

GENERALIZED ROTATIONAL HYPERSURFACES OF CONSTANT MEAN CURVATURE IN THE EUCLIDEAN SPACES, II

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Among various basic local differential geometric invariants of a given hypersurface in the euclidean $(n + 1)$ -space, $M^n \subset E^{n+1}$, the *mean curvature*, i.e. the trace of the second fundamental form, is certainly one of the simplest numerical invariants with important geometric meaning, namely, the first variation of “area”. Therefore, *complete hypersurfaces of constant mean curvatures* in E^{n+1} naturally constitute a nice family of simple global geometric objects that certainly deserve special attention. Especially, those *closed* ones can be considered as natural generalizations of “*soap bubbles*” and the problem of such generalized soap bubbles in the euclidean spaces has been attracting the attention of differential geometers since Euler and Monge.

This paper is the second part of a systematic study of hypersurfaces of constant mean curvatures in E^{n+1} which are of *generalized rotational types*, succeeding a previous paper with the same title.

Historically, the study of rotational surface of constant mean curvature in E^3 can be traced back to the work of Delaunay in 1841 [8], in which he discovered a beautiful way of constructing the *generating curves* of such rotational surfaces, namely, the trace of a focus by rolling a given conic section on the axis.

In the paper preceding this one [I], it was proposed to generalize the study of rotational surfaces in the setting of *invariant hypersurfaces* of a given isometric transformation group (G, E^{n+1}) with *two-dimensional orbit space*, namely, *hypersurfaces of generalized rotational types*. Such isometric transformation groups, (G, E^{n+1}) , of cohomogeneity 2 were classified in [13] and they can be naturally divided into five types according to the geometric shape of their orbit spaces, which are linear cones of angle π/g , $g = 1, 2, 3, 4$ and 6 respectively. The case $g = 1$ consists of the usual rotational transformation groups, $(O(n), E^{n+1})$, and it was proved in [16] that the generating curves of $O(n)$ -invariant hypersurfaces of constant mean curvatures can again be obtained by *rolling construction*, a straightforward generalization of the above mentioned theorem of Delaunay. The next case of $g = 2$ consists of those transformation groups $(O(p) \times O(q), E^{p+q})$, $p, q \geq 2$, and this is exactly the case studied in the paper preceding this one [I]. One of the main results of [I] is the existence of

infinitely many non-congruent $O(p) \times O(q)$ -invariant immersions of $S^{p+q-1} \rightarrow E^{p+q}$ with constant mean curvature 1 for each given pair of $p, q \geq 2$. The purpose of this paper is to extend the results of [I] not only to all the remaining cases of cohomogeneity two orthogonal transformation groups with $g = 3, 4$ and 6 , but also to all cases of *rank 2 isoparametric foliations*. Roughly speaking, isoparametric foliations of euclidean spaces are a kind of “geometric abstraction” of the orbital foliations of the isotropy representations of symmetric spaces. In particular, the family of rank 2 isoparametric foliations of euclidean spaces constitutes a geometric generalization of the family of orbital foliations of cohomogeneity 2 orthogonal transformation groups classified in [13], and hence, *foliated hypersurfaces* of such a given foliation are a kind of further generalization of those generalized rotational types proposed in [I].

Let \mathcal{F} be a given rank 2 isoparametric foliation of E^{n+2} , x_0 an arbitrarily chosen generic point, and $\mathcal{F}(x_0)$ the leaf passing through x_0 . Then the normal plane of $\mathcal{F}(x_0)$ at x_0 , $E_{x_0}^2$, intersects perpendicularly with every leaf of \mathcal{F} and has an induced transformation group of Coxeter type [20] (cf. §2A). Let C_0 be a Weyl chamber of the above associated Coxeter group, $(W, E_{x_0}^2)$, and M^{n+1} is a foliated hypersurface in E^{n+2} . Then C_0 is a wedge region of angle π/g , $g = 1, 2, 3, 4$ or 6 , which intersects every leaf of \mathcal{F} exactly once, and hence, a *foliated hypersurface* M^{n+1} is uniquely determined by its *generating curve*, $M^{n+1} \cap C_0$. In §2 we shall show that the generating curves of foliated hypersurfaces of constant mean curvature h are characterized by a specific ODE (cf. §2C). We state the main results of this paper in terms of the geometric behavior of the global solution curves of the specific ODE $(*, \mathcal{F})$ of §2C.

THEOREM A. *Let $\gamma = \{\gamma(s) = (r(s), \theta(s)); -\infty < s < +\infty\}$ be a global solution curve of $(*, \mathcal{F})$ parametrized by its arc length s . Then there exist two constants $d_1(h, \mathcal{F})$ and $d_2(h, \mathcal{F})$ such that*

$$\begin{aligned} \lim_{s \rightarrow -\infty} r(s) \cdot \sin \theta(s) &= d_1(h, \mathcal{F}), \\ \lim_{s \rightarrow +\infty} r(s) \sin\left(\frac{\pi}{g} - \theta(s)\right) &= d_2(h, \mathcal{F}), \end{aligned}$$

namely, every global solution curve has the following two straight lines

$$\begin{aligned} l_1 &= \{(r, \theta); r \sin \theta = d_1(h, \mathcal{F})\}, \\ l_2 &= \left\{(r, \theta); r \sin\left(\frac{\pi}{g} - \theta\right) = d_2(h, \mathcal{F})\right\} \end{aligned}$$

as its asymptotic lines.

THEOREM B. *Every global solution curve, γ , can hit the boundary line $\theta = 0$ (resp. $\theta = \pi/g$) at most once. There is a well-defined winding number of γ , $n(\gamma)$, such that the total change of direction of γ is given as follows*

$$\sigma_\gamma(+\infty) - \sigma_\gamma(-\infty) = (2n(\gamma) - 1)\pi + \frac{\pi}{g}$$

where $\sigma_\gamma(s)$ denotes the directional angle of γ at $\gamma(s)$ and $\sigma_\gamma(\pm\infty)$ are the limiting values of $\sigma_\gamma(s)$ as $s \rightarrow \pm\infty$.

REMARK. By our definition, each instance of the solution curve γ hitting ∂C_0 will contribute 1 to the winding number $n(\gamma)$. Therefore, it is not difficult to show that $n(\gamma) \geq$ the number of boundary points on γ .

THEOREM C. (i) *To each integer $k \geq 0$, there exists a global solution curve γ with $n(\gamma) = k$ and not hitting ∂C_0 .* (ii) *To each $k \geq 1$, there exists a global solution curve γ with $n(\gamma) = k$ and hitting ∂C_0 exactly once at the line $\theta = 0$ (resp. $\theta = \pi/g$).* (iii) *To each $k \geq 2$, there exists a global solution curve γ with $n(\gamma) = k$ and hitting ∂C_0 twice.*

THEOREM D. *There exist infinitely many non-congruent \mathcal{F} -foliated immersions of $S^{n+1} \rightarrow E^{n+2}$ with constant mean curvature 1, for each rank two isoparametric foliation, \mathcal{F} , of E^{n+2} with $g = 2, 3, 4$ or 6 .*

REMARKS. (i) In the case $g = 2$, the above four theorems constitute the main results proved in [I]. In this paper, we shall provide a unified proof of the above results which cover all cases of rank two isoparametric foliations with $g = 2, 3, 4$ or 6 .

(ii) In the special case of $g = 2$, there exist two auxiliary functions (denoted by I and J in [I]) which are *automatically monotonic* along any given solution curve. These two auxiliary functions together with their *monotonicity* play a crucial role throughout the whole proof of [I]. However, in the cases $g = 3, 4$ or 6 , the corresponding auxiliary functions are *no longer monotonic*! It is exactly the lack of such monotonicity for the cases $g = 3, 4$ and 6 that makes it necessary to seek a different approach other than that of [I], especially in the proof of Theorem A.

(iii) The above four theorems describe some of the major geometric features of the “*generating curves*” of foliated hypersurfaces of constant mean curvatures. Conceptually, they constitute an extension of the classical theorem of Delaunay to considerably greater generality, which also happens to include many varieties of new examples of “*soap bubbles*”.

(iv) The techniques developed in this paper can also be applied to the study of soap bubbles in symmetric spaces, cf. [12].

2. Isoparametric foliations of the Euclidean spaces and foliated hypersurfaces of constant mean curvature.

A. Isoparametric Foliations of the Euclidean Spaces.

An *isoparametric hypersurface* in a space of constant curvature (i.e., a euclidean, spherical or hyperbolic Riemannian manifold) is, by definition, a level surface of an isoparametric function, namely, a function $f: M^{n+1}(c) \rightarrow \mathbf{R}$ satisfying Δf and $|\nabla f| \equiv 0 \pmod{f}$. Geometrically, the level surfaces of such an isoparametric function constitutes a “*parallel*” *foliation of hypersurfaces of constant mean curvatures*. We shall call such a foliation a *codimension one isoparametric foliation of $M^{n+1}(c)$* . Such nice geometric structures were first studied by B. Segre and É. Cartan. In a series of papers [3, 4, 5, 6] of É. Cartan, he was particularly fascinated by the profound depth of the spherical case and its mysterious connection with the Lie group theory. This subject was somehow forgotten until it was revived by a sequence of recent papers by Nomizu [19], Münzner [18], Ozeki-Takeuchi [21], Ferus-Karcher-Münzner [9]. Higher codimensional generalization of isoparametric foliations was proposed in a recent paper of Terng [22], namely, a submanifold N^n in $M^{n+k}(c)$ is called isoparametric if its normal bundle is flat and the principal curvatures in the directions of any parallel normal vector field are constant; an *isoparametric foliation of $M^{n+k}(c)$* is, by definition, a *parallel foliation by isoparametric submanifolds and their focal varieties*. We refer to [22] for a general theory as well as some basic theorems on isoparametric foliations in E^{n+k} .

A typical example of isoparametric foliation (which, in fact, motivates its definition, cf. [22]) is the orbital foliation of the adjoint action of a compact connected Lie group G on its Lie algebra \mathfrak{g} . It follows from the maximal torus theorem of É. Cartan that the above orbital foliation has the following basic geometric properties:

- (i) the principal orbit type is G/T ,
- (ii) the Lie algebra of a given maximal torus, T , i.e. a Cartan subalgebra \mathfrak{h} , intersects every G -orbit perpendicularly,
- (iii) the Weyl group $W = N(T)/T$ acts on \mathfrak{h} as a group generated by reflections and $\mathfrak{g}/G \cong \mathfrak{h}/W$.

The above result was generalized by É. Cartan himself to the setting of the isotropy representation of a symmetric space $M = G/K$, namely,

- (i) a normal plane, \mathfrak{h} , to an arbitrary *principal K -orbit* in \mathfrak{p} automatically intersects all K -orbits perpendicularly

(ii) there is an induced Coxeter group (W, \mathfrak{h}) such that $\mathfrak{p}/K \cong \mathfrak{h}/W$.

Let M^n be a given *isoparametric submanifold* in E^{n+k} and E_x^k be the normal plane of M^n at a point $x \in M^n$. Choose an arbitrary base point x_0 on M^n . Then $E_{x_0}^k$ intersects M^n perpendicularly at a finite number of points and, moreover, there is a unique parallel translation $\Pi_{x_1, x_2}: E_{x_0}^k = E_{x_1}^k \rightarrow E_{x_2}^k = E_{x_0}^k$ for each given pair of points $x_1, x_2 \in E_{x_0}^k \cap M^n$, thus generating an induced transformation group $(W, E_{x_0}^k)$. It was proved in [22] that $(W, E_{x_0}^k)$ is automatically a group generated by reflections, called the *associated Coxeter group of $M^n \subset E^{n+k}$* . It follows from a theorem of Chevalley [7] that the ring of W -invariant polynomials of $E_{x_0}^k \simeq \mathbf{R}^k$ is a *free algebra* generated by k homogeneous generators, say $\{u_i; 1 \leq i \leq k\}$. It was proved in [22] that each u_i extends uniquely to a homogeneous polynomial on $E^{n+k} \simeq \mathbf{R}^{n+k}$ which is constant on M^n and moreover,

$$\mathbf{u} = (u_1, \dots, u_k): E^{n+k} \rightarrow \mathbf{R}^k$$

defines an “*isoparametric map*” whose level surfaces form an isoparametric foliation of E^{n+k} . As one can see from the above brief account of isoparametric foliations, it is exactly a kind of “geometric abstraction” of the orbital foliations of these isotropy representations of symmetric spaces.

B. Foliated Submanifolds of a Given Isoparametric Foliation.

For a given codimension k isoparametric foliation on E^{n+k} , let (W, \mathbf{R}^k) be its associated Coxeter group and $p: E^{n+k} \rightarrow \mathbf{R}^k/W$ be the associated projection map. By definition, the above, map, p , is a *Riemannian submersion* of a particularly simple kind, namely, in the terminology of O’Neill [20], the submersion tensor A is identically zero and the second fundamental form tensor T is constant along each generic fiber. A foliated submanifold of E^{n+k} is, by definition, a submanifold which consists of a suitable subcollection of leaves. Hence, a foliated submanifold, M^{n+d} , intersects with the normal section, \mathbf{R}^k , transversally and is uniquely determined by the above intersection, $M^{n+d} \cap \mathbf{R}^k$.

The tangent bundle of each isoparametric submanifold M^n in E^{n+k} has a splitting as the orthogonal direct sum of the subspaces of equiprincipal-directions:

$$TM^n = \bigoplus \sum \{E_j; \dim E_j = m_j, 1 \leq j \leq g\}.$$

The integral submanifold of E_j passing $x_0 \in M^n \cap \mathbf{R}^k$ is a round sphere of dimension m_j which intersects \mathbf{R}^k at the antipodal point of x_0 , say x_j , and moreover, $\{\Pi_{x_0 x_j}; 1 \leq j \leq g\}$ are exactly those generating reflections

of W . We shall call m_j the marked multiplicity of the j th reflection hyperplane, H_j , and call the above structure a *Coxeter group with marked multiplicities*. Observe that the projection map $p: E^{n+k} \rightarrow \mathbf{R}^k/W$ is a Riemannian submersion with $A \equiv 0$ and T completely determined by the above Coxeter group with marked multiplicities. Therefore, it forms a *complete set of invariants* for the *local geometry* of the given isoparametric foliation. In particular, it is quite straightforward to write down the *second fundamental form of a foliated submanifold*, $\bar{M}^{n+d} \subset E^{n+k}$, in terms of the second fundamental form of its image, $\bar{M}^d \subset \mathbf{R}^k/W$, and the following set of distance functions, namely,

$$\left\{ \rho_j(x) = \begin{array}{l} \text{the distance between } x \text{ and the} \\ j\text{th hyperplane } H_j, 1 \leq j \leq g \end{array} \right\}$$

where $x \in \bar{M}^d \subset \mathbf{R}^k/W$ and \mathbf{R}^k/W is identified with a chosen *Weyl chamber* of (W, \mathbf{R}^k) .

C. Rank Two Isoparametric Foliations and Foliated Hypersurfaces of Constant Mean Curvatures.

In this paper, we shall mainly exploit those *rank two isoparametric foliations* on euclidean spaces to construct families of examples of *constant mean curvature hypersurfaces*. For a given rank two isoparametric foliation on E^{n+2} , its associated Coxeter group is a dihedral group, $(W; \mathbf{R}^2)$, whose fundamental domain, \mathbf{R}^2/W , is a wedge with its angle equal to π/g , $g = 1, 2, 3, 4$ or 6 [cf. 13]. Since the cases $g = 1$ and 2 have already been treated in [16] and [I] respectively, we shall only discuss here the remaining cases of $g = 3, 4$ and 6 as follows:

(i) If $g = 3$, then it was proved by É. Cartan [4, 5] that there are exactly four cases with *uniform marked multiplicities* 1, 2, 4 and 8 respectively. They are the orbital foliations of $(\mathrm{SO}(3), \mathbf{R}^5)$, $(\mathrm{SU}(3), \mathbf{R}^8)$, $(\mathrm{Sp}(3), \mathbf{R}^{14})$ and (F_4, \mathbf{R}^{26}) respectively.

(ii) If $g = 6$, then it was proved by Abresch [1] that they must also be with *uniform marked multiplicities* 1 or 2.

(iii) The case $g = 4$ is certainly the most involved and also the most interesting one. There exist a large family of *non-orbital examples* obtained by means of orthogonal representations of Clifford algebras [9] and the exact extent of possibilities of marked multiplicities is still not yet completely determined. We quote here the following theorem of Abresch [1]:

THEOREM 2.1 [Abresch]. *Let $m_- \leq m_+$ be the marked multiplicities of the two generating reflection lines of a given rank two isoparametric foliation*

with $g = 4$. Then they satisfy one of the following three conditions:

$$(4A): \quad m_+ + m_- + 1 \text{ is divisible by } 2^K := \min\{2^\sigma \mid 2^\sigma > m_-\},$$

$$(4B1): \quad m_- \text{ is a power of 2 and } 2m_- \text{ divides } m_+ + 1,$$

$$(4B2): \quad m_- \text{ is a power of 2 and } 3m_- = 2 \cdot (m_+ + 1).$$

Let M^{n+1} be a foliated hypersurface of constant mean curvature in E^{n+2} , $C_0 \cong \mathbf{R}^2/W$ be a chosen Weyl chamber of its associated Coxeter group (W, \mathbf{R}^2) and $\gamma = M^{n+1} \cap C_0$ be the generating curve of M^{n+1} . Then one has the following characterizing ODE of γ , namely,

LEMMA 2.1. *In terms of the polar coordinates (r, θ) , $0 \leq r < \infty$, $0 \leq \theta \leq \pi/g$, the above generating curve, $\gamma = C_0 \cap M^{n+1}$, of a foliated hypersurface of constant mean curvature h is characterized by the following ODE $(*, \mathcal{F})$:*

(i) the case $g = 3$: $k = 1, 2, 4, 8$

$$(*)_3 \quad \frac{d\sigma}{ds} = (3k + 1)h + \frac{k}{r} \left\{ \frac{\cos \sigma}{\sin \theta} - \frac{\sin(\sigma + \pi/6)}{\cos(\theta + \pi/6)} - \frac{\sin(\sigma - \pi/6)}{\cos(\theta - \pi/6)} \right\},$$

(ii) the case $g = 6$: $k = 1, 2$

$$(*)_6 \quad \frac{d\sigma}{ds} = (6k + 1)h + \frac{k}{r} \left\{ \frac{\cos \sigma}{\sin \theta} - \frac{\sin \sigma}{\cos \theta} - \frac{\sin(\sigma + \pi/6)}{\cos(\theta + \pi/6)} - \frac{\sin(\sigma - \pi/6)}{\cos(\theta - \pi/6)} \right. \\ \left. - \frac{\sin(\sigma + \pi/3)}{\cos(\theta + \pi/3)} - \frac{\sin(\sigma - \pi/3)}{\cos(\theta - \pi/3)} \right\},$$

(iii) the case $g = 4$: $\{k, l\} = \{m_+, m_-\}$, (cf. Theorem 1.1)

$$(*)_4 \quad \frac{d\sigma}{ds} = (2k + 2l + 1)h + \frac{k}{r} \left\{ \frac{\cos \sigma}{\sin \theta} - \frac{\sin \sigma}{\cos \theta} \right\} \\ - \frac{l}{r} \left\{ \frac{\sin(\sigma + \pi/4)}{\cos(\theta + \pi/4)} + \frac{\sin(\sigma - \pi/4)}{\cos(\theta - \pi/4)} \right\},$$

where s is the arclength parameter of γ and σ is the angle between the positive x -direction and the tangent direction of γ .

Proof. Let $v(r, \theta)$ be the n -dimensional volume of the isoparametric submanifold passing through the point (r, θ) . Then it is not difficult to see that

$$v(r, \theta) = c_0 \cdot \prod_{i=1}^g \rho_i(r, \theta)^{m_i},$$

where $c_0 =$ a suitable constant, $\rho_i(r, \theta)$ is the distance between (r, θ) and the i th reflection line, and m_i is its marked multiplicity. Moreover, straightforward computation will show that

$$(n+1)h = \frac{d\sigma}{ds} - \frac{d}{dn} \ln v(r, \theta)$$

and hence the characterizing ODE $(*)_g$ for each of the specific cases of $g = 3, 4$ and 6 .

3. Some analytical lemmas and the proof of Theorems A and B.

A. The Analysis of the ODE $(*, \mathcal{F})$ in the Neighborhood of ∂C_0 .

At a boundary point of C_0 , say $(r, 0)$ or $(r, \pi/g)$, the ODE $(*, \mathcal{F})$ becomes singular because the denominator of one term tends to zero. However, one has the following lemma of existence and uniqueness on solution curves of $(*, \mathcal{F})$ passing through a boundary point:

LEMMA 3.1. *To each point $P = (r, 0)$ (resp. $Q = (r, \pi/g)$), $r > 0$, there exists a unique solution curve of $(*, \mathcal{F})$ passing through P (resp. Q) which forms a perpendicular cusp point at P (resp. Q) and depends on r analytically.*

We refer to Proposition 5 of [I] and Proposition 1 of [11] for a proof of the above Lemma 3.1.

From now on, we shall always assume, for simplicity of notation, that $h = 1$. We shall denote the unique solution curve passing through $(b, 0)$ (resp. $(b, \pi/g)$) by $\gamma_{(b,0)}$ (resp. $\gamma_{(b,\pi/g)}$) and moreover, use $\gamma_{(b,0)}^+$ (resp. $\gamma_{(b,0)}^-$) to denote the segment of $\gamma_{(b,0)}$ which consists of points after (resp. before) the cusp point $(b, 0)$. For a given $b > 0$, it follows from Lemma 3.1 that there exists a sufficiently small $\delta(b) > 0$ and two coordinate systems $\{(u, y)\}$ and $\{(v, y)\}$, of the $\delta(b)$ -collar neighborhood of the segment $\{(r, 0); b \leq r < \infty\}$ such that

$$0 \leq y = r \sin \theta \leq \delta(b)$$

and $\gamma_{(r,0)}^-$ (resp. $\gamma_{(r,0)}^+$) is exactly given by $u = r$ (resp. $v = r$) for all $b \leq r < \infty$.

In such a $\delta(b)$ -collar region, the value of $r \sin \theta$ is very small and hence the curvature, $d\sigma/ds$, of a solution curve inside this region is dominated by the term $k \cos \sigma / r \sin \theta$. Therefore, unless γ is, in fact, the curve $u = r_0$, a solution curve γ will make a rapid turn back as it approaches the boundary $\theta = 0$, as indicated in the following Figure 1.

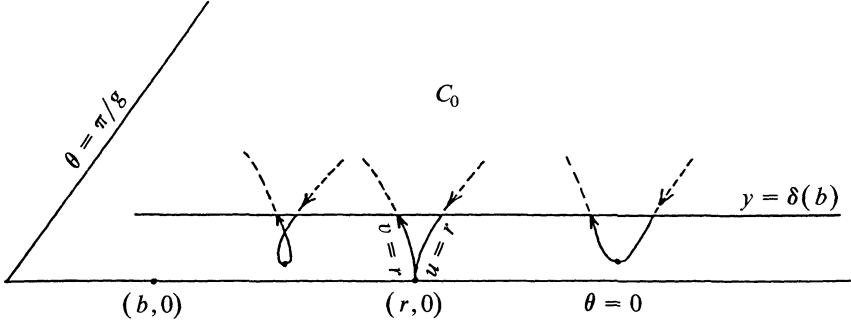


FIGURE 1

LEMMA 3.2. Let γ_t , $0 < t \leq 1$, be a continuous family of solution curves of $(*, \mathcal{F})$, $u(t)$ (resp. $v(t)$) be the u -coordinate (resp. v -coordinate) of the entrance (resp. exit) point of γ_t and $(\hat{u}(t), y(t))$ be the (u, y) -coordinate of the unique y -minimum point of γ_t . If $y(t) > 0$ for all $0 < t \leq 1$ and as $t \rightarrow 0$

$$\lim y(t) = 0, \quad \lim \hat{u}(t) = r_0$$

then the directions of γ_t at the y -minimum point, $0 < t \leq 1$, must be the same and $\lim u(t) = \lim v(t) = r_0$ as $t \rightarrow 0$.

Proof. Since $y(t)$ is assumed to be positive for all $0 < t \leq 1$, it follows directly from the continuity of γ_t that the direction of γ_t at the y -minimum point must be the same.

Let t_n be an arbitrary sequence with $\lim t_n = 0$. Without loss of generality, one may assume that, as $n \rightarrow \infty$,

$$\lim u(t_n), \quad \lim v(t_n), \quad \lim \sigma_n \quad \text{and} \quad \lim \sigma'_n \quad \text{all exist}$$

where σ_n and σ'_n are respectively the entrance and exit direction of γ_{t_n} . Then it follows from the assumptions $\lim y(t) = 0$, $\lim \hat{u}(t) = r_0$, and Lemma 3.1 that

$$\lim u(t_n) = \lim v(t_n) = r_0$$

and $\lim \sigma_n$ (resp. $\lim \sigma'_n$) must be equal to the entrance (resp. exit) direction of $\gamma_{(r_0,0)}$. For otherwise, those γ_{t_n} with large n should be a U -shaped curve with the bottom part coasting very closely along the singular boundary $\theta = 0$.

Such a solution curve is impossible because the dominating large value of the term $k \cos \sigma / r \sin \theta$ will make any solution curve sharply turn away from the line $\theta = 0$.

B. The Analysis of Two Auxiliary Functions.

In analyzing the geometric behavior of global solution curves of $(*, \mathcal{F})$ for the special case of $g = 2$ in [I], there are two auxiliary functions, denoted by I and J , which play the key role throughout the discussion of [I]. One of the crucial properties of these two auxiliary functions of [I] is their monotonicity along any given solution curve (cf. Proposition 4 of [I]). In this subsection, we shall again introduce the following two auxiliary functions and study their properties along certain specific segments of solution curves.

DEFINITION. For $g = 3, 4$ and 6 , set $n = 3k, 2k + 2l$ and $6k$ respectively; and let $k' = k$ if $g = 3$ or 6 , $k' = l$ if $g = 4$. Define

$$I = \left[r \cos \left(\theta + \frac{\pi}{2} - \frac{\pi}{g} \right) \right]^{k'} \sin \left(\sigma + \frac{\pi}{2} - \frac{\pi}{g} \right) - \frac{n+1}{k'+1} \left[r \cos \left(\theta + \frac{\pi}{2} - \frac{\pi}{g} \right) \right]^{k'+1},$$

$$J = [r \sin \theta]^k \cos \sigma + \frac{n+1}{k+1} [r \sin \theta]^{k+1} = y^k \cos \sigma + \frac{n+1}{k+1} y^{k+1}.*$$

As we shall see, the behavior of γ as $s \rightarrow -\infty$ is governed through an analysis of J , in the same way that I governs the behavior of γ as $s \rightarrow +\infty$. Henceforth we shall only exhibit our lemma for J .

Since $\dot{y} = \sin \sigma$, we have

$$\frac{dJ}{ds} = y^k \dot{y} \left[(n+1) + \frac{k \cos \sigma}{y} - \dot{\sigma} \right],$$

where the dot denotes differentiation with respect to s . Thus we see that J is a first integral of the equation

$$\dot{\sigma} = (n+1) + \frac{k \cos \sigma}{y},$$

*In the ensuing discussion, both rectangular coordinates (x, y) and polar coordinates (r, θ) will be used as convenient.

obtained from the ODE $(*, \mathcal{F})$ by removing the terms contributed by all but one factor in the volume function (see the proof of Lemma 2.1). These equations were thoroughly studied in [16]. In particular, the asymptotic line as $s \rightarrow -\infty$ stated in Theorem A is a solution of the above equation with $\dot{\sigma} = 0$.

LEMMA 3.3. *Along a given solution curve of ODE $(*, \mathcal{F})$:*

(i)

$$\frac{dJ}{ds} = \begin{cases} ky^k \cdot \frac{\sin \sigma [3 \cos \theta \sin \sigma + \sin \theta \cos \sigma]}{r(\frac{1}{2} + \cos 2\theta)} & \text{for } g = 3; \\ y^k \cdot \frac{\sin \sigma [((k+l)\cos 2\theta + l) \sin \sigma + l \sin 2\theta \cos \sigma]}{r \cos 2\theta \cos \theta} & \text{for } g = 4; \\ ky^k \cdot \frac{\sin \sigma \left[3 \frac{\cos 4\theta}{2} + \frac{\cos 2\theta}{2} + \frac{1}{4} \sin \sigma + \sin 4\theta + \frac{\sin 2\theta}{2} \cos \sigma \right]}{r \cos \theta (\cos 2\theta + \frac{1}{2})(\cos 2\theta - \frac{1}{2})} & \text{for } g = 6. \end{cases}$$

(ii) *The ranges of non-monotonicity of J are given as follows: $(dJ/ds) < 0$ if and only if $\pi - \bar{\alpha} < \sigma < \pi$ or $\alpha < \sigma < 0$, where*

$$\tan \bar{\alpha}(\theta) = \begin{cases} 1/3 \tan \theta & \text{for } g = 3; \\ \frac{\sin 2\theta}{[(k+l)\cos 2\theta + l]} & \text{for } g = 4; \\ \frac{\left[\sin 4\theta + \frac{\sin 2\theta}{2} \right]}{3 \left[\frac{\cos 4\theta}{2} + \frac{\cos 2\theta}{2} + \frac{1}{4} \right]} & \text{for } g = 6. \end{cases}$$

Proof. (i) For $g = 3$, compare the expression for dJ/ds in Remark (ii) with the ODE $(*, \mathcal{F})$; we have

$$\frac{dJ}{ds} = \frac{ky^k \dot{y}}{r} \left\{ \frac{\sin(\sigma + \pi/6)}{\cos(\theta + \pi/6)} + \frac{\sin(\sigma - \pi/6)}{\cos(\theta - \pi/6)} \right\}.$$

The assertion follows by direct computation, noting that $\dot{y} = \sin \sigma$. For $g = 4$ or 6 , the proof is similar. (ii) follows easily from (i).

REMARK. For $g = 3, 4$ and 6 ,

$$J \geq - \left[\frac{k}{n+1} \right]^k \frac{1}{k+1}.$$

The lower bounds are attained by the straight lines $y = k/(n + 1)$, which will be shown to be the asymptotic lines of Theorem A.

Let us now consider a segment of a fixed solution curve γ of the ODE $(*, \mathcal{F})$ which contains a local maximum for y_γ at $s = s_1$ followed by a local minimum for y_γ at $s = s_0$. We will allow $\gamma(s_0)$ to be a boundary point, viz. $y_\gamma(s_0) = 0$. Referring to Figure 2, we define certain functions of y for $y_\gamma(s_0) \leq y \leq y_\gamma(s_1)$. Set $s_0 > s(y) > s_1 > s'(y)$ so that $y_1 \geq y(s) = y(s') \geq y_0$. Let $\alpha(y) = \sigma(s) - \pi$ and $\alpha'(y) = \pi - \sigma(s')$. Set $J(y) = J(s)$, $J'(y) = J(s')$. We define s' , α' , J' only when s' is in one-to-one correspondence with y , which means that they are defined where $0 \leq \alpha' \leq \pi/2$.

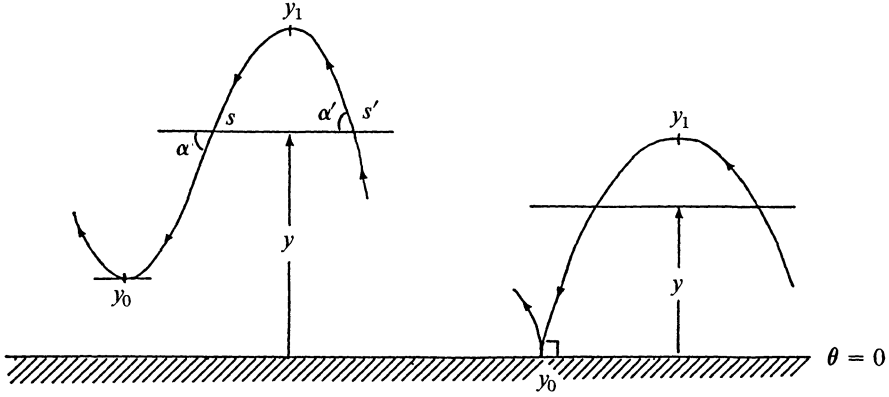


FIGURE 2

We have come to the important cancellation lemma that helps to overcome the *non-monotonicity* of J .

LEMMA 3.4. *For $y < y_1$ and when s' is defined,*

$$J(y) > J'(y), \quad \alpha(y) > \alpha'(y).$$

Proof. First observe that $J = (n + 1)/(k + 1) y^{k+1} - \cos \alpha \cdot y^k$, and a similar relation holds for J' , α' . Hence the two inequalities in the lemma are equivalent. Now

$$J(y) - J'(y) = \int_y^{y_1} \left[-\frac{dJ}{dy} \right]_s - \left[-\frac{dJ}{dy} \right]_{s'} dy.$$

It suffices to show that the integrand is positive.

Recall Lemma 3.3(i). For $g = 3$,

$$\left[-\frac{dJ}{dy} \right]_s = ky^k \frac{[3 \cos \theta \sin \alpha + \sin \theta \cos \alpha]}{r(\frac{1}{2} + \cos 2\theta)},$$

$$\left[-\frac{dJ}{dy} \right]_{s'} = ky^k \frac{[-3 \cos \theta' \sin \alpha' + \sin \theta' \cos \alpha']}{r'(\frac{1}{2} + \cos 2\theta')}.$$

It is clear that $r < r'$ and $\theta > \theta'$, so that the denominator of $[-dJ/dy]_s$ is greater than the denominator of $[-dJ/dy]_{s'}$. We need only show that

$$[3 \cos \theta \sin \alpha + \sin \theta \cos \alpha] - [-3 \cos \theta' \sin \alpha' + \sin \theta' \cos \alpha'] > 0.$$

But

the last quantity $> 3 \cos \theta \sin \alpha + \cos \alpha \sin \theta - \sin \theta'$

$$> \{3 \cos \theta \sin \alpha + \cos \alpha \sin \theta\} - \sin \theta, \quad \text{since } \theta > \theta',$$

$$> \min\{3 \cos \theta - \sin \theta, \sin \theta - \sin \theta'\} \geq 0 \quad \text{for } 0 \leq \theta \leq \frac{\pi}{3},$$

since the minimum of the sum of the positive terms occur either at $\alpha = 0$ or $\alpha = \pi/2$.

For $g = 4$, the same argument reduces the assertion to showing that

$$[((k+l) \cos 2\theta + l) \sin 6 + l \sin 2\theta \cos \alpha] \\ - [-((k+l) \cos 2\theta' + l) \sin \alpha' + l \sin 2\theta' \cos \alpha'] > 0.$$

The last quantity $> \min\{(k+l) \cos 2\theta + l - l \sin 2\theta, l \sin 2\theta - l \sin 2\theta'\} \geq 0$ for $0 \leq \theta \leq \pi/4$.

For $g = 6$, we need to show that

$$\left[3 \left(\frac{\cos 4\theta}{2} + \frac{\cos 2\theta}{2} + \frac{1}{4} \right) \sin \alpha + \left(\sin 4\theta + \frac{\sin 2\theta}{2} \right) \cos \alpha \right] \\ - \left[-3 \left(\frac{\cos 4\theta'}{2} + \frac{\cos 2\theta'}{2} + \frac{1}{4} \right) \sin \alpha' \right. \\ \left. + \left(\sin 4\theta' + \frac{\sin 2\theta'}{2} \right) \cos \alpha' \right] > 0.$$

The last quantity $> \min\left\{ 3 \left(\frac{\cos 4\theta}{2} + \frac{\cos 2\theta}{2} + \frac{1}{4} \right) - \left(\sin 4\theta + \frac{\sin 2\theta}{2} \right), \right. \\ \left. \left(\sin 4\theta + \frac{\sin 2\theta}{2} \right) - \left(\sin 4\theta' + \frac{\sin 2\theta'}{2} \right) \right\}$

$$\geq 0 \quad \text{for } 0 \leq \theta \leq 25^\circ.$$

Although 25° falls short of the full range for θ , this estimate suffices for our purpose, as it is impossible for a local maximum of y to occur with the specified orientation, and with $\theta > 25^\circ$. In fact, the present case compares favorably with the minimal equation ($h = 0$) for the same group of transformations, where such behavior is established in [10] (cf. Proposition 1, especially paragraph (b)).

As a simple but important consequence of the fact $\alpha(y) > \alpha'(y)$, we have the following

COROLLARY 3.4. *There is a local minimum for y_γ at s_2 with $s_2 < s_1$ and $y_\gamma(s_2) > y_\gamma(s_0)$.*

Proof. Since s_1 is a local maximum of y_γ , clearly there is an $s_2 < s_1$ such that either $y_\gamma(s_2) = 0$ or s_2 is a local minimum of y_γ , and furthermore that all points on γ between s_1 and s_2 are neither local y -minima nor boundary points.

Let us assume $y_\gamma(s_2) = 0$, or $y_\gamma(s_2)$ is a local minimum but with $y_\gamma(s_2) < y_\gamma(s_0)$. In either case, we can find an s' such that $s_0 > s_1 > s > s_2$ and $y_\gamma(s_0) = y_0 = y_\gamma(s')$. But then $\alpha(y_0) = 0$, $\alpha(y_0) = \sigma(s) - \pi > 0$. This contradicts Lemma 3.4 and proves the corollary.

C. The Proof of Theorems A and B.

In order to establish Theorem A, namely, that all global solution curves of the ODE $(*, \mathcal{F})$ are asymptotic to the line $y = k/(n+1)$ as $s \rightarrow -\infty$ (and the corresponding statement for $s \rightarrow +\infty$), let us first record a lemma which is, in essence, covered in [I].

LEMMA 3.5. *Suppose γ is a solution curve of the ODE $(*, \mathcal{F})$ such that, as $s \rightarrow -\infty$, y_γ remains bounded, $x_\gamma \rightarrow +\infty$, and $J \leq 0$. Then γ is asymptotic to $y = k/(n+1)$ as $s \rightarrow -\infty$.*

Proof (sketch). Recall that the ODE $(*, \mathcal{F})$ takes the form

$$\dot{\sigma} = C + \frac{k \cos \sigma}{y} + \phi(x, y, \sigma)$$

where C is a constant and $\phi(x, y, \sigma) \rightarrow 0$ as $x \rightarrow +\infty$ and y bounded.

Now

$$J = \frac{C}{k+1} y^{k+1} + \cos \sigma \cdot y^k, \quad \text{hence} \quad \frac{dJ}{ds} = y^k \dot{y} \phi(x, y, \sigma).$$

If γ behaves as in the hypothesis, then for a sufficiently large s_0 , $s > s_0$ implies that $0 < y < M$ and $|\phi(x, y, \sigma)| < \delta/M^{k+1}$ for some prescribed $\delta > 0$. Whence

$$\begin{aligned} |J_\gamma(s) - J_\gamma(s_0)| &= \left| \int_{s_0}^s \phi y^k \dot{y} ds \right| = \left| \int_{y(s_0)}^{y(s)} \phi y^k dy \right| \\ &< \frac{\delta}{M^{k+1}} M^k |y(s) - y(s_0)| \leq \delta. \end{aligned}$$

Thus the curve γ is closely approximated by a curve defined by $J = J_\gamma(s_0) \leq 0$, which is an x -translational invariant, slowly oscillating curve; when $J_\gamma(s_0) = -[k/(n+1)]^k(k+1)$, i.e. the infimum for J , this curve is the line $y = k/(n+1)$.

Moreover, when s is very large and negative, θ_γ is very small. By Lemma 3.3(ii), the range of σ for which $(dJ/ds) < 0$ appears only for a glimpse.

Let us assume, contrary to the desired conclusion, that γ is not asymptotic to the line $y = k/(n+1)$. Then we can remove segments of γ including parts where $(dJ/ds) < 0$, and apply the estimate in the proof of Theorem 1 [I] to obtain a contradiction; for details, we refer the reader to that paper.

In [I], by virtue of the monotonicity of the analogous auxiliary function J , one can easily see that $|J - \text{negative constant}| \leq \delta$ eventually ($s \rightarrow -\infty$) for *all* solution curves. Here we shall first prove this for a certain class of solution curves, and then extend the conclusion of the theorem to all curves by continuity, noticing that asymptoticity is a *closed property* relative to the deformation of solutions.

LEMMA 3.6. *Consider a solution curve γ such that for one value s_0 of s , $y_\gamma(s_0)$ is a local minimum of y_γ with $\sigma_\gamma(s_0) = \pi$, or $y_\gamma(s_0) = 0$, and also $J_\gamma(s_0) \leq 0$. γ is asymptotic to $y = k/(n+1)$ as $s \rightarrow -\infty$.*

Proof. First we argue that there is a local maximum of y at some $\gamma(s_1)$ with $s_1 < s_0$. Assume the contrary; then as $s \rightarrow -\infty$, either y increases without bound or γ approaches a horizontal line asymptotically from below. The latter possibility is dismissed since solution curves with large x and bounded y are oscillatory. For the former possibility, $J \rightarrow +\infty$ as $s_0 > s \rightarrow -\infty$. However, note that $dJ/ds \geq 0$ for $\pi < \sigma < 3\pi/2$. Also, the situation depicted in Figure 3 is prohibited by the differential equation. Together it means that J would decrease as $s_0 > s \rightarrow -\infty$, which is a contradiction.

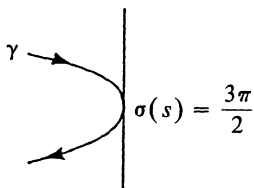


FIGURE 3

We are now placed in the hypothesis of Lemma 3.4. Apply Corollary 3.4 and the above argument repeatedly. We obtain a succession of y -local minima $y_0 \leq y_2 \leq y_4 \leq \dots$ as $s \rightarrow -\infty$; and, in between, a succession of y -local maxima y_1, y_3, y_5, \dots . Let s_i be the corresponding parameter values to these points. Note also the points $\gamma(\bar{s}_i)$ where $s_{i-1} > \bar{s}_i > s_i$ (i an odd integer) and $y(\bar{s}_i) = y(s_{i+1})$.

By Lemma 3.3(ii), $J(s_{i-1}) > J(\bar{s}_i) > J(s_i)$. By Lemma 3.4, $J(\bar{s}_i) > J(s_{i+1})$. Hence $0 \geq J_0 \geq J_{2k+1}$ for all k . This implies that as $s \rightarrow -\infty$, $J \leq 0$ and y remains bounded. It is also obvious that $x \rightarrow +\infty$. The required asymptoticity of γ now follows from Lemma 3.5.

Observe that $r = 1$ is a solution of the ODE $(*, \mathcal{F})$. It is part of a global solution curve Γ_2 with a point where $y = 0$; as an immediate corollary of the above lemma, we have

LEMMA 3.7. *The solution curve Γ_2 is asymptotic to the line $r \sin \theta = k/(n+1)$ as $s \rightarrow -\infty$, and to the line $r \sin(\pi/g - \theta) = k'/(n+1)$ as $s \rightarrow +\infty$.*

A feature new to the cases $g = 3, 4$ or 6 is the possible appearance of *points of inflection* on a solution curve, where $\dot{y} = 0$ and $\ddot{y} = 0$. In the previous case $g = 2$ of [I], the last condition would force the solution curve to be the horizontal line. We restrict our attention to those points where $\sigma = \pi$ and $\dot{\sigma} = 0$.

LEMMA 3.8. (i) *The above mentioned type of point of inflection along any solution curve occurs on the locus defined by*

$$\begin{cases} y = \frac{k}{(3k+1)} \left\{ 1 - \frac{2y^2}{3x^2 - y^2} \right\} & \text{for } g = 3 \\ y = \frac{k}{(2k+2l+1)} - \frac{2y^2 l}{(2k+2l+1)(x^2 - y^2)} & \text{for } g = 4 \\ y = \frac{k}{(6k+1)} \left\{ 1 - \frac{2y}{3x^2 - y^2} - \frac{6y^2}{x^2 - 3y^2} \right\} & \text{for } g = 6 \end{cases}$$

(ii) *Given any point on the locus, there exists a unique solution curve having this point as a point of inflection.*

(ii) *The set of solution curves possessing a point of inflection of the above type is closed.*

(iv) *A solution curve having a point of inflection of this type is asymptotic to $y = k/(n + 1)$ as $s \rightarrow -\infty$.*

Proof. (i) The locus is obtained by putting $\sigma = \pi$, $\dot{\sigma} = 0$ into ODE $(*, \mathcal{F})$. (ii) is immediate and (iii) follows from (i) and (ii).

For (iv) let $\gamma(s_0)$ be the point of inflection. It is easy to see that y_γ is in fact increasing with increasing s near $\gamma(s_0)$, hence there is a y -local minimum or a point with $y = 0$ at some $\gamma(s_1)$ with $s_0 > s_1$. Then apply Lemma 3.6.

We shall now use crucially the continuity principles established in §3A. Note that they cover the circumstance under which some of the solution curves may contain boundary points during a continuous change of initial conditions.

LEMMA 3.9. *Let $\{\gamma_t: 0 \leq t \leq 1\}$ be a continuous family of global solution curves with $\gamma_0 = \Gamma_2$ (cf. Lemma 3.7). Then γ_1 is asymptotic to $y = k/(n + 1)$ as $s \rightarrow -\infty$.*

Proof. Let $\Sigma = \{t \in (0, 1): \gamma_t \text{ is asymptotic to } y = k/(n + 1) \text{ as } s \rightarrow -\infty\}$. Σ is open. Indeed, let $t_0 \in \Sigma$. Then on γ_{t_0} we can find a y -local minimum $\gamma_{t_0}(s_0)$ with $\sigma(s_0) = \pi$, $\dot{\sigma}(s_0) < 0$, and $J(s_0) < 0$. These are clearly open conditions, so that such points can be found on curves in a neighborhood of γ_{t_0} . Lemma 3.6 then implies the asymptoticity of these neighboring curves.

Now if γ_1 has a point of inflection, then Lemma 3.8(iv) will give its asymptoticity. Otherwise there is an interval $(1 - \varepsilon, 1]$ on which no curve has a point of inflection, by Lemma 3.8(iii). Furthermore, since Σ is open, one may assume without loss of generality that all the curves in $(1 - \varepsilon, 1)$ are asymptotic. Let $t \in (1 - \varepsilon, 1)$ and $s(t)$ be parameters at which $\gamma_t(s(t))$ are y -local minima of γ_t with $J_{\gamma_t}(s(t)) < 0$. If $s(t)$ is unbounded as $t \rightarrow 1$, then we see that γ_1 is governed by Lemma 3.5 and therefore asymptotic. So we assume $s(t) \rightarrow s_1 < \infty$; three possibilities arise:

(a) $\dot{\sigma}_{\gamma_t}(s(t)) \rightarrow 0$. $\gamma_1(s_1)$ is a point of inflection of γ_1 , contrary to assumption.

(b) $\dot{\sigma}_{\gamma_1}(s(t))$ is bounded and $\rightarrow 0$. $\gamma_1(s_1)$ is a y -local minimum with $J < 0$. γ_1 is asymptotic by Lemma 3.6.

(c) $\dot{\sigma}_{\gamma_1}(s(t))$ is unbounded. $\gamma_1(s_1)$ is a boundary point with the correct orientation, and so γ_1 is asymptotic, again by Lemma 3.6.

Now *every* global solution curve can be linked to Γ_2 through a one-parameter family, and Theorem A is therefore proved.

The first statement of Theorem B, namely that a solution curve cannot contain two points on the same boundary, is a direct consequence of Corollary 3.4. As in [I], we define σ_γ so that it has a jump of $+\pi$ at a boundary point (cusp). By Theorem A, $\lim_{s \rightarrow +\infty} \sigma_\gamma(s) = 2n\pi + (\pi/g)$, whilst $\lim_{s \rightarrow -\infty} \sigma_\gamma(s) = \pi$.

DEFINITION. The integer n associated with a solution curve is called its *winding number*.

Theorem B follows.

4. Evolution of global solution curves and the proof of Theorems C and D.

A. Examples of Some Solution Curves and Their Winding Numbers.

EXAMPLE 4.1. Let Γ_0 be the unique global solution curve with $\theta(0) = \pi/6$, $\pi/12$, and μ for $g = 3, 6, 4$ respectively, where

$$\frac{\cot \mu - \tan \mu}{\cot(\mu - \pi/4) - \tan(\mu - \pi/4)} = -\frac{l}{k};$$

$\sigma(0) = \theta(0) + \pi/2$; $r(0) = \delta$. For sufficiently small δ , there is a small $s < 0$ such that $\Gamma_0(s)$ is a y -local minimum with $\sigma = \pi$, $J < 0$. Therefore no loops are formed for $s < 0$. Similar behavior is observed relative to the boundary $\theta = \pi/g$. Hence $n(\Gamma_0) = 0$.

EXAMPLE 4.2. As $\delta \rightarrow 0$ in the last example, the solution curve tends to γ_0 with $r(0) = 0$ and $n(\gamma_0) = 1$. Let γ be a curve with a point on the boundary $\theta = 0$. Deforming γ through a one-parameter family of curves with cusps towards γ_0 , we obtain a curve Γ_1 with a cusp on $\theta = 0$ and $n(\Gamma_1) = 1$. Similarly we get a Γ'_1 with a cusp on $\theta = \pi/g$ and $n(\Gamma'_1) = 1$.

EXAMPLE 4.3. The curve Γ_2 of Lemma 3.7 has exactly two cusps and $n(\Gamma_2) = 2$. The segment between the cusps represents an embedded hypersphere of constant mean curvature. By the theorem of Alexandrov [2], Γ_2 is the unique curve with this property.

LEMMA 4.4. *For every integer $N \geq 0$, there exists a global solution curve γ with $n(\gamma) \geq N$.*

Proof. We give a proof for $g = 3$. The other cases are similar. For $r > 9$ and $\pi/2 \leq \theta \leq \pi/4$,

$$\begin{aligned} |\dot{\sigma} - (3k + 1)| &< \frac{k}{r} \left(\left| \frac{\cos \sigma}{\theta} \right| + \left| \frac{\sin(\sigma + \pi/6)}{\cos(\theta + \pi/6)} \right| + \left| \frac{\sin(\sigma - \pi/6)}{\cos(\theta - \pi/6)} \right| \right) \\ &< \frac{k}{r} \left\{ \frac{1}{\sin 15^\circ} + \frac{1}{\cos 75^\circ} + \frac{1}{\cos 15^\circ} \right\} \leq k. \end{aligned}$$

Consider a solution curve γ with $r(0) = R$ and $\theta(0) = \pi/6$. Choose R so large that $R > 9$ and $(2k + 1)/2R \geq (2N - 1)\pi + \pi/3$.

By Theorem A, there exist $s_1 < 0$ such that $\theta(s_1) = \pi/12$, and $s_2 > 0$ such that $\theta(s_2) = \pi/4$. Then

$$\begin{aligned} [\Delta\sigma]_{s_1}^{s_2} &= \int_{s_1}^{s_2} \dot{\sigma} ds \geq \int_{s_1}^{s_2} (2k + 1) ds \geq (2k + 1)2R \sin \frac{\pi}{12} \\ &\geq \frac{(2k + 1)}{2} R \geq (2N - 1)\pi + \frac{\pi}{3}. \end{aligned}$$

Since the situation of Figure 3 is untenable on any part of the curve γ , it cannot “unwind” beyond $s = s_1$ and $s = s_2$. Hence $n(\gamma) \geq N$.

B. Deformation of Global Solution Curves.

We shall now obtain all the curves stated in Theorem C by deformation of known examples, following the method of §3 [I], which we summarize below. It is important to know that the winding number cannot be changed except by a transition through boundary points. The behavior of neighboring curves near a cusp is portrayed in Figure 4.

(1) We start with a curve with a large winding number N guaranteed by Lemma 4.4. It may have no cusps, one cusp, or two cusps.

(2) If it has no cusps, then by deforming it through a one-parameter family towards Γ_0 , we obtain as an intermediate curve a γ with $n(\gamma) = N$ and γ contains one or two cusps.

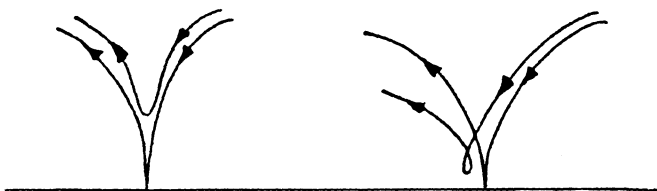


FIGURE 4

(3) If γ has one cusp, by deforming it through a one-parameter family of curves, each having one cusp, towards Γ_1 or Γ'_1 , we obtain a γ' with $n(\gamma') = N$ and two cusps.

(4) Deform γ' further along curves having one cusp towards Γ_1 or Γ'_1 , we obtain γ'' with $n(\gamma'') = N - 1$ and one cusp.

(5) We may obtain curves with two cusps and a smaller winding number by repeating (3).

(6) Let γ be a curve with one cusp, then by a small deformation near the cusp, we get a curve γ' with no cusp and $n(\gamma') = n(\gamma) - 1$.

These deformations, together with the examples 4.1, 4.2, 4.3 and 4.4, prove Theorem C. The segment of a solution curve between two cusps generates an immersion of S^{n+1} into E^{n+2} with constant mean curvature 1. Examples with unequal winding numbers are clearly non-congruent. This proves Theorem D.

For the topological types of the hypersurfaces generated by other types of curves, we refer the reader to [I].

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Received May 19, 1986 and in revised form November 19, 1986.

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