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TIME CHANGES FOR $\mathbf{R}^n$ FLOWS AND SUSPENSIONS

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We examine properties of flows on compact metric spaces under multi-parameter real actions (i.e., the acting group is $\mathbf{R}^n$), and the impact of changes of action (time changes) and conjugacies to other flows. These concepts are carefully studied in the case of $\mathbf{R}^n$-suspensions and a basic framework for the internal structure of suspensions is developed. The structure of minimal $\mathbf{R}^n$ flows which are conjugate to an equicontinuous flow is examined using suspensions, and these flows are shown to be either weak-mixing or equicontinuous under suitable assumptions. Basic properties for the cohomology of time changes are also developed, with the major result indicating when time changes on equicontinuous minimal flows are cohomologous to constant time changes, and the structure of those time changes which themselves yield equicontinuous flows.

1. Introduction. A *flow* $(X, T, \cdot)$ is a compact metric space X acted upon by a topological group T. The action of $t \in T$ on $x \in X$ is denoted by $x \cdot t$. In this paper we are mainly concerned with the properties of, and relationships between flows which are conjugate. Time changes and conjugacies have been investigated for real flows; our object is to develop an appropriate theory for flows where the acting group is $\mathbf{R}^n$. We will concentrate on those areas where the development of the theory for $\mathbf{R}^n$ is different to that for the reals $\mathbf{R}$ and merely mention situations where there is a straightforward extension.

Section 2 is concerned with developing a framework for the study of $\mathbf{R}^n$ suspensions and time changes of them. Section 3 studies those flows which are time changes of equicontinuous flows (so called harmonizable flows [5]). This also motivates the investigation of uniformly continuous time changes which turn out to preserve equicontinuous type conditions. Section 4 is concerned with cohomology of $\mathbf{R}^n$ cocycles and relations with suspensions and time changes.

Two flows $(X, T, \cdot)$ and $(Y, T, *)$ are said to be *conjugate* if there exists a homeomorphism h from X onto Y which takes orbits onto orbits i.e. $h(x \cdot T) = h(x) * T$. For convenience we will always take $Y = X$ and we are only concerned with the case $T = \mathbf{R}^n$ for some $n \geq 1$. We will also assume throughout that both actions $\cdot$ and $*$ are free. It is clear from the definition that we can find functions θ and ϕ from $X \times \mathbf{R}^n$ to $\mathbf{R}^n$ which are related to the conjugacy map h by $h(x \cdot \theta(x,t)) = h(x) * t$ and $h(x \cdot t) = h(x) * \phi(x,t)$. Furthermore θ and ϕ satisfy (i) $\theta(x, t+s) = \theta(x,t) + \theta(x \cdot \theta(x,t), s)$ (the time change equation), (ii) $\phi(x, t+s) = \phi(x,t) + \phi(x \cdot t, s)$ (the cocycle equation), (iii) $\phi(x, \theta(x,t)) = \theta(x, \phi(x,t)) = t$. Condition (iii) can be interpreted as θ and ϕ being "inverses" of one another in the sense that for each $x \in X$, $\phi(x, \cdot)$ and $\theta(x, \cdot)$ are inverses of each other. Thus $\phi(x, \cdot)$ and $\theta(x, \cdot)$ are one-to-one and onto $\mathbf{R}^n$.

1.1. PROPOSITION. *Let θ and ϕ be the maps defined above. For each $x \in X$, $\theta(x, \cdot)$ and $\phi(x, \cdot)$ are homeomorphisms from $\mathbf{R}^n$ onto itself. Furthermore, θ and ϕ are continuous as functions from $X \times \mathbf{R}^n$ onto $\mathbf{R}^n$.*

Proof. Let $\varepsilon > 0$ and let D_ε be the open ball of radius ε about the origin in $\mathbf{R}^n$. Choose $x \in X$ and a local section S for $(X, \mathbf{R}^n, *)$ with $h(x) \in S^*$, the relative interior of S. Now, for $\delta > 0$ sufficiently small, $S^* * D_\delta$ is open and we can find an open neighborhood U of x with $h(U) \subset S^* * D_\delta$. Thus we can find $m > 0$ such that $x \cdot \overline{D}_m \subset U$ and so $h(x \cdot \overline{D}_m) \subset S^* * D_\delta$. Now $h(x \cdot \overline{D}_m)$ is a compact connected set and the local section property guarantees that the projection of this set onto S is compact, connected and countable, hence $h(x)$. Thus $h(x \cdot \overline{D}_m) \subset h(x) * D_\delta$ and so $\phi(x, \overline{D}_m) \subset D_\delta$ which proves that $\phi(x, \cdot)$ is continuous. Applying the argument to h^{-1} yields the continuity of $\theta(x, \cdot)$.

To prove joint continuity, fix $x_0 \in X$ and $\varepsilon > 0$. If S_δ is the surface of the ball D_δ, then $h(x_0 \cdot D_\varepsilon)$ does not intersect $h(x_0 \cdot S_{2\varepsilon})$ so choose disjoint open sets U_0 and U_1 of these sets respectively. We can choose a neighborhood V of x_0 such that $h(V \cdot D_\varepsilon) \subset U_0$ and $h(V) * \phi(x_0, S_{2\varepsilon}) \subset U_1$. Choose $K > 0$ such that $\sup\{\|\phi(x,s)\|;\ s \in S_{2\varepsilon}\} \leq K$. Suppose we can find $x \in V$ and $t \in D_\varepsilon$ such that $\|\phi(x,t)\| > K$. Since $\phi(x_0, \cdot)$ is a homeomorphism and $\phi(x_0, 0) = 0$, $\phi(x_0, S_{2\varepsilon})$ is a closed surface in $\mathbf{R}^n$ containing the origin. Since $\phi(x, 0) = 0$, $\phi(x, D_\varepsilon)$ must intersect this surface. Let $t \in D_\varepsilon$ and $s \in S_{2\varepsilon}$ be such that $\phi(x,t) = \phi(x_0, s)$. Now $h(x \cdot t) \in U_0$ but $h(x \cdot t) = h(x) * \phi(x,t) = h(x) * \phi(x_0, s) \in U_1$ which is a contradicton. Thus $\phi(x,t)$ is bounded on $V \times D_\varepsilon$ and we deduce that ϕ is continuous on $X \times D_\delta$ for some $\delta > 0$.

Next note that $\phi(\cdot, t)$ is continuous for any fixed t. For we can write $t = NT$ where $T \in D_\delta$ and observe that

$$\phi(x,t) = \phi(x, NT) = \sum_{i=0}^{N-1} \phi(x \cdot (iT), T)$$

which is a sum of continuous functions in x.

Finally suppose $x_n \to x$ and $t_n \to t$. Then $\phi(x_n, t_n) = \phi(x_n, t_n - t) + \phi(x_n \cdot (t_n - t), t) \to 0 + \phi(x, t)$.

These remarks motivate the following definition.

1.2. DEFINITION. Let $(X, \mathbf{R}^n, \cdot)$ be a flow. (i) A *time change* of the flow is a continuous map θ: $X \times \mathbf{R}^n \to \mathbf{R}^n$ such that $\theta(x, t + s) = \theta(x, t) + \theta(x \cdot \theta(x, t), s)$ ($x \in X$; $t, s \in \mathbf{R}^n$) and $\theta(x, \cdot)$ is a homeomorphism from $\mathbf{R}^n$ onto itself for each $x \in X$. A time change θ gives a new action $*$ of $\mathbf{R}^n$ on X defined by $x * t = x \cdot \theta(x, t)$ called the *time changed action* and the resulting flow will be denoted by $(X, \mathbf{R}^n_\theta, *)$ or just $(X, \mathbf{R}^n_\theta)$. (ii) A *cocycle* is a continuous map ϕ: $X \times \mathbf{R}^n \to \mathbf{R}^n$ such that $\phi(x, t + s) = \phi(x, t) + \phi(x \cdot t, s)$ ($x \in X$; $t, s \in \mathbf{R}^n$) and $\phi(x, \cdot)$ is a homeomorphism from $\mathbf{R}^n$ onto itself for each $x \in X$. (Note that this invertibility requirement is not always imposed—see, for example, [7]. It is convenient in what follows to distinguish between a *cocycle* and a function from $X \times \mathbf{R}^n$ into $\mathbf{R}^n$ which *satisfies the cocycle equation*.)

1.3. REMARK. In the case of conjugate flows, the map θ determined by the conjugacy h is the time change required on the original flow to make h an isomorphism. The cocycle ϕ produced by h gives a new action on $(X, \mathbf{R}^n, *)$ via $\phi(h^{-1}(x), t)$ which makes it isomorphic to $(X, \mathbf{R}^n, \cdot)$.

The following proposition indicates that the duality between the time change and cocycle of a conjugacy holds in general. By the *inverse* of a time change or a cocycle ρ we mean $\rho(x, t)^{-1} = \rho_x^{-1}(t)$, where $\rho_x(\cdot) = \rho(x, \cdot)$ is the homeomorphism from $\mathbf{R}^n$ to $\mathbf{R}^n$.

1.4. PROPOSITION. *The inverse of a time change is a cocycle and the inverse of a cocycle is a time change.*

Proof. We show that the inverse of a cocycle is a time change. The converse is similar. Let ϕ on $X \times \mathbf{R}^n$ be a cocycle on the flow $(X, \mathbf{R}^n, \cdot)$ and define the inverse map $\theta(x, t) = \phi_x^{-1}(t)$ where ϕ_x is the homeomorphism $\phi(x, \cdot)$ from $\mathbf{R}^n$ onto $\mathbf{R}^n$. Note that $\theta(x, \phi(x, t)) = \phi(x, \theta(x, t)) = t$. Thus

$$\begin{aligned}\phi(x, \theta(x, t + s)) = t + s &= \phi(x, \theta(x, t)) \\ &\quad + \phi(x \cdot \theta(x, t), \theta(x \cdot \theta(x, t), s)) \\ &= \phi(x, \theta(x, t) + \theta(x \cdot \theta(x, t), s))\end{aligned}$$

so that $\theta(x, t + s) = \theta(x, t) + \theta(x \cdot \theta(x, t), s)$. So we only need to show that θ is continuous.

Suppose $x_n \to x$ and $t_n \to t$. Obviously if $\theta(x_n, t_n)$ converges, it must converge to $\theta(x, t)$. So suppose $\|\theta(x_n, t_n)\| \to \infty$. Define $u_n = \phi(x_n, \theta(x, t))$. Then $u_n \to t$ and $\theta(x_n, u_n) = \theta(x, t)$. Let $\varepsilon_n = t_n - u_n$ and choose $0 \leq a \leq 1$. Now $\phi(x_n, \theta(x_n, u_n + a\varepsilon_n)) = u_n + a\varepsilon_n$ which runs from u_n to t_n as a runs from 0 to 1. Thus as $\theta(x_n, \cdot)$ is a homeomorphism (for x_n fixed), $\{\theta(x_n, u_n + a\varepsilon_n); 0 \leq a \leq 1\}$ is a connected curve running from $\theta(x_n, u_n)$ to $\theta(x_n, t_n)$. Now for each n large enough choose a_n such that $\|\theta(x_n, u_n + a_n\varepsilon_n)\| = \|\theta(x, t)\| + 1$. We can obviously do this since $\theta(x_n, u_n) = \theta(x, t)$ and $\|\theta(x_n, t_n)\| \to \infty$. Choosing a subsequence, assume that $\theta(x_n, u_n + a_n\varepsilon_n) \to T$ say. Then

$$u_n + a_n\varepsilon_n = \phi(x_n, \theta(x_n, u_n + a_n\varepsilon_n)) \to \phi(x, T)$$

but $u_n \to t$ so $\varepsilon_n \to 0$ and thus $u_n + a_n\varepsilon_n \to t$. Thus $T = \theta(x, t)$, but $\|T\| = \|\theta(x, t)\| + 1$ which is a contradiction.

2. $\mathbf{R}^n$ suspensions. R. Ellis introduced the necessary ideas for suspensions with general acting groups in [**6**]. We sketch the procedure for $\mathbf{R}^n$ flows. Consider a transformation group $(K, \mathbf{Z}^n, \cdot)$ with K compact metric and ϕ an $\mathbf{R}^n$ valued function on $K \times \mathbf{Z}^n$ which satisfies the cocycle equation. Define actions of $\mathbf{Z}^n$ and $\mathbf{R}^n$ on $K \times \mathbf{R}^n$ by

$$m(k, t)s = (k \cdot m, t + s - \phi(k, m)) \qquad (m \in \mathbf{Z}^n;\ t, s \in \mathbf{R}^n;\ k \in K).$$

These actions yield a bitransformation group $(\mathbf{Z}^n, K \times \mathbf{R}^n, \mathbf{R}^n)$ and $(K \times \mathbf{R}^n/\mathbf{Z}^n, \mathbf{R}^n)$ becomes an $\mathbf{R}^n$ transformation group by identifying $\mathbf{Z}^n$ orbits of the bitransformation group. In an algebraic sense such flows "are" suspensions, but one wants the phase space to be compact metric and one intuitively feels that suspension flows should have nice behaviour over fundamental domains. This requires additional assumptions on ϕ which allow it to be extended to a cocycle on the identity suspension i.e. the suspension generated by the map id: $K \times \mathbf{Z}^n \to \mathbf{R}^n$ defined by $\mathrm{id}(k, m) = m$ $(k \in K;\ m \in \mathbf{Z}^n)$. We will denote this suspensison by $(K_{\mathrm{id}}, \mathbf{R}^n)$ and refer to it as the *constant one suspension*. Its elements will be denoted (k, u) where $k \in K$ and $u \in [0, 1)^n$.

We say that a continuous map ϕ: $K \times \mathbf{Z}^n \to \mathbf{R}^n$ is a *cocycle for a suspension* on $(K, \mathbf{Z}^n)$ if

(i) $\phi(k, p + q) = \phi(k, p) + \phi(k \cdot p, q)$ $(p, q \in \mathbf{Z}^n;\ k \in K)$

(ii) ϕ can be extended to a continuous map Φ: $K \times \mathbf{R}^n \to \mathbf{R}^n$ (not necessarily a cocycle) such that for every $k \in K$, (a) $\Phi(k, \cdot)$ is a homeomorphism from $\mathbf{R}^n$ onto itself, and (b) $\Phi(k, m + v) = \Phi(k, m) + \Phi(k \cdot m, v)$ $(m \in \mathbf{Z}^n;\ v \in [0, 1)^n)$.

We will show that the above conditions are not as restrictive as they appear. (See also [**8**].)

2.1. REMARK. Essentially (i) and (ii) require that ϕ can be extended to a cocycle $\bar{\phi}$ from $K_{\mathrm{id}} \times \mathbf{R}^n$ onto $\mathbf{R}^n$. To see this define $\bar{\phi}((k,u),t) = \Phi(k,t+u) - \Phi(k,u)$. We will denote the resulting transformation group by $(K_\phi, \mathbf{R}^n)$. In this case K_ϕ is a compact metric space. For if π: $K \times \mathbf{R}^n \to K_\phi$ is the canonical projection then, by (ii) (b), $\pi(K \times B) = K_\phi$ where $B = \Phi(K \times [0,1]^n)$ which is a compact set in $\mathbf{R}^n$. Thus we only need to check that K_ϕ is Hausdorff. This follows if for any sequence $\{k_i\}$ in K, $\phi(k_i, m_i) \to \infty$ as $\|m_i\| \to \infty$. But otherwise we can find sequences $\{k_i\}$ in K and $\{m_i\}$ in $\mathbf{Z}^n$ with $\phi(k_i, m_i) \to v, k_i \to k$, and then if θ is the inverse time change of $\bar{\phi}$,

$$m_i = \theta\big((k_i,0), \bar{\phi}((k_i,0),m_i)\big) = \theta((k_i,0), \phi(k_i,m_i)) \to \theta((k,0),v)$$

which is obviously false.

It will be convenient to use the notation (k,u) for points of K_ϕ ($k \in K$; $u \in [0,1)^n$) and to regard the action over each base point as given by Φ i.e. $(k,u) = (k,0) \cdot \Phi(k,u)$.

2.2. PROPOSITION. *Let $(K_\phi, \mathbf{R}^n)$ be a suspension. If θ is the inverse time change to the cocycle $\bar{\phi}$ of the previous lemma, then $(K_\phi, \mathbf{R}^n, \cdot)$ is isomorphic to $(K_{\mathrm{id}}, \mathbf{R}^n_\theta, *)$. Hence every $\mathbf{R}^n$ suspension is a time change of the constant one suspension. Moreover, any time change of $(K_\phi, \mathbf{R}^n)$ is a suspension over $(K, \mathbf{Z}^n)$.*

Proof. It is clear that the map π from $(K_\phi, \mathbf{R}^n)$ to $(K_{\mathrm{id}}, \mathbf{R}^n_\theta)$ defined by $\pi(k,u) = (k,u)$ (where (k,u) represents the appropriate point in each suspension) is a homeomorphism. Let $t \in \mathbf{R}^n$. $\pi((k,u) \cdot t) = (k \cdot m, v)$ where $\Phi(k,u) + t = \phi(k,m) + \Phi(k \cdot m, v) = \Phi(k, m+v)$, $m \in \mathbf{Z}^n$ and $v \in [0,1)^n$. So $t = \bar{\phi}((k,u), m+v-u)$ and thus $\theta((k,u),t) = m + v - u$. Thus $\pi(k,u) * t = (k,u) \cdot \theta((k,u),t) = (k \cdot m, v)$. So π is an isomorphism.

Now a time change of any suspension must be a time change of the constant one suspension over the same base. We only need to show that a time change θ of the constant one suspension on $(K, \mathbf{Z}^n)$ is a suspension on $(K, \mathbf{Z}^n)$. Let ϕ be the inverse cocycle to θ. Then $\phi|_{K \times \mathbf{Z}^n}$ is clearly a cocycle for a suspension and, by the above, $(K_\phi, \mathbf{R}^n)$ is isomorphic to $(K_{\mathrm{id}}, \mathbf{R}^n_\theta)$.

We have thus proved

2.3. THEOREM. *An $\mathbf{R}^n$ flow is a suspension on a base $(K, \mathbf{Z}^n)$ if and only if it is a time change of the constant one suspension on $(K, \mathbf{Z}^n)$.*

2.4. PROPOSITION. *Let ϕ be a continuous map from $K \times \mathbf{Z}^n$ to $\mathbf{R}^n$ such that $\phi(k, p+q) = \phi(k,p) + \phi(k \cdot p, q)$ $(p, q \in Z^n;\ k \in K)$. Then ϕ is a cocycle for a suspension on $(K, \mathbf{Z}^n)$ if and only if*

(i) *for each $k \in K$, $\phi(k,m) = 0$ implies $m = 0$ $(m \in \mathbf{Z}^n)$.*

(ii) *any bounded region in $\mathbf{R}^n$ contains $\mathbf{Z}^n$-bounded points of $\{\phi(k,m);\ m \in \mathbf{Z}^n\}$, uniformly in k,*

(iii) *for each $k \in K$, if we choose a triangulation Θ of the set $\{\phi(k,m);\ m \in \mathbf{Z}^n\}$ such that for each $m \in \mathbf{Z}^n$ any two points in $\{\phi(k, m+f);\ f = \sum_{i=1}^n \alpha_i e_i$ where α_i is either 0 or $1\}$ are connected by an edge, and if $t \in \mathbf{R}^n$, then t belongs to some member of Θ.*

Proof. Assume that we have a function ϕ satisfying (i), (ii) and (iii). Fix $k \in K$ and consider $\{\phi(k,m);\ m \in \mathbf{Z}^n\}$. The cocycle equation and (i) ensure that all these points are distinct and (ii) and (iii) ensure that they form a grid like set. We will inductively construct a collection of solids which only overlap along appropriate common faces and which cover $\mathbf{R}^n$. For $m \in \mathbf{Z}^n$, let $\|m\| = \max_{1 \le i \le n} |m_i|$.

Suppose that for each $m \in \mathbf{Z}^n$ with $\|m\| \le N-1$ we have constructed a solid whose (not necessary piecewise linear) boundary includes the points $\{\phi(k, m+f);\ f = \sum_{i=1}^n \alpha_i e_i$ where each α_i is either 0 or $1\}$. Furthermore these solids overlap only along appropriate common faces and their union is simply connected and contains no other points of $\phi(k, \mathbf{Z}^n)$, i.e. no points other than $\{\phi(k,m): |m_i| \le N-1$ or $m_i = N\}$. In other words we have constructed a "tiling" of $\mathbf{R}^n$ with each "tile" based at $\phi(k,m)$ for $\|m\| \le N-1$. We can obviously do this for $N = 1$. However, provided we construct them in the correct order, the solids to be added for $\|m\| = N$ each have at least one face in common with an existing solid. Thus these can be constructed to satisfy the required conditions by taking care to include no points of $\phi(k, \mathbf{Z}^n)$ except the appropriate boundary points as indicated above (using (ii)) and so that no "holes" are left between a newly added solid and the existing ones. Using (iii), this procedure can be done in such a way that the solids will eventually cover $\mathbf{R}^n$. The continuity of ϕ and uniformity in (ii) enables us to do this process for each different orbit so that the tiling depends continuously on K.

We now select a collection of homeomorphisms $\{A_l;\ l \in K\}$ such that A_l maps $[0,1)^n$ onto the "half open" solid with "vertices" $\{\phi(l,f);\ f = \sum_{i=1}^n \alpha_i e_i$ and each α_i is either 0 or $1\}$ in such a way that $A_l(0) = 0$, the maps A_l agree on common faces, and so that $\{A_l; l \in K\}$ depends continuously on K. We now define $\Phi(k,t) = \phi(k,m) + A_{k\cdot m}(u)$ where $t \in R^n$, $k \in K$, $m \in \mathbf{Z}^n$, $u \in [0,1)^n$ and $t = m + u$. It is clear that Φ satisfies the required conditions and one can show continuity from the construction.

Conversely, suppose that ϕ is a cocycle for a suspension. Since $\Phi(k,\cdot)$ is a homeomorphism from $\mathbf{R}^n$ onto $\mathbf{R}^n$ which extends ϕ, we immediately have (i) and (ii) (see 2.1). Consider $\{\phi(k,m); m \in \mathbf{Z}^n\}$ and choose a triangulation Θ of the right type. It is easy to see that the union of these solids, P say, is a simply connected set in $\mathbf{R}^n$. Suppose $P \neq \mathbf{R}^n$ and choose a component C in $\Phi^{-1}(k, \mathscr{C}P)$. Now if $\mathscr{F}$ is the set of planar faces of the simplices in Θ, then $C \cap \bigcup_{Q\in\mathscr{F}} \Phi^{-1}(k,Q) = \varnothing$. If C is unbounded, then one can argue from the required form of Θ to a contradiction of the continuity of $\Phi(k,\cdot)$. But if C is bounded $\Phi(k,C)$ is bounded and separated from the rest of $\mathscr{C}P$ which contradicts P being simply connected. Thus $P = \mathbf{R}^n$ and so (iii) holds.

2.5. REMARK. The three conditions in Proposition 2.4 are natural, ensuring that ϕ is not degenerate and that its range is a "grid" in $\mathbf{R}^n$. (i) and (ii) imply that $\phi(k,\cdot)$ is a homeomorphism from $\mathbf{Z}^n$ onto $\phi(k,\mathbf{Z}^n)$. Thus 2.4 justifies the definition of a cocycle for a suspension.

In real suspensions we have linear behaviour in each fundamental domain. If we wish to maintain this for $\mathbf{R}^n$ suspensions, we would need to ensure that the homeomorphisms A_k can be chosen to be piecewise linear. Essentially this requires the vertices of each solid to be the vertices of a non-degenerate polytope. In this case we can choose a suitable triangulation of this solid and define A_k as a linear map over each triangle from the corresponding triangulation of $[0,1]^n$. However, this apparently requires more stringent conditions on the original function ϕ.

An approach to characterising suspensions is in terms of continuous functions from the phase space to the circle group K. In the case of real actions, a flow is a constant suspension exactly when we can find an eigenfunction f for it (since then $f^{-1}(1)$ is a global section). For nonconstant suspensions the situation is a little more complex. P. D. Humphries noted in [7] that if f is any continuous function from X to K then there is a unique function ϕ_f: $X \times \mathbf{R} \to \mathbf{R}$ satisfying the cocycle equation such that $f(x\cdot t) = \exp(2\pi i \phi_f(x,t)) f(x)$. We then have

2.6. LEMMA. $(X, \mathbf{R})$ *is a suspension if and only if there exists a function* f: $X \to K$ *such that* $\phi_f(x,t) \neq 0$ *for* $t \neq 0$. (*Such functions* f *will be called pseudo-eigenfunctions and* ϕ_f *a pseudo-eigenvalue.*)

Proof. If the condition holds then for each x, $\phi_f(x,\cdot)$ is either increasing or decreasing and one can argue that it is in fact a homeomorphism. Thus ϕ_f is a cocycle and under the action given by the inverse time change f becomes an eigenfunction. Thus $(X, \mathbf{R})$ is a suspension.

Conversely if we have a suspension $(X_g, \mathbf{R})$ over (X, σ) say, then the function $f(x,u) = \exp(2\pi iu/g(x))$ $(0 \le u < g(x))$ is a pseudo-eigenfunction since $\phi_f((x,u),t) = v/g(\sigma^m x) - u/g(x) + m$ where v and m satisfy $u + t = \sum_{i=0}^{m-1} g(\sigma^i x) + v$, $0 \le v < g(\sigma^m x)$, and $m \in \mathbf{Z}$.

We now extend this approach to $\mathbf{R}^n$ actions. Let $(X, \mathbf{R}^n)$ be a flow and $f\colon X \to K^n$ continuous with an induced function $\phi_f\colon X \times \mathbf{R}^n \to \mathbf{R}^n$ satisfying the cocycle equation as in the real case. ϕ_f exists by considering the components of f and interpreting $f(x \cdot t) = \exp(2\pi i\phi_f(x,t)) \cdot f(x)$ as $f_j(x \cdot t) = \exp(2\pi i(\phi_{f_j})(x,t))f_j(x)$, and using the contractibility of $\mathbf{R}^n$. (See [7].) As in the real case we call f a *pseudo-eigenfunction* if $\phi_f(x,t) \neq 0$ for $t \neq 0$.

2.7. THEOREM. *$(X, \mathbf{R}^n)$ is a suspension if and only if it has a pseudo-eigenfunction $f\colon X \to \mathbf{R}^n$.*

Proof. Suppose we have a pseudo-eigenfunction f. Fix $x \in X$. Observe that the condition $\phi_f(x,t) \neq 0$ for $t \neq 0$ immediately implies that $\phi_f(x,\cdot)$ is one-to-one. We will prove that $\phi_f(x,\cdot)$ is actually a homeomorphism onto $\mathbf{R}^n$. Then ϕ_f is a cocycle and if θ is its inverse time change, we have $f(x \cdot \theta(x,t)) = \exp(2\pi it)f(x)$ which makes f an eigenfunction (into K^n) for $(X, \mathbf{R}^n_\theta)$. This implies that $(X, \mathbf{R}^n_\theta)$ is a constant suspension (see Theorem 3.3) and the result follows by Theorem 2.3. Let $S_\varepsilon = \{t \in \mathbf{R}^n;\ \|t\| = \varepsilon\}$ and $B_\varepsilon = \{t \in \mathbf{R}^n;\ \|t\| \le \varepsilon\}$. Since $\phi_f(x,\cdot)$ is one-to-one, $\phi_f(x, S_\varepsilon)$ is a closed surface with the origin in its interior and $\phi_f(x, S_\varepsilon)$ shrinks to the origin continuously as $\varepsilon \to 0$. So given $\varepsilon > 0$, for each $y \in X$ we can find $m_y > 0$ with $B_{m_y} \subset \phi_f(y, B_\varepsilon)$. The compactness of X guarantees that $m = \inf_{y \in X} m_y > 0$ so that $B_m \subset \phi_f(y, B_\varepsilon)$ for every $y \in X$. Now let $t \in \mathbf{R}^n$ and write $t = lT$ where $T \in B_m$ and $l \in \mathbf{Z}^+$. Inductively construct $s_i \in B_\varepsilon$ $(0 \le i \le l-1)$ such that

$$\phi_f\left(x \cdot \left(\sum_{i=0}^{k-1} s_i\right), s_k\right) = T \quad (0 \le k \le l-1).$$

Now $\phi_f(x, \sum_{i=0}^{l-1} s_i) = t$ and so $\phi_f(x,\cdot)$ is onto $\mathbf{R}^n$. Also if $t, s \in \mathbf{R}^n$ with $s - t = v$ say and $v \in B_m$, then if $\phi_f(x,t') = t$ we can find $u \in B_\varepsilon$ with $\phi_f(x, t' + u) = s$. Thus $\|\phi_f^{-1}(x,t) - \phi_f^{-1}(x,s)\| = \|u\| \le \varepsilon$. We deduce that $\phi_f^{-1}(x,\cdot)$ is continuous and so $\phi_f(x,\cdot)$ is a homeomorphism.

Conversely if $(X, \mathbf{R}^n)$ is a suspension, then by Theorem 2.3 it is a time change of the constant suspension and any eigenfunction into K^n for the constant suspension becomes a pseudo-eigenfunction for $(X, \mathbf{R}^n)$.

3. Harmonizable flows and uniformly continuous time changes. A flow $(X, \mathbf{R}^n, \cdot)$ is called *harmonizable* if it can be time changed to an equicontinuous flow. J. Egawa **[5]** used a study of uniformly continuous time changes to show that every minimal harmonizable real flow is either equicontinuous or weak mixing. I. U. Bronstein **[1]** investigated the real case by using strong connectedness assumptions on the phase space. We generalise to $\mathbf{R}^n$ flows by analysing harmonizable flows in the context of suspensions. To illustrate this approach we will first give the suspension proof for the real case. Note that we use the notation $(X_f, \mathbf{R})$ for the real suspension of the discrete flow (X, σ) under a continuous suspending function f: $X \to (0, \infty)$ rather than the cumbersome $(X_\phi, \mathbf{R}, \cdot)$ where ϕ is the induced cocycle $\phi(x, m) = \sum_{i=0}^{m-1} f(\sigma^i x)$.

3.1. LEMMA. *If a suspension $(X_f, \mathbf{R})$ is equicontinuous then the discrete base flow (X, σ) is equicontinuous.*

Proof. Let d be a metric on X_f and regard X as embedded in X_f. Given $\delta > 0$ and $\eta > 0$ (sufficiently small and satifying $\eta < \frac{1}{2}\inf_{x \in X} f(x)$) there exists $\varepsilon > 0$ such that for any $x \in X$ the ball of radius ε about x in X_f is contained in the neighbourhood $\{y \in X; d(x, y) < \delta\} \cdot (-\eta, \eta)$. Let $\xi > 0$ be such that if $d(x, y) < \xi$ then $d(x \cdot t, y \cdot t) < \varepsilon$ for all $t \in \mathbf{R}$ and $x, y \in X_f$. Let $x, y \in X$ be such that $d(x, y) < \xi$ and let $t_n \in \mathbf{R}$ be such that $x \cdot t_n = \sigma^n x$. Now since $d(x \cdot t_n, y \cdot t_n) < \varepsilon$, $d(y \cdot (t_n + \eta_n), \sigma^n x) < \delta$ where $y \cdot (t_n + \eta_n) \in X$ and $|\eta_n| < \eta$. It remains to show that $y \cdot (t_n + \eta_n) = \sigma^n y$ for each $n \in \mathbf{Z}$. Since $t_0 + \eta_0 = 0$ (obviously), let n be the first integer (say positive) such that $y \cdot (t_n + \eta_n) \neq \sigma^n y$. Then we can find s_n with $t_{n-1} + \eta_{n-1} < s_n < t_n + \eta_n$ such that $y \cdot s_n = \sigma^n y$ and, as above, $x \cdot (s_n + \eta') \in X$ for some $|\eta'| < \eta$. But since $\eta' < \frac{1}{2}\inf_{x \in X} f(x)$, $s_n = t_n - \eta'$ or $s_n = t_{n-1} - \eta'$ which is a contradiction.

3.2. THEOREM (*J. Egawa* **[5]**, *I. U. Bronstein* **[1]**). *Let $(X, \mathbf{R})$ be a minimal harmonizable real flow. Then $(X, \mathbf{R})$ is either equicontinuous or weak mixing.*

Proof. If $(X, \mathbf{R})$ is not weak mixing then it is isomorphic to a constant suspension. For if g is an eigenfunction from X onto the unit circle, then $Y = g^{-1}(1)$ is a global section for X (i.e. a local section, intersected by every orbit, which swells under a small open time interval to an open set). The first return map σ to Y makes $(X, \mathbf{R})$ isomorphic to a suspension built over the discrete flow (Y, σ) by the constant suspending function 1. Since $(X, \mathbf{R})$ can be time changed to an equicontinuous flow, $(Y_f, \mathbf{R})$ is equicontinuous for some suspending function f. Thus (Y, σ) is equicontinuous by Lemma 3.1 and so $(X, \mathbf{R})$ (which is isomorphic to $(Y_1, \mathbf{R})$) is also equicontinuous.

We next obtain an $\mathbf{R}^n$ version of this theorem. The ideas of this section are closely related to the analysis of D. F. De Riggi and N. G. Markley in [2] and [3]. We use K^n to denote the n torus.

3.3. THEOREM. *Let $(X, \mathbf{R}^n)$ be a free minimal flow with a locally free maximal equicontinuous factor. Then $(X, \mathbf{R}^n)$ is a constant suspension.*

Proof. Let the maximal equicontinuous factor be $(Z, \mathbf{R}^n)$ and let h be a homomorphism from $(X, \mathbf{R}^n)$ onto $(Z, \mathbf{R}^n)$. Since $(Z, \mathbf{R}^n, \cdot)$ is minimal and equicontinuous we can find a continuous group homomorphism θ: $\mathbf{R}^n \to Z$ such that $z \cdot t = z\theta(t)$. Let $\chi_1, \dots, \chi_n$ be non-trivial independent characters on Z and define H: $X \to K^n$ by $H(x)_j = \chi_j(h(x))$. Also let $T_1, \dots, T_n$ be linear maps from $\mathbf{R}^n$ to $\mathbf{R}$ such that $e^{2\pi i T_j(t)} = \chi_j(\theta(t))$. Then $H(x \cdot t)_j = e^{2\pi i T_j(t)} H(x)_j$. If we choose a basis $t_1, \dots, t_n$ for $\mathbf{R}^n$ then we can write $\{T_j\}$ as a matrix Υ and $H(x \cdot t)_j = e^{2\pi i (\Upsilon t)_j} H(x)_j$. In fact we can choose the basis so that Υ has the form $T_{ij} = 0$ for $j > i$ and $T_{ij} = T_{ii}$ for $j \leq i$. For initially choose t_1 such that $\chi_1\theta(t_1) \neq 1$, then $t_2 \in \ker(T_1)$ with $\chi_2\theta(t_2) \neq 1$, $t_3 \in \ker(T_1) \cap \ker(T_2)$ with $\chi_3\theta(t_3) \neq 1$ and so on. This gives Υ a lower triangular form with non zero diagonal entries. Now, starting with t_n, replace each t_k in turn by $t_k + \alpha_{k+1}t_{k+1} + \cdots + \alpha_n t_n$. Clearly the new vector t_k still satisfies the required conditions for the triangular form of the matrix and we choose $\alpha_{k+1}, \dots, \alpha_n$ to satisfy

$$\begin{pmatrix} T_{k+1,k+1} & & & & \\ \cdots & \cdots & & 0 & \\ T_{n-1,k+1} & T_{n-1,k+2} & \cdots & T_{n-1,n-1} & \\ T_{n,k+1}L & T_{n,k+2} & \cdots & T_{n,n-1} & T_{n,n} \end{pmatrix} \begin{pmatrix} \alpha_{k+1} \\ \vdots \\ \alpha_{n-1} \\ \alpha_n \end{pmatrix}$$

$$= \begin{pmatrix} T_{k+1,k+1} & - & T_{k+1,k} \\ \vdots & & \\ T_{n-1,n-1} & - & T_{n-1,k} \\ T_{n,n} & - & T_{n,k} \end{pmatrix}$$

This gives the required form to the matrix Υ.

Now $\Gamma = H^{-1}(1, \ldots, 1)$ is clearly a global section and it is the base for a constant suspension, for if $xt \in \Gamma$ where $x \in \Gamma$ then $\Upsilon t \in Z^n$ and so each $t_i \in (1/\Upsilon_{ii})Z$ which gives the appropriate Z^n action on Γ. Thus $(X, \mathbf{R}^n)$ is isomorphic to $(\Gamma_k, \mathbf{R}^n)$ for constant suspending function

$$k = \left(\frac{1}{T_{11}}, \frac{1}{T_{22}}, \ldots, \frac{1}{T_{nn}}\right).$$

3.4. REMARK. In the terminology of [2], the condition that we have a locally free maximal equicontinuous factor is equivalent to the flow $(X, \mathbf{R}^n)$ having *zero shear*. Note that we can partially extend Theorem 3.3 in the sense that if we have a factor free in p directions—say $(K^p, \mathbf{R}^p)$—then we have an $\mathbf{R}^p$ suspension over a base flow $(X/K^p, \mathbf{R}^{n-p})$ (cf. Theorem 2.1 of [3]). We simply obtain a homomorphism $f\colon X \to K^p$ and use $f^{-1}(1)$ with an appropriate $p \times p$ submatrix in the above proof.

3.5. LEMMA. *Let ϕ be a cocycle for a suspension on $(Y, \mathbf{Z}^n)$. If $(Y_\phi, \mathbf{R}^n)$ is equicontinuous, then $(Y, \mathbf{Z}^n)$ is equicontinuous.*

Proof. Using the notation of Lemma 3.1, we note, as in that proof, that $\|\phi(x, m) - \phi(y, m)\| < 2\eta$ and so, in $(Y, \mathbf{Z}^n)$, $d(x \cdot m, y \cdot m) < \delta$ for all $m \in \mathbf{Z}^n$ which gives the result.

Note that for minimal real flows, any non-trivial equicontinuous factor is locally free. Thus the following theorem generalizes Theorem 3.3.

3.6. THEOREM. *Let $(X, \mathbf{R}^n)$ be a harmonizable minimal flow with locally free maximal equicontinuous factor (i.e. $(X, \mathbf{R}^n)$ has zero shear). Then $(X, \mathbf{R}^n)$ is equicontinuous.*

Proof. By Theorem 3.3 $(X, \mathbf{R}^n)$ is a constant suspension, $(Y_k, \mathbf{R}^n)$, on $(Y, \mathbf{Z}^n)$ say. Thus $(Y_k, \mathbf{R}^n)$ can be time changed to an equicontinuous flow which is also a suspension on $(Y, \mathbf{Z}^n)$ by Theorem 2.3. The result follows by Lemma 3.5.

3.7. REMARK. The following example shows that the requirement that the maximal equicontinuous factor is locally free is necessary for $\mathbf{R}^n$ harmonizable flows.

Let $(K^2, \mathbf{R})$ be a minimal equicontinuous flow on the 2-torus. Let (K, σ) be a minimal circle rotation and choose a suspending function f such that $(K_f, \mathbf{R})$ is weakly mixing. Now $(K^2 \times K_f, \mathbf{R}^2)$ is neither weakly mixing $((K^2, \mathbf{R}^2)$ with a trivial second real action is an equicontinuous factor) nor equicontinuous (it has proximal points). However it is harmonizable via the time change $\theta((x, y, z), (t_1, t_2)) = (t_1, \theta_1(z, t_2))$ where θ_1 is the time change from $(K_f, \mathbf{R})$ to $(K_1, \mathbf{R})$.

It is well known that time changes destroy equicontinuity type properties in general. However, in [5] the good behaviour of *uniformly continuous* real time changes was exploited. Below we investigate this property for $\mathbf{R}^n$ flows.

3.8. LEMMA. *Suppose θ is a time change and ϕ is its inverse cocycle. Then θ is uniformly continuous if and only if ϕ is uniformly continuous.*

Proof. Suppose, for example, that θ is uniformly continuous. Then $\|\phi(x, t) - \phi(y, t)\| = \|\phi(y\theta(y, u), \varepsilon)\|$ where we write $t = \theta(x, u)$ and $\varepsilon = \theta(x, u) - \theta(y, u)$, and the result follows immediately.

3.9. THEOREM. *Let $(X, \mathbf{R}^n, \cdot)$ and $(Y, \mathbf{R}^n, *)$ be conjugate flows via a conjugacy h with associated time change θ.*

(1) *If θ is uniformly continuous, then $(X, \mathbf{R}^n, \cdot)$ is equicontinuous (distal) if and only if $(Y, \mathbf{R}^n, *)$ is equicontinuous (distal).*

(2) *If both flows are minimal and equicontinuous, then θ is uniformly continuous.*

Proof. (1) Suppose $(X, \mathbf{R}^n, \cdot)$ is equicontinuous and let $\varepsilon > 0$ be given. Choose $\delta_1 > 0$ such that $d_X(x, y) < \delta_1$ implies $d_Y(h(x), h(y)) < \varepsilon$ for $x, y \in X$; $\delta_2 > 0$ and $\eta > 0$ such that $\|s\| < \eta$ and $d_X(x, y) < \delta_2$ implies $d_X(x, y \cdot s) < \delta_1$ for $x, y \in X$ and $s \in \mathbf{R}^n$; and $\delta_3 < \delta_2$ such that $d_X(x, y) < \delta_3$ implies $\|\theta(x, t) - \theta(y, t)\| < \eta$ for all $t \in \mathbf{R}^n$ and $x, y \in X$. Choose $\delta_4 < \delta_3$ such that $d_X(x, y) < \delta_4$ implies $d_X(x \cdot t, y \cdot t) < \delta_3$ for all $t \in \mathbf{R}^n$ and $x, y \in X$, and finally choose $\delta > 0$ such that $d_Y(h(x), h(y)) < \delta$ implies $d_X(x, y) < \delta_4$ for all $x, y \in X$.

Now suppose $x, y \in X$ with $d_X(x, y) < \delta_4$. Then for all $t \in \mathbf{R}^n$, $d_X(x \cdot \theta(x, t),\ y \cdot \theta(x, t)) < \delta_3$ and $\|\theta(x, t) - \theta(y, t)\| < \eta$ so $d_X(x \cdot \theta(x, t),\ y \cdot \theta(y, t)) < \delta_1$ and $d_Y(h(x) * t, h(y) * t) < \varepsilon$. Thus if $x, y \in Y$ and $d_Y(x, y) < \delta$ then $d_Y(x * t, y * t) < \varepsilon$ for all $t \in \mathbf{R}^n$. The converse holds by Lemma 3.8.

Next suppose $(X, \mathbf{R}^n, \cdot)$ is distal and let $x, y \in X$ be such that $h(x)$ and $h(y)$ are proximal in $(Y, \mathbf{R}^n, *)$. So we can find $\{t_n\} \subset \mathbf{R}^n$ with $d_X(x \cdot \theta(x, t_n), y \cdot \theta(y, t_n)) \to 0$. Let $\theta(x, t_n) - \theta(y, t_n) = \varepsilon_n$ and assume $\varepsilon_n \to \varepsilon$. ($\varepsilon_n$ is bounded.) Now clearly $d_X(x \cdot \theta(x, t_n), y \cdot (-\varepsilon) \cdot \theta(x, t_n))$ $\to 0$ so $x \cdot \varepsilon = y$. But now $(Y, \mathbf{R}^n, *)$ has a pair of proximal points on the same orbit which contradicts the assumption that it is free. The converse is by Lemma 3.8 again.

(2) The proof of this (partial) converse to (1) for $\mathbf{R}^n$ flows follows exactly the proof for real flows (Theorem 1 of [5]).

3.10. REMARK. In fact, it is not necessary for h to be one-to-one in the above theorem (provided, of course, that θ still exists with the right properties). This observation allows us to deduce a proof of Theorem 3.6 along the lines used in [5]. For suppose $(X, \mathbf{R}^n)$ is minimal equicontinuous and conjugate via H (with corresponding time change θ) to a flow $(Y, \mathbf{R}^n)$ with zero shear i.e. whose maximal equicontinuous factor, $(E, \mathbf{R}^n)$ say, is locally free. If π is the projection from $(Y, \mathbf{R}^n)$ onto $(E, \mathbf{R}^n)$ then $\pi \circ H$ satisfies the conditions described above and so θ is uniformly continuous by Theorem 3.9(2). But then $(Y, \mathbf{R}^n)$ is equicontinuous by Theorem 3.9(1).

4. Cohomology of cocycles and isomorphism. In the study of flows obtained from a given flow by time changes, a natural question is to find necessary and sufficient conditions on two time changes which will ensure that the resulting flows are isomorphic. Two cocycles ϕ_1 and ϕ_2 on a flow $(X, \mathbf{R}^n, \cdot)$ are *cohomologous* (written $\phi_1 \sim \phi_2$) if there exists a continuous map g from X to $\mathbf{R}^n$ (which we call a *transfer* function) such that $\phi_1(x, t) = g(x \cdot t) - g(x) + \phi_2(x, t)$. It is obvious that $\sim$ defines an equivalence relation on the class of cocycles on $(X, \mathbf{R}^n, \cdot)$. Two time changes are said to be cohomologous if their inverse cocycles are cohomologous. The following lemmas are straightforward and their proofs are omitted.

4.1. LEMMA. *Suppose that θ_1 and θ_2 are cohomologous time changes on a flow $(X, \mathbf{R}^n, \cdot)$. Then $(X, \mathbf{R}^n_{\theta_1}, \cdot)$ and $(X, \mathbf{R}^n_{\theta_2}, *)$ are isomorphic.*

4.2. COROLLARY. *Let ϕ_1 and ϕ_2 be cocycles for suspensions on $(X, \mathbf{Z}^n)$. If $\phi_1 \sim \phi_2$ as $\mathbf{Z}^n$-cocycles then the suspensions $(X_{\phi_1}, \mathbf{R}^n)$ and $(X_{\phi_2}, \mathbf{R}^n)$ are isomorphic.*

4.3. REMARK. $\phi_1 \sim \phi_2$ can be replaced in 4.2 by ϕ_1 *weakly cohomologous* to ϕ_2 i.e. $\phi_1(x, m) = g(x \cdot m) - g(m) + \phi_2(\alpha x, m)$ $(x \in X; m \in \mathbf{Z}^n)$ where α is an automorphism of $(X, \mathbf{Z}^n)$.

4.4. LEMMA. *Let θ be a time change on a flow $(X, \mathbf{R}^n)$. Then θ is cohomologous to a constant time change (i.e. At for some appropriate invertible constant $n \times n$ matrix A) if and only if we can find a continuous function g: $X \to \mathbf{R}^n$ such that $\theta(x, t) = g(x \cdot \theta(x, t)) - g(x) + At$.*

4.5. PROPOSITION. *Let* $(X, \mathbf{R}^n)$ *be a minimal flow and* ϕ_1 *and* ϕ_2 *be cocycles on the flow. Then* ϕ_1 *is cohomologous to* ϕ_2 *if and only if* $\phi_1 - \phi_2$ *is uniformly bounded on* $X \times \mathbf{R}^n$.

Proof. Only sufficiency is non-trivial so suppose $\phi_1 - \phi_2$ is uniformly bounded. Form the space $X \times \mathbf{R}^n$ and define actions R_t and L_u on it by $R_t(x, s) = (x \cdot t, s + \phi_1(x, t) - \phi_2(x, t))$ and $L_u(x, s) = (x, s + u)$. It is clear that these are commuting $\mathbf{R}^n$ actions. Now, by the assumption, $R_{\mathbf{R}^n}(x, s)$ is bounded in $X \times \mathbf{R}^n$ so choose a minimal set M in the closure of this orbit. Let π_1 be the projection of M onto X (since X is minimal). Now $\pi_1^{-1}(x)$ is one point, $(x, f(x))$ say. For if (x, a) and $(x, a + \delta) \in M$, then $M = L_\delta(M)$ which contradicts the compactness of M. Thus π_1 is actually an isomorphism and so f is continuous. Also

$$\begin{aligned}(x \cdot t, f(x, t)) &= \pi^{-1}(x \cdot t) = R_t(\pi^{-1}(x)) \\ &= (x \cdot t, f(x) + \phi_1(x, t) - \phi_2(x, t))\end{aligned}$$

so that f is the required transfer function.

The previous two results give

4.6. COROLLARY. *Let* $(X_1, \mathbf{R}^n)$ *be a minimal flow and* θ *a time change of it. Then* θ *is cohomologous to a constant time change* At *if and only if* $\theta - At$ *is uniformly bounded on* $X \times \mathbf{R}^n$.

If ϕ_1 and ϕ_2 are cocycles on $X \times \mathbf{R}^n$ then they are also cocycles on $X \times \mathbf{Z}^n$ in the sense that they satisfy the cocycle equation. Prima facie cohomology of ϕ_1 and ϕ_2 depends on this interpretation but we can now deduce that the difference is illusory if $(X, \mathbf{R}^n)$ is minimal. This also gives an alternate proof of Corollary 4.2 if $(X, \mathbf{Z}^n)$ is minimal.

4.7. COROLLARY. *Let* $(X, \mathbf{R}^n)$ *be minimal. Then if* ϕ_1 *and* ϕ_2 *are cocycles on* $X \times \mathbf{R}^n$, $\phi_1 \sim \phi_2$ *if and only if* $\phi_1 \sim \phi_2$ *as cocycles on* $X \times \mathbf{Z}^n$.

Proof.

$$\begin{aligned}\|\phi_1(x, t) - \phi_2(x, t)\| \leq &\|\phi_1(x, m) - \phi_2(x, m)\| \\ &+ \|\phi_1(x \cdot m, u) - \phi_2(x \cdot m, u)\|\end{aligned}$$

where $t \in \mathbf{R}^n$, $m \in \mathbf{Z}^n$, $u \in [0, 1)^n$ and $t = m + u$. Thus $\phi_1 - \phi_2$ is uniformly bounded if $\phi_1 \sim \phi_2$ as cocycles on $X \times \mathbf{Z}^n$.

For any cocycles ϕ_1 and ϕ_2, $\phi_1 - \phi_2$ uniformly bounded implies that $(\|\phi_1 - \phi_2\|/\|\phi_2\|)(t) \to 0$ as $\|t\| \to \infty$ since $\|\phi_2(x, t)\| \to \infty$. The next lemma shows that these conditions are equivalent for a class of cocycles which includes the uniformly continuous ones.

4.8. LEMMA. *Let $(X, \mathbf{R}^n)$ be a flow and let ϕ_1 and ϕ_2 be cocycles on the flow such that $\|\phi_i(x,t) - \phi_i(y,t)\|$ is bounded independent of t for $i = 1$ or 2. If $(\|\phi_1 - \phi_2\|/\|\phi_2\|)(t) \to 0$ as $\|t\| \to \infty$ then $\phi_1 - \phi_2$ is uniformly bounded on $X \times \mathbf{R}^n$.*

Proof. For any function ϕ on $X \times \mathbf{R}^n$ satisfying the cocycle equation we have

$$\phi(x, mt) = m\phi(x,t) + \sum_{k=1}^{m-1} \{\phi(x \cdot t, kt) - \phi(x, kt)\} \qquad (x \in X; t \in \mathbf{R}^n; m \in \mathbf{Z}).$$

Now applying this formula to $\phi_3 = \phi_1 - \phi_2$ and choosing K_1, K_2 and K_3 such that

$$\sup_{t \in \mathbf{R}^n} \|\phi_i(x,t) - \phi_i(y,t)\| \le K_i \qquad (i = 1, 2, \text{ or } 3)$$

we get

$$\frac{\|\phi_3(x, mt)\|}{\|\phi_2(x, mt)\|} \ge \frac{m\|\phi_3(x,t)\| - (m-1)K_3}{m\|\phi_2(x,t)\| + (m-1)K_2} \ge \frac{\|\phi_3(x,t)\| - K_3}{\|\phi_2(x,t)\| + K_2}.$$

Since, for any $t \neq 0$, the left hand side of this inequality tends to zero as $m \to \infty$ we must have $\|\phi_3(x,t)\| \le K_3$ for all $x \in X$ and $t \in \mathbf{R}^n$.

Note that in the proof it is adequate to have $(\|\phi_1 - \phi_2\|/\|\phi_2\|)(t) \to 0$ if $|t_i| \to \infty$ for every $i = 1, \dots, n$ since then we deduce that $\|\phi_1 - \phi_2\|$ is uniformly bounded over all t with all non-zero components and so over all t by continuity.

As an application of these results we will deduce a theorem for minimal $\mathbf{R}^n$ flows established by J. Egawa in [4] for real flows.

4.9. THEOREM. *Let $(X, \mathbf{R}^n)$ be an equicontinuous minimal flow and suppose θ is a uniformly continuous time change on X. Then θ is cohomologous to a constant time change.*

Proof. We use the notation $v_j(t)$ for the vector which has $t \in \mathbf{R}$ as its jth component and all other entries zero. Let ϕ be the inverse cocycle of θ and define $F_t^{ij}(x) = \phi_i(x, v_j(t))/t$. $\{F_t^{ij}; t \ge 1\}$ is equicontinuous because ϕ is uniformly continuous.

$$\phi_i(x, v_j(t)) = \sum_{k=0}^{[t]-1} \phi_i(x \cdot v_j(k), v_j(1)) + \phi_i(x \cdot v_j([t]), v_j(t - [t]))$$

so that

$$\|F_t^{ij}\| \leq \frac{|[t]|+1}{|t|} \sup_{\substack{x\in X\\ t\leq 1}} |\phi_i(x, v_j(t))|$$

and thus $\{F_t^{ij}; t \geq 1\}$ is uniformly bounded. Also,

$$F_m^{ij}(x) = \frac{1}{m}\sum_{k=0}^{m-1} \phi_i(x \cdot v_j(k), v_j(1))$$

and since the action is equicontinuous $\lim F_m^{ij}(x)$ exists as $m \to \infty$. It is easy to deduce that $\lim F_t^{ij}(x)$ exists as $t \to \infty$. Hence $F_t^{ij}(x)$ converges uniformly to a continuous real valued function a_{ij} on X. Noting that X is connected one can show that each a_{ij} is actually a constant. Let A be the matrix (a_{ij}). Use the norm $\|a\| = \sum_{i=1}^n |a_i|$ on $\mathbf{R}^n$. Then

$$\frac{\|\phi(x,t) - At\|}{\|At\|}$$

$$\leq \|A^{-1}\| \sum_{j=1}^{n}\sum_{i=1}^{n} \left|\phi_i(x \cdot (t_1,\ldots,t_{j-1},0,\ldots,0), v_j(t_j)) - \alpha_{ij}t_j\right| / |t_j|$$

for some appropriate matrix norm. Thus $\|\phi(x,t) - At\|/\|At\| \to 0$ as every $|t_j| \to \infty$. Noting the comment at the end of Lemma 4.8 and applying Proposition 4.5 gives the result.

4.10. COROLLARY. *Let $(X, \mathbf{R}^n)$ be an equicontinuous minimal flow and let θ be a time change of it. Then $(X, \mathbf{R}^n_\theta)$ is equicontinuous if and only if $(X, \mathbf{R}^n_\theta)$ is isomorphic to a constant time change of $(X, \mathbf{R}^n)$.*

Proof. Use Theorem 3.9(2), Theorem 4. 9 and Lemma 4.1.

We also clearly have

4.11. COROLLARY. *Let ϕ be a cocycle for a suspension on a minimal equicontinuous flow $(X, \mathbf{Z}^n)$. Then $(X_\phi, \mathbf{R}^n)$ is equicontinuous if and only if ϕ is cohomologous to a constant cocycle i.e. if and only if $(X_\phi, \mathbf{R}^n)$ is isomorphic to a contant suspension on $(X, \mathbf{Z}^n)$.*

4.12. REMARK. Corollary 4.11 has some rather surprising consequences for particular examples. Consider the 2-torus obtained by suspending a circle with minimal rotation by a constant function. Let Γ be any global section (so in general Γ may be a curve of degree (m, n) for any $0 \leq m, n < \infty$). Let σ: $\Gamma \to \Gamma$ be the first return map on Γ and let t: $\Gamma \to R$ be the first return time on Γ. Thus the original flow is realised as a suspension $(\Gamma_t, \mathbf{R})$. Now $(\Gamma_1, \mathbf{R})$ is equicontinuous by Lemma 3.1. Thus t is cohomologous to a constant.

Note that Corollary 4.11 classifies equicontinuous time changes of an equicontinuous flow as constant time changes. However, even in the real case, it is an open question in general as to which pairs of these constant time changes give isomorphic flows i.e. what conditions on the matrices will ensure isomorphism of the corresponding flows.

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