) = x \otimes 1 + 1 \otimes x - b(x \otimes x), \qquad ab = 2.$$

Then $H_a^* = H_b$.

EXAMPLES. Let $a = 2$. Then $H_2 \cong R[x]/(x^2 - 2x)$. Let G throughout denote the cyclic group of order 2 with generator σ. Then $RG \to H_2$ via the map $\sigma \mapsto 1 - x$, is an isomorphism.

Let $a = 1$. Then $H_1 \cong R[x]/(x^2 - x) \cong Re_0 + Re_1, e_0, e_1$ pairwise orthogonal idempotents, via $x \mapsto e_0$, $1 - x \mapsto e_1$. Then H_1 is isomorphic to the dual of the group ring RG, where $e_i(\sigma^j) = \delta_{ij}$.

For a less-standard example, let $R = Z[\sqrt{2}]$, let $a = \sqrt{2}$. Then $H_{\sqrt{2}} \cong R[x]/(x^2 - \sqrt{2}x)$, which is self-dual.

PROPOSITION 1.1. *The isomorphism classes of free rank* 2 *Hopf R-algebras form a poset under the relation*: $H_a < H_b$ *if there is an injective map from H_a to H_b. This poset is isomorphic to the poset under inclusion of the principal ideals of R containing the principal ideal* (2).

Proof. Suppose c, d are in R and c divides d, $d = ce$. Then we have an injective map from $H_d = R[x]/(x^2 - dx)$ to $H_c = R[y]/(y^2 - cy)$ via $x \mapsto ey$. This is easily seen to be a Hopf algebra map, and an isomorphism if e is invertible in R. Since by definition of H_d, d divides 2, the proposition is clear.

Note in particular that $H_2 = RG$, hence, in particular, G embeds in H_a for any a, where σ maps to $1 - bx$, $ab = 2$ and H_a in turn embeds in $H_1 = RG^*$.

COROLLARY 1.2. *If (2) is a maximal ideal of R, then the only free rank 2 Hopf algebras over R are RG and its dual. If 2 is invertible in R then the only free rank 2 Hopf algebra over R is RG (which is isomorphic to its dual).*

More generally, if (2) is maximal and the class number of R is odd, then the only rank 2 Hopf algebras (free or not) are RG and its dual ([**20**], p. 21).

DEFINITION. If H is a Hopf R-algebra and S an R-algebra, S is an H-object if there is an algebra map α: $S \to S \otimes H$ such that $(\alpha \otimes 1)\alpha = (1 \otimes \Delta)\alpha$ and $(1 \otimes \varepsilon)\alpha = \mathrm{id}$. If H, S are finitely generated projective R-modules, S is a Galois H-object if γ: $S \otimes S \to S \otimes H$ by $\gamma(s \otimes t) = (s \otimes 1)\alpha(t)$ is an isomorphism.

If S is an H-object, H^* acts on S via a measuring, and [**4**, Theorem 9.3] S is a Galois H-object if and only if the map j: $S\#H^* \to \mathrm{End}_R(S)$, $j(s\#f)(t) = sf(t)$, is an isomorphism. Just as with classical Galois extensions, this isomorphism points the way to constructing non-trivial Azumaya R-algebras by altering the multiplication in $S\#H^*$ by an appropriate two-cocycle. The construction has been developed by Sweedler [**19**] and Yokogawa [**22**].

The trivial Galois H-object is H itself, with $\alpha = \Delta$.

If S is a Galois H-object, S has normal basis if S is isomorphic to H as an H^*-module. Kreimer [**15**] and Nakajima [**16**] have obtained a classification of Galois H-objects with normal basis, subsequently extended by Hurley [**14**] to classify such objects for any prime p, not just $p = 2$. For our purposes we need the following version.

Suppose 2 is not a zero divisor in R, and $H = H_a$, $ab = 2$. Denote by $U_c(R)$ the units of R which are congruent to 1 modulo cR.

PROPOSITION 1.3. *The set (actually group) of isomorphism classes (as R-algebras and H_a^*-modules) of Galois H_a-objects, $NB(R, H_a)$, is isomorphic to $U_{b^2}(R)/U_b(R)^2$.*

The relationship, given by Hurley, is as follows: Given u in $U_{b^2}(R)$, set $e = (u-1)/b^2$, and set

$$H_a(e) = R[t]/(t^2 - at - e).$$

The H_a-comodule structure of $H_a(e)$ is given as follows: if $H_a = R[x]$, then α: $H_a(e) \to H_a(e) \otimes H_a$ is given by

$$\alpha(t) = t \otimes 1 + 1 \otimes x - b(t \otimes x). \tag{1.4}$$

If $H_b = R[f]$ with $f^2 - bf = 0$, $\langle f, x \rangle = -1$ then f acts on $H_a(e)$ by

$$ft = -1 + bt, \qquad f1 = 0. \tag{1.5}$$

In particular, $\sigma = 1 - af$ acts on t by

$$\sigma(t) = (1 - af)t = t - a(-1 + bt) = a + t - 2t = a - t,$$

as it must if σ is to be an automorphism of $H_a(e)$.

As Kreimer and Hurley show, any Galois H_a-object with normal basis is isomorphic, via an R-algebra, H_b-module isomorphism, to $H_a(e)$ for some e with $1 + b^2e = u$ in $U_{b^2}(R)$.

2. Crossed products. As noted in the introduction, any Azumaya algebra over a ring of integers of a number field is split by a Galois object. In fact:

PROPOSITION 2.1. *Let R be the ring of integers of a number field K, and suppose* $\mathrm{Br}(R) \neq (0)$. *Then $S = R[i]$ is a non-trivial Galois RG-object, G cyclic of order* 2, *which splits* $\mathrm{Br}(R)$.

Proof. Since $\mathrm{Br}(R) \neq (0)$, R has a real embedding. Thus $z^2 + 1$ has no roots in R, and so $S = R[i] \cong R[z]/(z^2 + 1) \cong R[x]/(x^2 + 2x + 2)$ is a non-trivial Galois RG-object with normal basis.

Now the map from $\mathrm{Br}(R)$ to $\mathrm{Br}(K)$ is 1-1 by [**1**] and fits into the diagram of 1-1 maps (cf. [**17**], p. 78)

$$\begin{array}{ccc} \mathrm{Br}(R) & \to & \bigcup\limits_{v \text{ real}} \mathrm{Br}(K_v) \\ \downarrow & & \downarrow \\ \mathrm{Br}(K) & \to & \bigcup\limits_{v \text{ real}} \mathrm{Br}(K_v) \times \bigcup\limits_{p \text{ finite}} \mathrm{Br}(K_p) \end{array}$$

where K_v, K_p denote completions of K at the real infinite, resp. finite primes of K. Let $L = K[i]$, the quotient field of S. Then L is totally imaginary, so at every real prime v of K, the map $\mathrm{Br}(K_v) \to \mathrm{Br}(K_v \otimes L)$ is zero. Thus the composite map $\mathrm{Br}(R) \to \mathrm{Br}(K) \to \mathrm{Br}(L)$ in the commutative diagram

$$\begin{array}{ccc} \mathrm{Br}(R) & \to & \mathrm{Br}(K) \\ \downarrow & & \downarrow \\ \mathrm{Br}(S) & \to & \mathrm{Br}(L) \end{array}$$

is zero. But the map $\mathrm{Br}(S) \to \mathrm{Br}(L)$ is also 1-1 by ([**9**]). Thus $\mathrm{Br}(R) \to \mathrm{Br}(S)$ is zero, completing the proof.

This result motivates the inquiry, which Azumaya algebras, or which Brauer classes, can be represented as crossed products.

Recall that if S is an H-object where we denote the comodule action $\alpha: S \to S \otimes H$ by $\alpha(s) = \sum_{(s)} s_{(1)} \otimes s_{(2)}$, and T is an H^*-object, then the smash product of S and T, $S\#T$, is defined as $S \otimes_R T$ with multiplication defined as follows: for s, s' in S, t, t' in T,

$$(s\#t)(s'\#t') = \sum_{(s')(t)} ss'_{(1)}\langle t_{(2)}, s'_{(2)}\rangle \#t_{(1)}t',$$

(e.g. Chase **[5]** p. 163 or Gamst and Hoechsman **[12]**). Then **[12]** $S\#T$ is an Azumaya R-algebra.

Suppose S, T are Galois objects for H_a, H_b, respectively, where $ab = 2$, and suppose S, T have normal basis. Then $S = H_a(e)$ for some e with $1 + b^2e \in U_{b^2}(R)$, and $T = H_b(d)$, $1 + a^2d \in U_{a^2}(R)$. In that case, $S\#T = R\{z,w\}$ with $z^2 - az - e = 0$, $w^2 - bw - d = 0$, and $zw + wz = bz + aw - 1$. For we have $S = R[t]$, $t^2 - at - e = 0$, $T = R[t]$, $y^2 - by - d = 0$, and $z = t\#1$, $w = 1\#y$ satisfy the claimed relations.

A smash product of the form $H_a(e)\#H_b(d)$ will be called a normal basis smash product.

The rest of this section is devoted to constructing examples of smash products, particularly normal basis smash products.

In determining whether or not a given smash product is a non-trivial element of the Brauer group, the following result is useful.

PROPOSITION 2.2. *Let R be the ring of integers of a number field K. Let $A = H_a(e)\#H_b(d)$. Then $A \otimes K$ is the quaternion algebra*

$$(1 + b^2e, 1 + a^2d)_K = K\{x,y\}$$

with $x^2 = 1 + b^2e$, $y^2 = 1 + a^2d$, $xy + yx = 0$. Thus A represents a non-trivial class of $\mathrm{Br}(R)$ *if and only if $1 + b^2e$ and $1 + a^2d$ are in $U(R)$ and for some real embedding $\iota: R \to \mathbf{R}$, $\iota(1 + b^2e)$ and $\iota(1 + a^2d)$ are both < 0.*

Proof. We have $A = R\{z,w\}$ with $z^2 - az - e = 0$, $w^2 - bw - d = 0$, $zw + wz = bz + aw - 1$. Set $x = bz - 1$, $y = aw - 1$. Then since $ab = 2$, $x^2 = 1 + b^2e$, $y^2 = 1 + a^2d$, $xy + yx = 0$. Thus

$$(1 + b^2e, 1 + a^2d)_R \subseteq A$$

and the two algebras become equal over K. The remainder of the proof follows from the fact that A is non-trivial in $\mathrm{Br}(R)$ if and only if $A \otimes K$ is non-trivial in $\mathrm{Br}(K)$, if and only if $A \otimes K \otimes^{\iota} \mathbf{R}$ is non-trivial in $\mathrm{Br}(\mathbf{R})$ for some real place ι of K.

EXAMPLES 2.3. Let $2 = ab$ in R, and e, d in R such that $1 + b^2e = -u^2$, $1 + a^2d = -v^2$, u, v units of R. Then

$$A = H_a(e)\#H_b(d)$$

is a separable maximal order in the usual quaternion algebra $H(K) = (-1, -1)_K$.

If $A = R\{z, w\}$, $z^2 = az + e$, $w^2 = bw + d$, then A embeds in $H(K)$ by

$$z \to \frac{iu+1}{b}, \qquad w \to \frac{jv+1}{a}.$$

This class of examples is of particular interest because for real quadratic fields K, maximal orders in $H(K)$ represent the only non-trivial class in $\mathrm{Br}(\mathcal{O}_K)$ except when $K = Q(\sqrt{m})$, $m \equiv 1 \pmod 8$. But before considering such examples, we illustrate (2.3) with two examples of Swan [**18**] over a quartic extension of Q.

EXAMPLES 2.4 (Swan). Let $R = Z[t]$, $t = (2 + \sqrt{2})^{1/2}$, a root of $x^4 - 4x^2 + 2$, the ring of integers of $Q(t) = Q(\zeta_{16}) \cap \mathbf{R}$.

Then $t^2 = 2 + \sqrt{2} = \sqrt{2}(\sqrt{2} + 1)$, hence $\sqrt{2} = (\sqrt{2} - 1)t^2$ is in R.

1. Set $a = b = \sqrt{2}$, then

$$A_1 = H_{\sqrt{2}}(-1)\#H_{\sqrt{2}}(-1) = R\{z, w\},$$

an Azumaya R-algebra which embeds in $H(K)$, the usual quaternion algebra over the quotient field of R, via

$$z \mapsto (1 + i)/\sqrt{2}, \qquad w \mapsto (1 + j)/\sqrt{2}.$$

2. Set $a = t$, $b = 2/t$, $e = -1$, $d = -2/t^2$. Then $1 + b^2e = 1 - 4/t^2 = -3 + 2\sqrt{2} = -(\sqrt{2} - 1)^2$, $1 + a^2d = 1 + t^2(-2/t^2) = -1$, so set

$$A_2 = H_t(-1)\#H_{2/t}(-2/t^2) = R\{x, y\},$$

where $x^2 = tx - 1$, $y^2 = 2y/t - 2/t^2$, $xy + yx = 2x/t + ty - 1$. Then A_2 embeds in $H(K)$ by

$$x \mapsto \frac{i(\sqrt{2} - 1) + 1}{2/t}, \qquad y \mapsto \frac{j+1}{t}.$$

The generators of the image of A_2 in $H(K)$ are, up to a unit of R, the same as those given by Swan.

These algebras provided examples of the failure of cancelation of projective modules [**18**] and of the failure of the Skolem-Noether theorem for Azumaya algebras [**6**], [**7**].

Here is another set of examples based on Example 2.3.

PROPOSITION 2.5. *Let $m = p$ or $2p$, p an odd prime $\equiv 3 \pmod 4$, or $m = 2$, and let $K = Q(\sqrt{m})$. Then the non-trivial class in* $\mathrm{Br}(\mathcal{O}_K)$ *is represented by a normal basis smash product.*

Proof. Since K has odd class number [11] and 2 ramifies in K, we have $2\mathcal{O}_K = b^2\mathcal{O}_K$, and we may find a in $\mathcal{O}_K$ with $ab = 2$ and $a\mathcal{O}_K = b\mathcal{O}_K$. Then setting $e = -a/b$, $d = -b/a$, $u = v = 1$, we obtain the Azumaya algebra

$$A = H_a\left(\frac{-a}{b}\right)\#H_b\left(\frac{-b}{a}\right)$$

which is a maximal order in the quaternion algebra $H(K)$, and hence represents the non-trivial class in $\mathrm{Br}(\mathcal{O}_K)$.

The same example may be obtained over $\mathcal{O}_K$, $K = Q(\sqrt{m})$, for any $m \equiv 2$ or $3 \pmod 4$ such that $2\mathcal{O}_K$ is the square of a principal ideal. For m composite < 100 this is the case for $m = 51$ and 66.

REMARK 2.6. Let $R = \mathcal{O}_m$, the ring of integers of $K = Q(\sqrt{m})$, and suppose $2R = b^2R$. Let $G = \mathrm{Gal}(K/Q) = \langle \sigma \rangle$ where if $\alpha = r + s\sqrt{m}$, $\bar{\alpha} = \sigma(\alpha) = r - s\sqrt{m}$. Then the Azumaya R-algebra A of Proposition 2.5 is G-normal in the sense of [8]. For $\pm 2 = a\bar{a}$, so $b = \pm\bar{a}$, and we may present $A = H_a(-a/b)\#H_b(-b/a)$ $(ab = 2)$ inside $H(K)$ as

$$A = \left\langle 1, \frac{i+1}{\bar{a}}, \frac{j+1}{a}, \text{product}\right\rangle.$$

Then there is a group homomorphism of G into $\mathrm{Aut}(A)$ extending the action of G on R, defined by $\sigma(i) = j$, $\sigma(j) = i$. While G extends to a group of automorphisms of A, A is not lifted from Z. Thus we have a collection of examples which illustrates the necessity of the hypothesis in Corollary 5.2 of [8] that R/R^G be a Galois extension.

For quadratic fields $K = Q(\sqrt{m})$, we cannot obtain crossed products involving rank 2 Hopf algebras other than $\mathcal{O}_K G$ and $(\mathcal{O}_K G)^*$ if $m \equiv 5 \pmod 8$, for $2\mathcal{O}_K$ is prime. For $m \equiv 1 \pmod 8$, we have the following result:

PROPOSITION 2.7. *Let $K = Q(\sqrt{m})$, $m \equiv 1 \pmod 8$, and let $w = (1+\sqrt{m})/2$. Then the non-trivial class in* $\mathrm{Br}(\mathcal{O}_K)$ *is represented by a normal basis smash product under the following circumstances*:

Suppose the fundamental unit $\varepsilon = r + sw$, $r, s > 0$ in $\mathcal{O}_K$, satisfies

$\varepsilon \equiv 3 \pmod 4$. *Then*

$$H_2(-2)\#H_1((-\varepsilon - 1)/4)$$

represents the non-trivial class in $\mathrm{Br}(\mathcal{O}_K)$.

Suppose $2\mathcal{O}_K = P\bar{P}$, $P \neq \bar{P}$, $P = (b)$, *so* $\pm 2 = b\bar{b}$. *Then*

$$H_b((-\varepsilon - 1)b^2/4)\#H_{2/b}((-\bar{\varepsilon} - 1)/b^2)$$

represents the non-trivial class in $\mathrm{Br}(\mathcal{O}_K)$ *if* $\varepsilon \equiv 3 \pmod{\bar{b}^2}$ *and* $\varepsilon\bar{\varepsilon} = 1$.

Proof. The hypotheses on ε are needed to insure that $(-\varepsilon - 1)/4$, $(-\varepsilon - 1)b^2/4$ and $(-\bar{\varepsilon} - 1)/b^2$ are in $\mathcal{O}_K$, and to satisfy the hypotheses of Proposition 2.2.

Examples include $m = 33, 57$. For $m = 33$, $\varepsilon = 19 + 8w$, $b = 2 + w$, $b^2 = 12 + 5w$, and $w \equiv 0 \pmod{b^2}$. For $m = 57$, $\varepsilon = 131 + 40w$, $b = 3 + w$, $b^2 = 16 + 7w$, and $w \equiv 0 \pmod{b^2}$. In each case both kinds of smash products occur.

No other examples occur for $m \equiv 1 \pmod 8$, $m < 100$. In all other cases, $\varepsilon \not\equiv 3 \pmod 4$ so computing $U_4(R)/U_2(R)^2$ one finds that no Galois H_1-objects with normal basis of the form $H_1(e)$ with $1 + 4e < 0$ exist; also in all other cases $\varepsilon\bar{\varepsilon} = -1$, so no examples of the second kind exist.

TABLE I

	$H_1\left(\frac{u-1}{4}\right)\#H_2(v-1)$ standard		$H_b\left(\frac{(u-1)b^2}{4}\right)\#H_{2/b}\left(\frac{v-1}{b^2}\right)$ non-standard		
m	u	v	b	u	v
2			w	-1	-1
3	$-\varepsilon^2$	-1	$1+w$	-1	-1
5					
6			$2+w$	-1	-1
7	$-\varepsilon^2$	-1	$3+w$	-1	-1
10					
11	$-\varepsilon^2$	-1	$3+w$	-1	-1
13					
14	$-\varepsilon$	-1	$4+w$	-1	-1
15	$-\varepsilon^2$	-1			
17					
19	$-\varepsilon^2$	-1	$13+3w$	-1	-1
21	$-\varepsilon^3$	-1			
22			$14+3w$	-1	-1
23	$-\varepsilon^2$	-1	$5+w$	-1	-1
26					
29					
30					
31	$-\varepsilon^2$	-1	$39+7w$	-1	-1

TABLE I (continued)

m	standard u	standard v	non-standard b	non-standard u	non-standard v
33	$-\varepsilon$	-1	$2+w$	$-19-8w$	$-19+8w$
34			$6+w$	-1	-1
35	$-\varepsilon^2$	-1			
37					
38			$6+w$	-1	-1
39					
41					
42					
43	$-\varepsilon^2$	-1	$59+9w$	-1	-1
46	$-\varepsilon$	-1	$156+23w$	-1	-1
47	$-\varepsilon^2$	-1	$7+w$	-1	-1
51	$-\varepsilon^2$	-1	$7+w$	-1	-1
53					
55					
57	$-\varepsilon$	-1	$3+w$	$-(131+40w)$	$-(131-40w)$
58					
59	$-\varepsilon^2$	-1	$23+3w$	-1	-1
61					
62	$-\varepsilon$	-1	$8+w$	-1	-1
65					
66			$8+w$	-1	-1
67	$-\varepsilon^2$	-1	$221+27w$	-1	-1
69	$-\varepsilon^2$	-1			
70					
71	$-\varepsilon^2$	-1	$59+7w$	-1	-1
73					
74					
77	$-\varepsilon^3$	-1			
78					
79	$-\varepsilon^2$	-1	$9+w$	-1	-1
82					
83	$-\varepsilon^2$	-1	$9+w$	-1	-1
85					
86			$102+11w$	-1	-1
87	$-\varepsilon^2$	-1			
89					
91	$-\varepsilon^2$	-1			
93	$-\varepsilon^3$	-1			
94	$-\varepsilon$	-1	$1464+151w$	-1	-1
95	$-\varepsilon$	-1			
97					

Table I describes all possible normal basis crossed products which represent the non-trivial class of $\mathrm{Br}(\mathcal{O}_K)$ for $K = Q(\sqrt{m})$, $m < 100$. That no others occur than are shown follows from explicit computations of the groups of Galois H-objects with normal basis (Proposition 1.3) using the table of fundamental units in [2], and application of Proposition 2.2.

3. Representing Brauer classes. Table I illustrates that normal basis smash products can be useful for describing Azumaya algebras and representing Brauer classes, but often such smash products may not be available. Here we explore this further.

(3.1) There are Brauer classes which are not representable as normal basis smash products but are representable as crossed products.

Our example is $K = Q(\sqrt{39})$, $\mathcal{O}_K = Z[\sqrt{39}\,]$. The ideal $2\mathcal{O}_K$ is the square of a non-principal ideal of $\mathcal{O}_K$, so the only free rank 2 Hopf algebras over $\mathcal{O}_K$ are $\mathcal{O}_K G$ and its dual, G cyclic of order 2. Thus if a normal basis smash product exists over $\mathcal{O}_K$ for the non-identity class in $\mathrm{Br}(\mathcal{O}_K)$, it is of the form $H_1(e)\#H_2(d)$ where $1 + 4e$ and $1 + d$ are negative units of $\mathcal{O}_K$, by Proposition 2.2. Now the fundamental unit of $\mathcal{O}_K$ is $25 + 4\sqrt{39}$, which has norm 1 and is congruent to 1 $(\mathrm{mod}\, 4\mathcal{O}_K)$, hence every negative unit of $\mathcal{O}_K$ is congruent to -1 $(\mathrm{mod}\, 4\mathcal{O}_K)$. Thus $1 + 4e$ cannot be a negative unit of $\mathcal{O}_K$, and no Galois $(RG)^*$-objects with normal basis exist.

Thus the non-trivial class in $\mathrm{Br}(\mathcal{O}_K)$ is not representable as a normal basis smash product. However, Fossum's example of **[10]** for $m = 39$ is a classical crossed product:

$$D = D(\mathcal{O}_L, G)_f$$

where $L = K(i)$, $\mathcal{O}_L = \mathcal{O}_K[u]$, $u = (\sqrt{m} + i)/2$, and $f\colon G \times G \to U(\mathcal{O}_L)$ is the normalized 2-cocycle with $f(\sigma, \sigma) = -1$ for $\mathrm{Gal}(L/K) = G = \langle \sigma \rangle$. Thus the non-trivial class in $\mathrm{Br}(\mathcal{O}_K)$ is representable as a crossed product.

For $K = Q(\sqrt{m})$, $m \equiv 3\ (\mathrm{mod}\, 4)$, the extension $L = K(i)$ always has ring of integers $\mathcal{O}_L = \mathcal{O}_K[u]$, $u = (\sqrt{m} + i)/2$, and $\mathcal{O}_L$ is a Galois extension of $\mathcal{O}_K$ with group $G = \mathrm{Gal}(L/K)$. Whenever the fundamental unit ε of $\mathcal{O}_K$ satisfies $\varepsilon \equiv \sqrt{m}\ (\mathrm{mod}\, 2\mathcal{O}_K)$ then $\mathcal{O}_L$ has a normal basis over $\mathcal{O}_K$ consisting of $(\varepsilon + i)/2$ and $(\varepsilon - i)/2$, as is easily seen. It is known that $\varepsilon \equiv \sqrt{m}\ (\mathrm{mod}\, 2\mathcal{O}_K)$ whenever m is prime **[21]**.

(3.2) There are Brauer classes not representable as rank 4 crossed products.

The first gap in Table I is for $m = 5$. We show

PROPOSITION 3.3. *The non-trivial class in* $\mathrm{Br}(\mathcal{O}_5)$ *is not a crossed product for any Galois object with respect to a Hopf algebra of rank* 2.

Proof. Recall **[19]** that if H is a Hopf R-algebra, S an H^*-object, and $f\colon H \otimes H \to S$ a normalized 2-cocycle, the crossed product $S\#_f H$ is defined to be $S \otimes_R H$ with multiplication given by

$$(s\#g)(t\#h) = \sum_{(g)(h)} s(g_{(1)}t)f(g_{(2)}, h_{(1)})\#g_{(3)}h_{(2)}$$

for s, t in S, $g, h \in H$. Suppose $H = H_b = R[x]$, $x^2 = bx$, $S = H_a(e)$ is a Galois H_a-object with normal basis $(ab = 2)$ and f: $H_b \otimes H_b \to S$. is defined by $f(1,1) = 1$, $f(1,x) = f(x,1) = 0$, $f(x,x) = d$ with $1 + a^2d \in U(R)$, then f is a cocycle and one sees easily that $H_a(e)\#_f H_b \cong H_a(e)\#H_b(d)$: the cocycle f in the smash product has the effect of altering the ring structure of H_b.

The symmetric character of $H_a(e)\#H_b(d)$ means that any crossed product $S\#_f H_b$ where S is a Galois H_a-object with normal basis is isomorphic to a crossed product $T\#_g H_a$ where T is a Galois H_b-object with normal basis $(ab = 2)$.

Now let $R = \mathcal{O}_5$. We first limit the posssible Hopf R-algebras.

Since R has class number 1, and $2R$ is a prime ideal of R, the only rank 2 Hopf algebras over R are RG and its dual ([**20**], Corollary to Theorem 3, page 21). Thus we need only look for Galois objects for RG and $(RG)^*$.

First suppose $H = H_2 = RG$. Then any Galois H-object S has the form $S = S_1 \oplus S_\sigma$, where $S_1 = R$ and S_σ is a rank one projective R-module. Since R has class number 1, S_σ is free, and $S = R + Rz$, $z^2 = a$, a unit of R. But then

$$S \cong H_2(a-1) = R[t]$$

with $t^2 - 2t - (a-1) = 0$, via the map sending t to $z + 1$. So S has normal basis. But then any crossed product $S\#_f H_2$ involving S is isomorphic to a classical crossed product $T\#H_1$ involving a Galois extension T of R with Galois group G and with normal basis.

Suppose then, that $H = H_1 = (RG)^*$. Then we are seeking a Galois extension T of R with group G of order 2. However, any non-trivial such T would have a field of quotients L which would be contained in K^+, the maximal abelian extension of $K = Q(\sqrt{5})$ which is unramified except at the infinite primes. But since no prime congruent to 3 (mod 4) divides the discriminant of K over Q, K^+ is the Hilbert class field of K ([**13**], p. VII-7, Remark), which is K itself since K has class number 1. That completes the proof.

Fossum [**10**] has an explicit representative for the non-trivial class of $\mathrm{Br}(R)$, namely, the free R-submodule of the usual quaternions over K generated by 1, $u = (1 + w + i + jw)/2$, j and uj, where $w = (1 + \sqrt{5})/2$.

If $S = R[j]$, it follows from Proposition 2.1 that $\mathrm{Br}(R) = \mathrm{Br}(S/R)$. Now Yokogawa ([**22**], A-7) has extended the Chase-Harrison-Rosenberg seven term cohomology sequence to Galois objects of Hopf algebras, to yield

$$\cdots \to H^2(H_1, S/R, U) \xrightarrow{\gamma} \mathrm{Br}(S/R) \to H^1(H_1, S/R, \mathrm{Pic}) \to \cdots$$

where γ is the crossed product map. Proposition 3.3 shows that when $R = \mathcal{O}_5$, $S = R[j]$, then $\gamma = 0$. Thus the non-trivial class of $\mathrm{Br}(S/R)$, represented by Fossum's algebra, gives rise to a non-trivial element of $H^1(H_1, S/R, \mathrm{Pic})$, a subquotient of $\mathrm{Pic}(SG)$.

References

[1] M. Auslander and O. Goldman, *The Brauer group of a commutative ring*, Trans. Amer. Math. Soc., **97** (1960), 367–409.

[2] Z. Borevich and I. Shafarevich, *Number Theory*, Academic Press, 1966.

[3] S. U. Chase, D. K. Harrison and A. Rosenberg, *Galois theory and Galois cohomology of commutative rings*, Memoirs Amer. Math. Soc., **52** (1965), 15–33.

[4] S. U. Chase and M. E. Sweedler, *Hopf Algebras and Galois Theory* Springer Lecture Notes in Mathematics 97 (1968).

[5] S. U. Chase, *On a Variant of the Witt and Brauer Groups*, Springer Lecture Notes in Math., **549** (1976), 148–187.

[6] L. N. Childs and F. R. DeMeyer, *On automorphisms of separable algebras*, Pacific J. Math., **23** (1967), 25–34.

[7] L. N. Childs, *On projective modules and automorphisms of central separable algebras*, Canad. J. Math., **21** (1969), 44–53.

[8] ______, *On normal Azumaya algebras and the Teichmuller cocycle map*, J. Algebra, **23** (1972), 1–17.

[9] ______, *Mayer-Vietoris sequences and Brauer groups of non-normal domains*, Trans. Amer. Math. Soc., **196** (1974), 51–67.

[10] R. M. Fossum, *The Noetherian Different of Projective Orders*, thesis, University of Michigan, 1965.

[11] A. Frohlich, *Central extensions, Galois groups and ideal class groups of number fields*, Amer. Math. Soc. Contemp. Math., **24** (1983).

[12] J. Gamst and K. Hoechsmann, *Quaternions generalises*, C. R. Acad. Sci. Paris, **269** (1969), 560–562.

[13] C. Herz, *Construction of Class Fields*, in *Seminar on Complex Multiplication*, Springer Lecture Notes in Math. **24** (1966).

[14] S. Hurley, *Galois objects with normal bases for free Hopf algebras of prime degree*, J. Algebra, (to appear).

[15] H. F. Kreimer, *Quadratic Hopf algebras and Galois extensions*, Amer. Math. Soc. Contemp. Math., **13** (1981), 353–361.

[16] A. Nakajima, *Free algebras and Galois objects of rank* 2, Math. J. Okayama Univ., **23** (1981), 181–187.

[17] M. Orzech and C. Small, *The Brauer Group of Commutative Rings*, New York, Marcel Dekker, 1975.

[18] R. G. Swan, *Projective modules over group rings and maximal orders*, Annals of Math., **76** (1962), 55–61.
[19] M. E. Sweedler, *Cohomology of algebras over Hopf algebras*, Trans. Amer. Math. Soc., **133** (1968), 205–239.
[20] J. T. Tate and F. Oort, *Group schemes of prime order*, Ann. Scient. Ec. Norm. Sup., (4e serie) **3** (1970), 1–21.
[21] K. S. Williams *The class number of* $Q(\sqrt{-p})$ *modulo* 4, *for* $p \equiv 3 \pmod 4$ *a prime*, Pacific J. Math., **83** (1979), 565–570.
[22] K. Yokogawa, *On* $S \otimes S$*-module structure of* S/R*-Azumaya algebras*, Osaka J. Math., **12** (1975), 673–690.

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