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RESTRICTION TO $\mathrm{GL}_2(\mathcal{O})$ OF SUPERCUSPIDAL REPRESENTATIONS OF $\mathrm{GL}_2(F)$

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Let F be a p-field with ring of integers $\mathcal{O}$ whose maximal prime ideal is $\mathfrak{p} = \omega\mathcal{O}$, and with finite residue field $\mathfrak{k} = \mathcal{O}/\mathfrak{p}$. Let $G = \mathrm{GL}_2(F)$ and let K be the subgroup $\mathrm{GL}_2(\mathcal{O})$ of G. In this paper we obtain the decomposition of the restriction to K of any irreducible supercuspidal representation of G. (The corresponding result for unitary representations, $G = \mathrm{PGL}_2$, and $\mathfrak{k}$ of characteristic $\neq 2$ was found by Silberger. Here we make no assumption on the characteristic of $\mathfrak{k}$.) It is well-known that any irreducible supercuspidal representation of G is admissible and hence decomposes as a direct sum of irreducible K-types, each of which appears with finite multiplicity. Here we show that, in fact, each of these irreducible components occurs with multiplicity one, and we give an explicit description of each component.

This work is based upon results of Kutzko, who proved that any irreducible supercuspidal representation of G is twist-equivalent to another such representation which, in turn, may be compactly induced from one of two compact-modulo-center subgroups of G.

Introduction. Let F be a p-field with ring of integers $\mathcal{O}$ whose maximal prime ideal is $\mathfrak{p} = \omega\mathcal{O}$, and with finite residue field $\mathfrak{k} = \mathcal{O}/\mathfrak{p}$. Let $G = \mathrm{GL}_2(F)$ and let K be the subgroup $\mathrm{GL}_2(\mathcal{O})$ of G. In this paper we obtain the decomposition of the restriction to K of any irreducible supercuspidal representation of G. (The corresponding result for unitary representations, $G = \mathrm{PGL}_2$, and $\mathfrak{k}$ of characteristic $\neq 2$ was found by Silberger in [**Si2**]. Here we make no assumption on the characteristic of $\mathfrak{k}$.) It is well-known that any irreducible supercuspidal representation of G is admissible and hence decomposes as a direct sum of irreducible K-types, each of which appears with finite multiplicity. Here we show that, in fact, each of these irreducible components occurs with multiplicity one, and we give an explicit description of each component.

This work is based upon results of Kutzko ([**K3**] and [**K4**]), who proved that any irreducible supercuspidal representation of G is twist-equivalent to another such representation which, in turn, may be compactly induced from one of two compact-modulo-center subgroups of G. I would like to thank Philip Kutzko, my thesis advisor, for his inspiration and guidance of this work. I would also like to express my appreciation to Paul Sally for his preliminary reading of and comments on this paper.

Some notation and facts. We need some notation in addition to that established above. We make the convention that for any subsets A, B, C, and D, $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ denotes the set of elements $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with a in A, b in B, c in C and d in D.

Let $\mathscr{k}$ be the residue field $\mathscr{O}/\mathscr{p}$ and let $|\mathscr{k}| = q$. The additive group F^+ has a filtration $\cdots \mathscr{p}^{-1} \supset \mathscr{O} = \mathscr{p}^0 \supset \mathscr{p}^1 \supset \mathscr{p}^2 \supset \cdots$ with $[\mathscr{p}^s : \mathscr{p}^t] = q^{t-s}$ for $t \geq s$. If $U = \mathscr{O} - \mathscr{p}$, the group of units in $\mathscr{O}$, then U has a corresponding filtration $U \supset U^1 \supset U^2 \supset \cdots$, where $U^i = 1 + \mathscr{p}^i$, and we have $[U : U^1] = q - 1$ and $[U^s : U^t] = q^{t-s}$ for $t \geq s \geq 1$.

Let P be the standard parabolic subgroup of G, and let N be its unipotent radical. (That is, $P = \begin{pmatrix} F^* & F \\ 0 & F^* \end{pmatrix}$ and $N = \begin{pmatrix} 1 & F \\ 0 & 1 \end{pmatrix}$, where F^* denotes the group of units of F.) Then N is isomorphic to the additive group F^+ and has a corresponding filtration $\cdots N_{-1} \supset N_0 \supset N_1 \supset \cdots$, where $N_t = \begin{pmatrix} 1 & \mathscr{p}^t \\ 0 & 1 \end{pmatrix}$. Let $P_0 = P \cap \mathrm{Gl}_2(\mathscr{O}) = \begin{pmatrix} U & \mathscr{O} \\ 0 & U \end{pmatrix}$; then $N_0 = N \cap P_0$.

Our matrix groups have similar filtrations. If $M = M_2(\mathscr{O})$, then we have a filtration $M_0 = M \supset M_1 \supset M_2 \cdots$, where $M_i = \mathscr{p}^i M$ for each nonnegative integer i. If we let M' be the set of matrices in M which are upper triangular modulo $\mathscr{p}$, then M' also has a filtration given by

$$M_0' = M' \supset M_1' \supset M_2' \cdots, \quad \text{where} \quad M_i' = \begin{pmatrix} \mathscr{p}^{i_2} & \mathscr{p}^{i_1} \\ \mathscr{p}^{i_1+1} & \mathscr{p}^{i_2} \end{pmatrix}$$

$$\text{for } i_1 = [i/2] \text{ and } i_2 = [(i+1)/2],$$

where the brackets denote the greatest-integer function. The filtration of M defines a corresponding filtration $K_0 = K \supset K_1 \supset K_2 \cdots$ of $K = \mathrm{Gl}_2(\mathscr{O})$, where $K_i = I + M_i$, for each $i \geq 1$. We note that $K/K_1 \cong \mathrm{Gl}_2(\mathscr{k})$, so that $[K : K_1] = (q-1)^2 q(q+1)$, and that $[K_s : K_t] = [M_s : M_t] = q^{4(t-s)}$ if $t \geq s \geq 1$. We also consider the subgroup B of K consisting of those matrices in K which are upper triangular modulo $\mathscr{p}$. B has a filtration $B_0 = B \supset B_1 \supset B_2 \cdots$ corresponding to that of M', where $B_t = I + M_i'$, for each $i \geq 1$. We note that $[B : B_1] = [U : U^1]^2 = (q-1)^2$, and $[B_s : B_t] = [\mathscr{O} : \mathscr{p}]^2 = q^2$ for each $t \geq s \geq 1$, so this filtration of B is roughly twice as fine as that of K. Finally, we define the subgroups Z and Z' of G to be the center $Z(G)$ of G and the subgroup $\{\omega^i I \mid i \text{ an integer}\}$ of $Z(G)$, respectively, and the subgroup Z_0 of K to be the center $Z(K)$ of K.

At times in what follows we refer to conjugate groups of various groups. If J is a subgroup of G, we define the conjugate group J^γ for γ in G to be $\{\gamma j \gamma^{-1} \mid j \text{ in } J\}$. In particular, we frequently refer to conjugate groups J^w; here w is the Weyl element $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

We also frequently use the notation $P \backslash Q/R$. If P and R are subgroups of the group Q, then $P \backslash Q/R$ refers to a complete set of (P, R) double-coset representatives in Q.

Level and twist-equivalence. Given a representation σ of K or ZK (respectively, B or $Z'B$), because σ is locally constant, there is a minimal number n such that K_n (respectively, B_n) is contained in the kernel of σ. This integer n is called the K-level (respectively, B-level) of σ. For such a representation σ, it is clear that the restriction of σ to N_n (respectively, N_{n_1}) decomposes as a direct sum of identity representations. σ is defined to be cuspidal if its restriction to N_0 contains no nontrivial identity component, so that there is a minimal number m with $0 < m \leq n$ (respectively, $0 < m \leq n_1$) such that the restriction of σ to N_m contains a nontrivial identity component. In this case, the nonnegative integer $n - m$ (respectively, $n_1 - m$) is called the K-defect (respectively, B-defect) of σ.

Levels are defined similarly for a supercuspidal representation τ of G; however, in this case, because τ is admissible, it can contain no group K_n or B_n in its kernel. Here we define the K-level (respectively, B-level) of τ to be the minimal integer n such that the subspace of vectors fixed under τ by K_n (respectively, B_n) is nontrivial. It is a fact [**Bo**] that any supercuspidal representation of G has K-level at least 1 and B-level at least 2.

Let π and τ be representations of G. We say that π and τ are twist-equivalent if there is a quasicharacter χ of the multiplicative group F^* of F such that π is isomorphic to $\tau \otimes \chi \circ \det$. Twist-equivalence is an equivalence relation on the set of (equivalence classes) of representations of G, and if π and τ are twist-equivalent, then they share the same irreducible subspaces. Moreover, π is smooth (respectively, admissible, supercuspidal) if and only if τ is.

If π is a supercuspidal representation, we define the minimal level of π to be the minimum of the levels of all representations τ which are twist-equivalent to π. We say that π is of minimal level if its level is equal to its minimal level. In this paper, we obtain an explicit decomposition of the restriction to K of any supercuspidal representation π which is of minimal level; the remarks above show that there is no loss of generality in placing this added assumption on π.

Supercuspidal representations of $\mathrm{GL}_2(F)$. The results of this paper are based on work which appears in two papers of Kutzko ([**K3**] and [**K4**]). In the first, he proves that cuspidal representations of ZK or $Z'B$

compactly induce to supercuspidal representations of G, that such representations which are both irreducible and of defect 0 compactly induce irreducible representations, and finally, that each irreducible supercuspidal representation of G is induced uniquely in this fashion. In the second paper, he gives explicit descriptions of the inducing representations. The work of this paper is based upon these descriptions, and we begin with a recounting of them.

To commence, we shall, as previously indicated, fix an arbitrary irreducible supercuspidal representation (π, V), which we without loss of generality assume to be of minimal level l.

Level 1. Suppose first that $l = 1$, so that the subspace V_1 of V consisting of those vectors in V which are fixed under π by K_1 is nontrivial. Because K_1 is normal in K, V_1 is a K-subspace of V; let π_1 denote the restriction to V_1 of $\pi|_K$. If σ is any irreducible K-subrepresentation of π_1, then clearly $i_{N_1}(\sigma, 1) \neq 0$. If in addition we had $i_{N_0}(\sigma, 1) \neq 0$, then because $B_1 = K_1 N_0$, σ would contain a B-subrepresentation of level 1, but this is impossible because the B-level of π is at least 2. Thus we must have $i_{N_0}(\sigma, 1) \neq 0$, whence σ is a cuspidal representation of defect 0.

Let W be a subspace of V_1 where π_1 acts as σ. Then since the group Z is contained in the center $Z(G)$, W is in fact a ZK-subrepresentation of π. Thus if τ denotes the restriction to W of $\pi|_{ZK}$, τ is also irreducible and cuspidal of defect 0, and Kutzko's first paper shows that the compactly induced representation τ^G is irreducible. Since, by Frobenius reciprocity (which may be proved in this case as a corollary to Kutzko's generalized Mackey's theorem, [**K1**]), π is a subrepresentation of τ^G, this irreducibility implies that these two representations are in fact isomorphic.

Now we consider the case of supeı cuspidal representations of minimal level $l > 1$. These representations may be divided into two categories, depending upon whether they are compactly induced from a representation of ZK or of $Z'B$; those in the first category are called unramified, and those in the second, ramified (for reasons which will be seen later). Kutzko's descriptions of the inducing representations for these supercuspidal representations fall into three categories, depending upon whether the supercuspidal representation is unramified of even level, unramified of odd level, or ramified.

Level > 1. Suppose that the representation π has level $l > 1$. As in the case that $l = 1$, the subspace V_l of vectors in V which are fixed under π by K_l is a nontrivial finite-dimensional K-subspace of V, and we let π_l denote the restriction to V_l of $\pi|_K$. Let $l_1 = [l/2]$ and $l_2 = [(l + 1)/2]$, as above. Then the restriction of π_l is a representation of K_{l_2} which factors through K_{l_2}/K_l. This quotient group is isomorphic to the abelian group M/M_{l_1} under the mapping $(1+\omega^{l_2}A)/K_l \to A + M_{l_1}$. It follows that all of the irreducible subrepresentations of the restriction of π_l to K_{l_2} are one-dimensional. In fact, the existence of this isomorphism implies that every irreducible K_{l_2}-subrepresentation of π_l is of the form ψ_A for some A in M, where $\psi_A(k) = \psi(\omega^{-l}\operatorname{Tr}(A(k-I)))$, where ψ is a fixed character of F^+ which is trivial on $\mathcal{O}$ but not on $\mathfrak{p}^{-1}$. It is easy to show that $\psi_A = \psi_B$ if and only if $A - B$ lies in M_{l_1} and that for any k in K, the conjugate representation ψ_A^k is isomorphic to $\psi_{kAk^{-1}}$. The latter fact implies that π_l has an irreducible K_{l_2}-subrepresentation ψ_A if and only if it also contains a subrepresentation ψ_B for any matrix B in M which is K-similar to A.

We proceed by considering a matrix A such that ψ_A is a subrepresentation of the restriction of π_l to K_{l_2}. If $\bar{A}$ denotes the image of A under the canonical epimorphism of M onto $M_2(\mathfrak{k})$, and if $\bar{A}$ is similar to $\bar{B}$ in $M_2(\mathfrak{k})$, then A is K-similar to a matrix B in M with image $\bar{B}$, and ψ_B is also a K_{l_2}-subrepresentation of π_l. This means that we may without loss of generality assume that $\bar{A}$ is in Jordan canonical form. Let χ_A denote the characteristic polynomial of $\bar{A}$ in $M_2(\mathfrak{k})$. The cases that π is unramified or ramified correspond, respectively, to the cases that χ_A is irreducible or reducible.

Unramified case. In this case, there are $\bar{s}$ and $\bar{\Delta}$ in $\mathfrak{k}$ with $\chi_A(x) = x^2 - \bar{s}x + \bar{\Delta}$, an irreducible polynomial in $\mathfrak{k}[x]$ (so $\bar{\Delta} \neq 0$). In this case, $A = \left(\begin{smallmatrix} 0 & 1 \\ -\Delta & s \end{smallmatrix}\right) + \omega C$, for some preimages s and Δ in $\mathcal{O}$ of $\bar{s}$ and $\bar{\Delta}$, and some matrix C in M. It is easily shown that A is then K-similar to a matrix $B = \left(\begin{smallmatrix} 0 & 1 \\ -\Delta' & s' \end{smallmatrix}\right)$, where s' and Δ' are also preimages in $\mathcal{O}$ of $\bar{s}$ and $\bar{\Delta}$, so we without loss of generality assume that $C = 0$.

It is then easy to show that the stabilizer in K of ψ_A is the subgroup $U_E K_{l_1}$, where U_E denotes the group of units in the subalgebra $\mathcal{O}_E = \mathcal{O}[A]$ of M generated by A. (Note that if we let E be the subalgebra $F[A]$ of $M_2(F)$, then because χ_A is irreducible, E is a p-field which is unramified of degree 2 over F (see [S]), with ring of integers $\mathcal{O}_E$; this explains the terminology "unramified" used for this case.)

Unramified, even level case. First suppose that π has even level, so that $l = 2m$ $(m \geq 1)$, and ψ_A is a representation of K_m factoring through K_m/K_{2m} with stabilizer $U_E K_m$. In this case, because $U_E \cap K_m = U_E^m$, the restriction to U_E^m of ψ_A has an extension $\psi_A^{\sim}$ to U_E, and for each λ in

$\Lambda = (U_E/U_E^m)^\wedge$ we get a well-defined representation $\psi_{A,\lambda}$ of U_EK_m by defining $\psi_{A,\lambda}(uk) = \lambda(u)\psi_A^\sim(u)\psi_A(k)$ for each u in U_E and k in K_m.

Let σ be an irreducible K-subrepresentation of π_l such that the restriction of σ to K_m contains the subrepresentation ψ_A. Then by Frobenius reciprocity, σ is a subrepresentation of the induced representation ψ_A^K. However, by Clifford's theorem, ψ_A^K is isomorphic to the direct sum representation $\sum_{\lambda \in \Lambda} \psi_{A,\lambda}^K$, each of whose summands is an irreducible representation of K. Thus σ is isomorphic to $\psi_{A,\lambda}^K$ for some λ in Λ.

Let W be an irreducible subspace of V_l where π_l acts as σ. As in the level one case, W is a ZK-subspace of V, and if we let τ denote the restriction to W of $\pi|_{ZK}$, τ is irreducible. The fact that σ is isomorphic to $\psi_{A,\lambda}^K$ implies that τ is cuspidal, of level l and defect 0, and as before, we find that π is isomorphic to the compactly-induced representation τ^K.

Unramified, odd level case. Suppose next that π has odd level, so that $l = 2m + 1$ and ψ_A is a representation of K_{m+1} factoring through K_{m+1}/K_{2m+1}. As in the case above, ψ_A may be extended to U_EK_{m+1}, but in this case U_EK_m is the stabilizer of ψ_A in K. In this case we make use of the following filtration of subgroups:

$$\begin{tikzcd}[arrows=dash]
U_EK_m \arrow[dddd] \arrow[dr] & \\
& U_FU_E^1K_m \arrow[d] \\
& U_FU_E^1K_{m+1}N_m^w \arrow[d] \\
& \\
U_EK_{m+1} \arrow[r] & U_FU_E^1K_{m+1}
\end{tikzcd}$$

We note first that there exists an extension ψ_A' to $U_FU_E^1$ of the restriction of ψ_A to $U_FU_E^1 \cap K_{m+1} = U_E^{m+1}$. Then as above, each representation λ' in $(U_FU_E^1/U_E^{m+1})^\wedge$ determines a representation $\psi_{A,\lambda'}$ of $U_FU_E^1K_{m+1}$ which extends ψ_A. Each representation ψ_A' has in turn an extension $\psi_A^\sim$ to U_E, and for λ in $U_E^\wedge$ which extend λ', we get representations $\psi_{A,\lambda}$ which extend $\psi_{A,\lambda'}$ to U_EK_{m+1}. On the other hand, $K_{m+1}N_m^w = B_l^w$ is a subgroup of K_m for which the formula $\psi_A(k) = \psi(\omega^{-l}\operatorname{Tr} A(k - I))$ defines a representation, and since $U_FU_E^1 \cap B_l^w = U_E^{m+1}$, there are extensions $\psi_{A,\lambda',\gamma}$ of $\psi_{A,\lambda'}$ to $U_FU_E^1B_l^w$ corresponding to γ in $(N_m^w/N_{m+1}^w)^\wedge$. Each of these representations induces to a unique irreducible representation $\zeta_{A,\lambda'}$ of $U_FU_E^1K_m$ (independently of γ), and each induced representation $\zeta_{A,\lambda'}^{U_EK_m}$ is isomorphic to a direct sum of irreducible representations $\sum_{\lambda \text{ extending } \lambda'} \xi_{A,\lambda}$, each summand of which is completely determined as the complement in $\zeta_{A,\lambda'}^{U_EK_m}$ of the induced representation $\psi_{A,\lambda'}^{U_EK_m}$, for some extension λ of λ' to U_E, where distinct extensions λ determine nonisomorphic representations $\xi_{A,\lambda}$, and where each $\xi_{A,\lambda}$ is actually an extension of $\zeta_{A,\lambda'}$ to $U_E K_m$. It follows from Clifford's theorem that the induced representation $\psi_A^{U_E K_m}$ is isomorphic to q copies of the direct sum $\sum_{\lambda'} \sum_{\lambda \text{ extending } \lambda'} \xi_{A,\lambda}$, and that ψ_A^K is isomorphic to q copies of $\sum_{\lambda'} \sum_\lambda \xi_{A,\lambda}^K$, all of whose summands are irreducible.

Now if σ is an irreducible K-subrepresentation of π_l such that the restriction of σ to K_{m+1} contains ψ_A, then by Frobenius reciprocity, σ must be isomorphic to one of the representations $\xi_{A,\lambda}^K$. If, as before, we let W denote a subspace of V_l where π_l acts as σ, then W is a ZK-subspace of V, and if τ denotes the restriction to W of $\pi|_{ZK}$, then τ is irreducible. Because σ is isomorphic to $\xi_{A,\lambda}^K$ and $\xi_{A,\lambda}$ restricts to $\zeta_{A,\lambda'}$ on $U_F U_E^1 K_m$, τ is cuspidal, of level l and defect 0, and as before, we find that π is isomorphic to the compactly-induced representation τ^G.

Ramified case. Finally, we consider the ramified case. Here, for any subrepresentation ψ_A of the restriction to K_{l_2} of π_l, we have $\chi_A(x) = (x - \bar{a})(x - \bar{b})$, a reducible polynomial in $\mathscr{k}[x]$. Kutzko has shown **[K2]** that because π is supercuspidal, we cannot have $\bar{a} \neq \bar{b}$, so in fact, $\chi_A(x) = (x - \bar{a})^2$ and we can assume that $\bar{A} = \left(\begin{smallmatrix} \bar{a} & \delta \\ 0 & \bar{a} \end{smallmatrix}\right)$, where δ is either 0 or 1. Noting $\psi_{aI} = \chi_a \circ \det$, where χ_a is a character of U_{l_2}, we see that if $\bar{B} = \left(\begin{smallmatrix} 0 & \delta \\ 0 & 0 \end{smallmatrix}\right)$ and B is any preimage of $\bar{B}$ in M, then $\psi_A = \psi_B \otimes \chi_a \circ \det$. Thus if $\chi_a^{\sim}$ is an extension of χ_a to F^*, and $\xi = \pi \otimes \chi_a^{\sim} \circ \det$, then ξ is twist-equivalent to π and ψ_B is a K-subrepresentation of ξ. Since, as previously mentioned, twist-equivalent representations have corresponding irreducible subrepresentations on common subspaces, we without loss of generality assume in what follows that $\pi = \xi$ and $\bar{A} = \bar{B}$. Moreover, if $\delta = 0$, then ψ_A is trivial on K_{l-1}, contradicting the fact that $V_{l-1} \neq 0$, so we have $\delta = 1$.

The form of $\bar{A}$ implies that A is K-similar to a matrix $\left(\begin{smallmatrix} 0 & 1 \\ -\Delta & s \end{smallmatrix}\right)$ with Δ and s in $\mathscr{p}$, so we assume that A is equal to this matrix. Then ψ_A is trivial on the subgroup B_{2l-2} of B, and if we let W_j be the space of vectors in V fixed under π by B_j, then $W_{2l-2} \neq 0$. (Recall that l must be at least 2.) Because B_j is normal in B, each space W_j is a B-subspace of V, and we let π_j' denote the restriction to W_j of $\pi|_B$. Then the restriction to B_{l-1} of π_{2l-2}' factors through the group B_{l-1}/B_{2l-2}, which is isomorphic to the abelian group M_{l-1}'/M_{2l-2}', and this implies that the representation decomposes as a direct sum of one-dimensional representations ψ_D where, as before, $\psi_D(b) = \psi(\omega^{-l} \operatorname{Tr} D(b - I))$ but now D must lie in M_1' and $\psi_{D_1} = \psi_{D_2}$ if and only if $D_1 - D_2$ lies in M_l'. Because $B_{l-1} \supset K_{l_2}$, there must be some D in M_1' such that the restriction $\psi_D|_{K_{l_2}}$ is equal to ψ_A.

This implies that $D - A \in M_{l_1}$, so that D is B-similar to $D' = \left(\begin{smallmatrix} 0 & 1 \\ -\Delta' & s' \end{smallmatrix}\right)$ where Δ' and s' lie in $\mathcal{O}$ and $\Delta' \equiv \Delta$ and $s' \equiv s$ (modulo $\wp^{l_1}$), and we can without loss of generality assume $D = D'$. Furthermore, since $\psi_A = \psi_D$ on K_{l_2}, we can assume that $A = D' = D$. Suppose that $\Delta \in \wp^2$. Then if $g = \left(\begin{smallmatrix} 0 & \omega^{-1} \\ 1 & 0 \end{smallmatrix}\right)$, gAg^{-1} lies in $\wp M$, so that ψ_A^g, also a subrepresentation of the restriction of π'_{2l-2}, is trivial on K_{l-1}. But this contradicts the fact that $V_{l-1} \neq 0$, so $\Delta \in \wp - \wp^2$.

This implies that the characteristic polynomial χ_A of $\bar{A}$ is an Eisenstein polynomial and hence is irreducible. This in turn implies that if $E = F[A]$, the subalgebra of $M_2(F)$ generated by A, then E is a p-field **[Se]**, ramified of degree 2 over F, and it is easy to show that $\wp_E^n = E \cap M'_n$ and $U_E^n = U_E \cap B_n$. This case is similar to the unramified even-level case in that we can show that the stabilizer of ψ_A in B is B_{l-1}. Defining an extension ψ'_A to U_E of the restriction of ψ_A to U_E^{l-1} as in that case, we get a representation $\psi_{A,\lambda}$ on $U_E B_{l-1}$ extending ψ_A for each representation λ in $(U_E/U_E^{l-1})^\wedge$. Clifford's theorem then implies that the induced representation ψ_A^B decomposes as a direct sum of nonisomorphic irreducible representations $\sum \psi_{A,\lambda}^B$, and as before, $\pi|_B$ must contain a subrepresentation σ isomorphic to $\psi_{A,\lambda}^B$ for some λ. If W is a subspace of V where $\pi|_B$ acts by σ, then σ has an extension τ to $Z'B$, and again we find that π is isomorphic to the compactly induced representation τ^G.

Decomposition of the representation. We can now begin to decompose the representation $\pi|_K$, as desired. In each case above, π is compactly induced from an irreducible representation τ of the subgroup L of G, where $L = ZK$ in the level 1 and unramified cases, and $L = Z'B$ in the ramified case. Using this fact we can find an initial decomposition of $\pi|_K$, as follows.

THEOREM 1. *If π is isomorphic to the compactly induced representation τ^G, where τ is a cuspidal representation of L (equal to either ZK or $Z'B$), then the restriction $\pi|_K$ is isomorphic to the direct sum of induced representations $\sum_{\eta \in K\backslash G/L} (\tau^\eta|_{K \cap L^\eta})^K$.*

Proof. Because π is admissible, we know **[Si1]** that the restriction $\pi|_K$ decomposes as a direct sum of irreducible K-types, each occurring with finite multiplicity. This means that $\pi|_K$ is completely determined by the intertwining numbers $i_K(\delta, \pi|_K)$, for irreducible representations δ of K, and it suffices to show that

$$i_K(\delta, \pi\,|_K) = i_K\left(\delta, \sum_{\eta \in K\backslash G/L} (\tau^\eta\,|_{K \cap L^\eta})^K\right)$$

$$= \sum_{\eta \in K\backslash G/L} i_K\left(\delta, (\tau^\eta\,|_{K\cap L^\eta})^K\right),$$

for each such δ. By using Kutzko's generalization of Mackey's theorem **[K1]**, we can prove a version of Frobenius reciprocity **[Ha]** which implies that $i_K(\delta, \pi\,|_K) = i_K(\delta, \tau^G\,|_K) = \sum_{\eta \in K\backslash G/L} i_{K\cap L^\eta}(\delta, \tau^\eta)$. On the other hand, using Frobenius reciprocity for compact groups, we find that $i_K(\delta, (\tau^\eta\,|_{K\cap L^\eta})^K) = i_{K\cap L^\eta}(\delta, \tau^\eta)$. □

The next lemmas allow us to write the direct sum in Theorem 1 more explicitly. Their proofs are quite simple and are omitted. For each integer $t \geq 0$, let $\eta_t = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \omega^t \end{smallmatrix}\right)$.

LEMMA 1. *If $L = ZK$ or $Z'B$, then $K \backslash G/L$ may be taken to be the set* $\{\eta_t\,|\,t = 0, 1, 2, \ldots\}$.

LEMMA 2. *If $L = ZK$, then $K \cap L^{\eta_t} = K \cap K^{\eta_t}$. If $L = Z'B$, then $K \cap L^{\eta_t} = K \cap B^{\eta_t}$.*

LEMMA 3. *If $t = 0$, then $K \cap K^{\eta_t} = K$ and $K \cap B^{\eta_t} = B$. If $t \geq 1$, then $K \cap K^{\eta_t} = K \cap B^{\eta_t} = P_0 K_t$, the set of matrices in K which are upper triangular modulo $\mathfrak{p}^t$.*

Recall that the representation τ of L extends a representation σ of a subgroup J of L, where $J = K$ when $L = ZK$ and $J = B$ when $L = Z'B$. This fact and the last three lemmas imply that the initial decomposition of $\pi\,|_K$ which is given in Theorem 1 may be rewritten as a countable direct sum as follows: $\pi\,|_K \cong \sigma^K \oplus \sum_{t \geq 1} (\sigma^{\eta_t}\,|_{P_0 K_t})^K$. The rest of this paper is devoted to proving that our decomposition of $\pi\,|_K$ is now complete; i.e., that each summand in the direct sum above is in fact irreducible. We proceed with a case-by-case analysis.

Level 1. Assume again that $l = 1$ so that, as seen above, π is compactly induced from the representation τ of ZK extending the representation σ of K, where σ is irreducible and cuspidal of defect 0. Because σ factors through K/K_1 and this group is isomorphic to $\mathrm{GL}_2(\mathfrak{k})$, σ determines an irreducible representation $\bar{\sigma}$ of the latter group. The fact that σ is cuspidal implies that $\bar{\sigma}$ is also: i.e., if $\overline{N}$ denotes the group of upper triangular matrices in $\mathrm{GL}_2(\mathscr{k})$, then $i_{\overline{N}}(\bar{\sigma}, 1) = 0$.

The irreducible cuspidal representations of $\mathrm{GL}_2(\mathscr{k})$ are well-known, so we have explicit formulas for the action of $\bar{\sigma}$. Specifically, $\bar{\sigma}$ has dimension $q-1$ and there exists a character ρ of $\mathscr{k}^*$ such that the following formulas hold for the character $\chi(\bar{\sigma})$ of $\bar{\sigma}$:

$$\chi(\bar{\sigma})\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} = 0, \quad \text{for all } c \text{ in } \mathscr{k} \text{ and } a \neq d \text{ in } \mathscr{k}^*,$$

$$\chi(\bar{\sigma})\begin{pmatrix} a & 0 \\ c & a \end{pmatrix} = -\rho(a), \quad \text{for all } c \text{ in } \mathscr{k}^* \text{ and } a \text{ in } \mathscr{k}^*, \quad \text{and}$$

$$\chi(\bar{\sigma})\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = (q-1)\rho(a), \quad \text{for all } a \text{ in } \mathscr{k}^*.$$

We use these formulas below to show that each summand $(\sigma^{\eta_t}|_{P_0K_t})^K$ is irreducible by proving that its intertwining number with itself is 1. Applying Mackey's theorem for compact groups to this representation, we find that

$$i_K\left(\left(\sigma^{\eta_t}|_{P_0K_t}\right)^K, \left(\sigma^{\eta_t}|_{P_0K_t}\right)^K\right) = \sum_{\alpha \in P_0K_t \backslash K/P_0K_t} i_{P_0K_t \cap (P_0K_t)^\alpha}\left(\sigma^{\eta_t}, (\sigma^{\eta_t})^\alpha\right).$$

The formulas above with the lemmas below allow us to compute this number. For each positive integer i, let $\alpha_i = \left(\begin{smallmatrix} 1 & 0 \\ \omega^i & 1 \end{smallmatrix}\right)$.

LEMMA 4. *For each positive integer t, $P_0K_t \backslash K/P_0K_t$ may be taken to be the set $\{I, w\} \cup \{\alpha_i | 0 < i < t\}$, where w denotes the Weyl element.*

Proof. If $\overline{P}$ denotes the set of upper triangular matrices in $\overline{G} = \mathrm{GL}_2(\mathscr{k})$, then it is well-known that $\overline{P}\backslash\overline{G}/\overline{P}$ may be taken to be $\{I, w\}$. Since K_1 is normal in K, this implies that for any k in K, there are elements p_1 and p_2 in P_0 such that p_1kp_2 is equal to either k_1 or wk_1, where k_1 lies in K_1. Moreover, if $k_1 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\gamma = \left(\begin{smallmatrix} 1 & 0 \\ ca^{-1} & 1 \end{smallmatrix}\right)$, then $k_1 \in \gamma P_0$, so that either γ or $w\gamma$ lies in the P_0K_t double coset of k in K. In the first case, if c has valuation i in F (so $i \geq 1$), then there exists u in U such that $ca^{-1} = \omega^i u$, so that $\gamma = \left(\begin{smallmatrix} 1 & 0 \\ 0 & u \end{smallmatrix}\right)\alpha_i\left(\begin{smallmatrix} 1 & 0 \\ 0 & u^{-1} \end{smallmatrix}\right)$. Thus if $i < t$, then α_i lies in the P_0K_t double coset of k in K, whereas if $i \geq t$, then I does. In the second case, since $w\gamma = (w\gamma w^{-1})w$ and $w\gamma w^{-1}$ lies in P_0, we see that w lies in the double coset of k in K. □

The proof of the following lemma is straightforward and is omitted.

LEMMA 5. *For all positive integers t and i with $i < t$, $P_0K_t \cap (P_0K_t)^{\alpha_i}$ is the set*

$$\left\{ \begin{pmatrix} w & x \\ \omega^t y & w - \omega^i x + \omega^{t-i} z \end{pmatrix} \mid w \in U,\ x, y, z \in \mathcal{O} \right\}.$$

Also, $P_0K_t \cap (P_0K_t)^w$ is the set Z_0K_t.

LEMMA 6. *In the case that π has level 1, for each positive integer t, $(\sigma^{\eta_t}|_{P_0K_t})^K$ is irreducible and has degree $q^{t+1} - q^{t-1}$.*

Proof. As previously stated, we prove the irreducibility of $(\sigma^{\eta_t}|_{P_0K_t})^K$ by computing its intertwining number with itself. Applying Lemmas 4 and 5 and the equation which precedes them, we have

$$\begin{aligned} i_K\left((\sigma^{\eta_t}|_{P_0K_t})^K, (\sigma^{\eta_t}|_{P_0K_t})^K\right) &= i_{P_0K_t}(\sigma^{\eta_t}, \sigma^{\eta_t}) + i_{Z_0K_t}(\sigma^{\eta_t}, (\sigma^{\eta_t})^w) \\ &\quad + \sum_{0<i<t} i_{P_0K_t \cap (P_0K_t)^{\alpha_i}}(\sigma^{\eta_t}, (\sigma^{\eta_t})^{\alpha_i}). \end{aligned}$$

We can compute the value of each summand using the character formula for $\bar{\sigma}$ given above. Specifically, if μ is a Haar measure on $\mathcal{O}$ normalized so that $\int_{\mathcal{O}} d\mu = 1$ and inducing Haar measure μ^* on $\mathrm{Gl}_2(\mathcal{O})$, then we have:

$$\begin{aligned} i_{P_0K_t}(\sigma^{\eta_t}, \sigma^{\eta_t}) &= \mu^*(P_0K_t)^{-1} \int_{P_0K_t} |\chi(\sigma^{\eta_t})(\beta)|^2 \, d\mu^*(\beta) \\ &= q^{t+2}(q-1)^{-2} \int_U \int_{\mathfrak{p}^t} \int_{\mathcal{O}} \int_U \left| \chi(\sigma^{\eta_t}) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right|^2 \\ &\qquad \times d\mu(a)\, d\mu(b)\, d\mu(c)\, d\mu(d) \\ &= q^2(q-1)^{-2} \int_U \int_{\mathcal{O}} \int_{\mathcal{O}} \int_U \left| \chi(\sigma^{\eta_t}) \begin{pmatrix} a & b \\ \omega^t c & d \end{pmatrix} \right|^2 \\ &\qquad \times d\mu(a)\, d\mu(b)\, d\mu(c)\, d\mu(d) \\ &= q^2(q-1)^{-2} \int_U \int_{\mathcal{O}} \int_U \left| \chi(\bar{\sigma}) \begin{pmatrix} \bar{a} & 0 \\ \bar{c} & \bar{d} \end{pmatrix} \right|^2 d\mu(a)\, d\mu(c)\, d\mu(d) \\ &= q^2(q-1)^{-2} \int_U \int_{\mathcal{O}} \int_{d+\mathfrak{p}} \left| \chi(\bar{\sigma}) \begin{pmatrix} \bar{d} & 0 \\ \bar{c} & \bar{d} \end{pmatrix} \right|^2 d\mu(a)\, d\mu(c)\, d\mu(d) \end{aligned}$$

(continues)

(continued)

$$= q(q-1)^{-2}\left[\int_U \int_{\not p} |(q-1)\rho(\bar{d})|^2 \, d\mu(c)\, d\mu(d) \right.$$

$$\left. + \int_U \int_U |\rho(\bar{d})|^2 \, d\mu(c)\, d\mu(d)\right]$$

$$= q(q-1)^{-2}\left[(q-1)^2 q^{-1} + (q-1)q^{-1}\right] \int_U |\rho(\bar{d})|^2 \, d\mu(d)$$

$$= q(q-1)^{-1} \int_U |\rho(\bar{d})|^2 \, d\mu(d) = \mu(U)^{-1} \int_U |\rho(\bar{d})|^2 \, d\mu(d)$$

$$= i_U(\rho \circ \operatorname{mod} \not p, \rho \circ \operatorname{mod} \not p) = 1.$$

Similar arguments show that:

$$i_{Z_0 K_t}\left(\sigma^{\eta_t}, (\sigma^{\eta_t})^{w}\right) = 0 \quad \text{and}$$
$$i_{P_0 K_t \cap (P_0 K_t)^{\alpha_i}}\left(\sigma^{\eta_t}, (\sigma^{\eta_t})^{\alpha_i}\right) = 0, \quad \text{for each } i \text{ with } 0 < i < t.$$

Thus $i_K((\sigma^{\eta_t}|_{P_0 K_t})^K, (\sigma^{\eta_t}|_{P_0 K_t})^K) = 1$, so that $(\sigma^{\eta_t}|_{P_0 K_t})^K$ is irreducible, as claimed.

Finally,

$$\begin{aligned} \deg\left(\sigma^{\eta_t}|_{P_0 K_t}\right)^K &= [K : P_0 K_t] \deg \sigma \\ &= [K : K_t][P_0 K_t : K_t]^{-1}(q-1) \\ &= \left[(q-1)^2 q^{4t-3}(q+1)\right]\left[(q-1)^2 q^{3t-2}\right]^{-1}[q-1] \\ &= (q-1)q^{t-1}(q+1). \end{aligned}$$ □

These lemmas show that we have found the desired decomposition of $\pi|_K$. (Note that $\sigma^K = \sigma$ in this case.) We have proved:

THEOREM 2. *If π has level 1, then the sum $\sigma \oplus \Sigma_{t \geq 1}(\sigma^{\eta_t}|_{P_0 K_t})^K$ is a complete decomposition of $\pi|_K$ into irreducible K-types. The summands are of degree $q-1$ and $(q-1)q^{t-1}(q+1) = q^{t+1} - q^{t-1}$, $t = 1, 2, \ldots,$ respectively.*

We now continue with the ramified and unramified cases.

Level > 1. We return to the case that π has level $l > 1$. As seen from Theorem 1 and the remarks following Lemma 3, we have $\pi|_K \cong \sigma^K \oplus \sum_{t \geq 1}(\sigma^{\eta_t}|_{P_0 K_t})^K$. When π is unramified, as in the level 1 case, σ is a representation of K, so that $\sigma^K = \sigma$ and is irreducible. We claim that σ^K is also irreducible in the ramified case. The following lemma enables us to prove this fact.

LEMMA 7. *Let π be ramified, and define E as above. Then $B = (B \cap B^w)U_E$.*

Proof. Let a and d lie in U and b and c lie in $\mathcal{O}$, so that $aI + bA$ lies in U_E. Let N be the norm $N_{E/F}(aI + bA)$, so that N lies in U_F. Then since

$$\begin{pmatrix} a & b \\ \omega c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ [\omega c(a + sb) + \Delta bd]N^{-1} & (ad - \omega bc)N^{-1} \end{pmatrix}(aI + bA)$$

and $B \cap B^w = Z_0 K_1$, $B = (B \cap B^w)U_E$, as claimed. □

LEMMA 8. *If π is ramified, then σ^K is irreducible.*

Proof. Since σ is a representation of B, we apply Mackey's induction-restriction theorem for compact groups to find that $i_K(\sigma^K, \sigma^K) = \sum_{\gamma \in B \setminus K/B} i_{B \cap B^\gamma}(\sigma, \sigma^\gamma)$. Noting that $B = P_0 K_1$, we see that by Lemma 4, we may take $B \setminus K/B$ to be the set $\{I, w\}$. It follows that $i_K(\sigma^K, \sigma^K) = i_B(\sigma, \sigma) + i_{B \cap B^w}(\sigma, \sigma^w) = 1 + i_{B \cap B^w}(\sigma, \sigma^w)$, so it suffices to show that the latter summand is equal to 0. To prove this, we use the fact that σ is induced from the representation $\psi_{A,\lambda}$ of $U_E B_{l-1}$. Again applying Mackey's theorem and using the result in Lemma 7, we find that:

$$\begin{aligned} i_{B \cap B^w}(\sigma, \sigma^w) &= i_{B \cap B^w}\left(\psi_{A,\lambda}^B, (\psi_{A,\lambda}^w)^{B^w}\right) \\ &= i_{B \cap B^w}\left(\left[\psi_{A,\lambda}|_{T_{l-1} \cap B^w}\right]^{B \cap B^w}, \left[\psi_{A,\lambda}^w|_{T_{l-1}^w \cap B}\right]^{B \cap B^w}\right) \\ &= \sum_{\gamma \in B \cap (T_{l-1})^w \setminus B \cap B^w / T_{l-1} \cap B^w} i_{B \cap (T_{l-1})^w \cap [B^w \cap T_{l-1}]^\gamma}(\psi^\gamma, \psi^w), \end{aligned}$$

where T_{l-1} denotes the set $U_E B_{l-1}$, and $\psi = \psi_{A,\lambda}$. Furthermore, $K_{l_2} \subset U_E B_{l-1} \cap B$ and K_{l_2} is normal in K, so $K_{l_2} \subset B \cap (U_E B_{l-1})^w$ and $K_{l_2} \subset [U_E B_{l-1} \cap B^w]^\gamma$ for each γ in the index set. This and the fact that

$\psi_{A,\lambda}|_{K_{l_2}} = \psi_A$ imply that

$$i_{B\cap B^w}(\sigma,\sigma^w) \le \sum_{\gamma\in B\cap(T_{l-1})^w\backslash B\cap B^w/T_{l-1}\cap B^w} i_{K_{l_2}}(\psi_A^\gamma,\psi_A^w).$$

Note that the factorization of elements of B given in Lemma 7 implies that we can take each of the indices γ to be of the form $\left(\begin{smallmatrix}1 & 0\\ \omega c & d\end{smallmatrix}\right)$ for some c and d in $\mathcal{O}$. This implies that $\gamma A\gamma^{-1} - wAw^{-1}$ does not lie in M_{l_1}, so $\psi_A^\gamma \neq \psi_A^w$ and $i_{K_{l_2}}(\psi_A^\gamma,\psi_A^w) = 0$ for each γ. Thus $i_{B\cap B^w}(\sigma,\sigma^w) = 0$, as claimed. □

It remains to investigate the summands $(\sigma^{\eta_t}|_{P_0K_t})^K$ for $t \ge 1$. To continue, we use the fact that σ is a representation of J (equal to K or B) which is induced from a representation ρ of the subgroup H of J (equal to $U_EK_{l_1}$ or U_EB_{l-1}, respectively, where E varies in the two cases). Since $\sigma = \rho^J$, Mackey's induction-restriction theorem allows us to argue that for each $t \ge 1$,

$$\begin{aligned}\left(\sigma^{\eta_t}|_{P_0K_t}\right)^K &\cong \left(\left(\rho^{\eta_t}\right)^{J^{\eta_t}}|_{P_0K_t}\right)^K \\ &\cong \sum_{\gamma\in P_0K_t\backslash J^{\eta_t}/H^{\eta_t}} \left(\rho^{\gamma\eta_t}|_{P_0K_t\cap H^{\gamma\eta_t}}\right)^K.\end{aligned}$$

This seems to yield a contradiction of our previous claim that each of the representations $(\sigma^{\eta_t}|_{P_0K_t})^K$ is irreducible. The apparent contradiction is resolved by the following lemma.

LEMMA 9. *Let J and H be defined as above. Then for each integer $t \ge 1$, $J^{\eta_t} = (P_0K_t)H^{\eta_t}$.*

Proof. Since $P_0K_t = (P_0^wK_t)^{\eta_t}$, it is equivalent to show that $J = (P_0^wK_t)H$. In both the ramified and unramified cases, $H \supset U_E$, and we prove the stronger result that $J = (P_0^wK_t)U_E$.

Let $g = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ lie in J. In the unramified case, $J = K$, and since $\det g \notin \mathfrak{p}$, not both a and b can lie in $\mathfrak{p}_F$. Since $\mathfrak{p}_E = \mathfrak{p}_F\mathcal{O}_E$ in this case, this means that $aI + bA$ lies in U_E. In the unramified case, $J = B$, so that a must lie in U_F, and again $aI + bA$ lies in U_E. Now define x and y in $\mathcal{O}$ by $xI + yA = (cI + dA)(aI + bA)^{-1}$. Then $g = \left(\begin{smallmatrix}1 & 0\\ x & y\end{smallmatrix}\right)(aI + bA)$, so $g \in P_0^wU_E$, as claimed. □

Applying Lemma 9 to the result which precedes it and noting that $K \cap H^{\eta_t} = K \cap K^{\eta_t} \cap H^{\eta_t} = P_0K_t \cap H^{\eta_t}$, we find that $(\sigma^{\eta_t}|_{P_0K_t})^K \cong (\rho^{\eta_t}|_{P_0K_t\cap H^{\eta_t}})^K = (\rho^{\eta_t}|_{K\cap H^{\eta_t}})^K$, for each $t \ge 1$. Thus it remains to investigate the K-representations $(\rho^{\eta_t}|_{K\cap H^{\eta_t}})^K$. In view of the previous work, it is natural to first find the level of each such representation.

LEMMA 10. *With H defined as above, for each $t \geq 1$, the representation $(\rho^{\eta_t}|_{K \cap H^{\eta_t}})^K$ has level $t + l$.*

Proof. Note that in both the ramified and unramified cases, if $r = t + l_2$, then $(K_r)^{\eta_t^{-1}} \subset K_{l_2} \subset H$. Thus for any γ in K, we have $K_r = K_r^\gamma \subset (K \cap K_{l_2}^{\eta_t})^\gamma \subset (K \cap H^{\eta_t})^\gamma$. It follows that when we apply Mackey's induction-restriction theorem, we find that

$$(\rho^{\eta_t}|_{K \cap H^{\eta_t}})^K|_{K_r} \cong \sum_{\gamma \in K_r \backslash K/K \cap H^{\eta_t}} \left((\rho^{\eta_t})^\gamma|_{K_r \cap (K \cap H^{\eta_t})^\gamma}\right)^{K_r}$$
$$\cong \sum_{\gamma \in K/K \cap H^{\eta_t}} \left(\rho^{\eta_t}|_{K_r}\right)^\gamma.$$

If ρ is ramified or if π is unramified and l is even, then ρ restricts to ψ_A on K_{l_2}. If π is unramified and l is odd, then $\rho = \xi_{A,\lambda}$ on $H = U_E K_{l_1}$ and restricts to $\zeta_{A,\lambda'} = (\psi_{A,\lambda',1})^{U_F U_E^1 K_{l_1}}$ on $U_F U_E^1 K_{l_1}$, and since $U_E K_{l_1}$ is the stabilizer of ψ_A, by Mackey's induction-restriction theorem, we have:

$$\rho|_{K_{l_2}} \cong \sum_{\alpha \in K_{l_2} \backslash U_F U_E^1 K_{l_2}/U_F U_E^1 B_l^w} \psi_{A,\lambda',1}^{\alpha}|_{K_{l_2}}$$
$$\cong \sum_{\alpha \in U_F U_E^1 K_{l_2}/U_F U_E^1 B_l^w} \left(\psi_{A,\lambda',1}|_{K_{l_2}}\right)^\alpha = q\psi_A.$$

Thus ρ restricts to $n\psi_A$ on K_{l_2}, where $n = 1$ or q, and hence ρ^{η_t} restricts to $n\psi_A^{\eta_t}$ on $K_{l_2}^{\eta_t} \supset K_r$. Thus $(\rho^{\eta_t}|_{K \cap H^{\eta_t}})^K|_{K_r} \cong \sum_{\gamma \in K/K \cap H^{\eta_t}} n(\psi_A^{\eta_t}|_{K_r})^\gamma$, so that $(\rho^{\eta_t}|_{K \cap H^{\eta_t}})^K$ restricts to the identity on a subgroup K_q of K_r $(q \geq r)$ if and only if $\psi_A^{\eta_t}|_{K_q} = 1$. Finally, because the level of ψ_A is l, a simple computation shows that $\psi_A^{\eta_t}|_{K_q} = 1$ if and only if $q \geq t + l \geq r$. The result follows. □

Now we fix t and further consider the representation $(\rho^{\eta_t}|_{K \cap H^{\eta_t}})^K$. Because it restricts to copies of the identity representation on K_{t+l}, its restriction to $K_{(t+l)_2} \subset K_{l_2}$ decomposes as a direct sum $\sum \psi'_C$, where for C in M, ψ'_C is defined on $K_{(t+l)_2}$ by $\psi'_C(k) = \psi(\omega^{-t-l}\operatorname{Tr} C(k - I))$. (Note that $\psi'_{C_1} = \psi'_{C_2}$ if and only if $C_1 - C_2$ lies in $M_{(t+l)_1}$.) We shall find a C in M such that ψ'_C is a subrepresentation of $(\rho^{\eta_t}|_{K \cap H^{\eta_t}})^K|_{K_{(t+l)_2}}$. Mackey's induction-restriction theorem implies that

$$(\rho^{\eta_t}|_{K \cap H^{\eta_t}})^K|_{K_{(t+l)_2}} \cong \sum_{\gamma \in \Gamma} \left(\rho^{\gamma\eta_t}|_{K_{(t+l)_2} \cap (H^{\eta_t})^\gamma}\right)^{K_{(t+l)_2}}$$
$$\cong \sum_{\gamma \in \Gamma} \left[\left(\rho^{\eta_t}|_{K_{(t+l)_2} \cap H^{\eta_t}}\right)^{K_{(t+l)_2}}\right]^\gamma,$$

where $\Gamma = K_{(t+l)_2} \backslash K/K \cap H^{\eta_t}$, so it suffices to find a C for which ψ'_C is a subrepresentation of $(\rho^{\eta_t}|_{K_{(t+l)_2} \cap H^{\eta_t}})^{K_{(t+l)_2}}$. By Frobenius reciprocity, this is the case if and only if $\psi'_C|_{K_{(t+l)_2} \cap H^{\eta_t}}$ is a subrepresentation of $\rho^{\eta_t}|_{K_{(t+l)_2} \cap H^{\eta_t}}$. We need the following lemma:

LEMMA 11. *In the case that π is ramified, so $H = U_E B_{l-1}$, $K_{(t+l)_2} \cap H^{\eta_t} = K_{(t+l)_2} \cap B_{l-1}^{\eta_t}$. If π is unramified, so $H = U_E K_{l_1}$, then $K_{(t+l)_2} \cap H^{\eta_t} = K_{(t+l)_2} \cap K_{l_1}^{\eta_t}$; furthermore, $K_{(t+l)_2} \cap K_{l_1}^{\eta_t} = K_{(t+l)_2} \cap (K_{l_2} N_{l_1}^w)^{\eta_t}$ in this case.*

Proof. We prove the second result; the first is proved similarly. Let $k \in K_{(t+l)_2} \cap H^{\eta_t}$ so that $k = (uk')^{\eta_t}$, where $u = wI + xA \in U_E$ and $k' \in K_{l_1}$. Writing this equation with matrices makes it clear that this implies that $w \in U^{l_1}$ and $x \in \mathfrak{p}^{l_1}$, so that $u \in K_{l_1}$ and hence $k \in K_{(t+l)_2} \cap K_{l_1}^{\eta_t}$, as claimed. Moreover, if $k = \left(\begin{smallmatrix} 1+a & b \\ c & 1+d \end{smallmatrix}\right)$, then a, b and d must lie in $\mathfrak{p}^{(t+l)_2} \subset \mathfrak{p}^{l_2}$ and c lies in $\mathfrak{p}^{t+l_1}$, so that $k^{\eta_t^{-1}}$ lies in $K_{l_2} N_{l_1}^w$, as claimed. □

Now for each $t \geq 1$, let

$$A_t = \begin{pmatrix} 0 & 1 \\ -\Delta\omega^{2t} & s\omega^t \end{pmatrix} = \omega^t \eta_t A_{\eta_t^{-1}}.$$

Then ψ'_{A_t} is the representation we seek:

LEMMA 12. *ψ'_{A_t} is a subrepresentation of $(\rho^{\eta_t}|_{K_{(t+l)_2} \cap H^{\eta_t}})^{K_{(t+l)_2}}$, in both the ramified and unramified cases.*

Proof. By the remarks preceding Lemma 11, we see that it is enough to show that $\psi'_{A_t}|_{K_{(t+l)_2} \cap H^{\eta_t}}$ is a subrepresentation of the restriction $\rho^{\eta_t}|_{K_{(t+l)_2} \cap H^{\eta_t}}$. The lemma above allows us to consider the latter representation more closely. In the case that π is ramified, $\rho = \psi_{A,\lambda}$ on $H = U_E B_{l-1}$, and ρ restricts to ψ_A on B_{l-1}. If π is unramified, then $H = U_E K_{l_1}$, and if l is even, $\rho = \psi_{A,\lambda}$ on H and restricts to ψ_A on $K_{l_2} = K_{l_2} N_{l_1}^w$. If π is unramified and l is odd, then $\rho = \xi_{A,\lambda}$ on H and restricts to $\zeta_{A,\lambda'} = (\psi_{A,\lambda',1})^{U_F U_E^1 K_{l_1}}$ on $U_F U_E^1 K_{l_1}$, so the restriction of ρ to $U_F U_E^1 B_l^w$ contains $\psi_{A,\lambda',1}$ as a subrepresentation. Thus the restriction of ρ to $B_l^w = K_{l_2} N_{l_1}^w$ again contains ψ_A. Applying Lemma 11, we find that $\psi_A|_{K_{(t+l)_2} \cap H^{\eta_t}}$ is a subrepresentation of $\rho|_{K_{(t+l)_2} \cap H^{\eta_t}}$ in all cases. Moreover, if $k \in$

$K_{(t+l)_2} \cap H^{\eta_t}$ then:

$$\begin{aligned}\psi'_{A_t}(k) &= \psi\big(\omega^{-t-l}\operatorname{Tr} A_t(k-I)\big)\\ &= \psi\big(\omega^{-t-l}\operatorname{Tr}\omega^t\eta_t A\eta_t^{-1}(k-I)\big) = \psi\big(\omega^{-l}\operatorname{Tr} A\eta_t^{-1}(k-I)\eta_t\big)\\ &= \psi\big(\omega^{-l}\operatorname{Tr} A\big(\eta_t^{-1}k\eta_t - I\big)\big) = \psi_A^{\eta_t}(k).\end{aligned}$$

Thus $\psi'_{A_t}|_{K_{(t+l)_2}\cap H^{\eta_t}}$ is a subrepresentation of the restriction $\rho^{\eta_t}|_{K_{(t+l)_2}\cap H^{\eta_t}}$, as claimed. □

Because ψ'_{A_t} is a subrepresentation of $(\rho^{\eta_t}|_{K_{(t+l)_2}\cap H^{\eta_t}})^{K_{(t+l)_2}}$ which is in turn a subrepresentation of $(\rho^{\eta_t}|_{K\cap H^{\eta_t}})^K|_{K_{(t+l)_2}}$, by Frobenius reciprocity it follows that $\psi_{A_t}'^K$ and $(\rho^{\eta_t}|_{K\cap H^{\eta_t}})^K$ must intertwine. We continue by finding the decomposition into irreducible components of the representation $\psi_{A_t}'^K$.

ψ'_{A_t} is a representation of $K_{(t+l)_2}$ factoring through $K_{(t+l)_2}/K_{t+l}$, and as above, its stabilizer in K is the subgroup $U(A_t)K_{(t+l)_1}$, where $U(A_t)$ denotes the group of units in the subalgebra $\mathcal{O}(A_t)$ of $M_2(\mathcal{O})$ generated by A_t. Note that $U(A_t) = (U_F U_E^t)^{\eta_t}$ if the field E is ramified over F, and $U(A_t) = (U_F U_E^{2t+1})^{\eta_t}$ if E is unramified over F, so that $U(A_t) \subset (U_F U_E^1)^{\eta_t} \subset U_E^{\eta_t}$ in either case. Thus if $\psi_A^{\sim}$ denotes, as before, an extension of the restriction $\psi_A|_{K_{l_2}\cap U_E}$ to U_E, then if $(\psi'_{A_t})^{\sim} = (\psi_A^{\sim})^{\eta_t}|_{U(A_t)}$, $(\psi'_{A_t})^{\sim}$ extends ψ'_{A_t} to $U(A_t)$.

Thus in the case that $t + l$ is even, as before, for each representation λ in $\Lambda = [U(A_t)/[U(A_t) \cap K_{(t+l)_2}]]^{\wedge}$, we get a corresponding extension $\psi'_{A_t,\lambda}$ of ψ'_{A_t} to its stabilizer in K, and Clifford's theorem implies that $\psi_{A_t}'^K$ is isomorphic to the direct sum of nonisomorphic irreducible representations $\Sigma_{\lambda\in\Lambda}\psi_{A_t,\lambda}'^K$.

In the case that $t + l$ is odd, the form of A_t implies that the formula $\psi'_{A_t}(k) = \psi(\omega^{-t-l}\operatorname{Tr} A_t(k - I))$ defines a representation on the group B_{t+l-1} which factors through $B_{t+l-1}/B_{2t+2l-2}$. The stabilizer in B of this representation is $U(A_t)B_{t+l-1}$, so this time, for each λ in $\Lambda = [U(A_t)/[U(A_t) \cap B_{t+l-1}]]^{\wedge}$, we get an extension $\psi'_{A_t,\lambda}$ of ψ'_{A_t} to its stabilizer in B, and we find that $\psi_{A_t}'^B$ is isomorphic to the direct sum of nonisomorphic irreducible representations $\Sigma_{\lambda\in\Lambda}\psi_{A_t,\lambda}'^B$. This implies that $\psi_{A_t}'^K$ is isomorphic to the direct sum $\Sigma_{\lambda\in\Lambda}\psi_{A_t,\lambda}'^K$; we show that these summands are also irreducible and nonisomorphic. The argument is similar to the one given earlier to show that σ^K is irreducible in the ramified case. As in that case, we find that for any λ_1 and λ_2 in Λ, we have:

$$\begin{aligned}i\big(\psi_{A_t,\lambda_1}'^K, \psi_{A_t,\lambda_2}'^K\big) &= i_B\big(\psi_{A_t,\lambda_1}'^B, \psi_{A_t,\lambda_2}'^B\big) + i_{B\cap B^w}\Big(\psi_{A_t,\lambda_1}'^B, \big(\psi_{A_t,\lambda_2}'^B\big)^w\Big)\\ &= \delta_{\lambda_1,\lambda_2} + i_{B\cap B^w}\Big(\psi_{A_t,\lambda_1}'^B, \big(\psi_{A_t,\lambda_2}'^w\big)^{B^w}\Big),\end{aligned}$$

so it suffices to show that the latter number is always equal to 0. We use the following lemma, which generalizes Lemma 7, and which is proved in the same way.

LEMMA 13. *For any integers* $l \geq 1$ *and* $t \geq 1$ (*and* $t = 0$, *in the case that* E *is ramified over* F), $B = (B \cap B^w)U(A_t)$.

This lemma and Mackey's induction-restriction theorem imply that

$$\psi'^{\,B}_{A_t,\lambda} \mid_{B \cap B^w} = \left(\psi'_{A_t,\lambda} \mid_{B^w \cap U(A_t)B_{t+l-1}}\right)^{B \cap B^w},$$

and since the lemma also implies that $B^w = (B \cap B^w)(U(A_t))^w$, we also have $(\psi'^{w}_{A_t,\lambda})^{B^w} \mid_{B \cap B^w} = (\psi'^{w}_{A_t,\lambda} \mid_{B \cap [U(A_t)B_{t+l-1}]^w})^{B \cap B^w}$. Thus

$$\begin{aligned} i_{B \cap B^w}&\left(\psi'^{\,B}_{A_t,\lambda_1}, \left(\psi'^{w}_{A_t,\lambda_2}\right)^{B^w}\right) \\ &= i\left(\left(\psi'_{A_t,\lambda} \mid_{B^w \cap U(A_t)B_{t+l-1}}\right)^{B \cap B^w}, \left(\psi'^{w}_{A_t,\lambda} \mid_{B \cap [U(A_t)B_{t+l-1}]^w}\right)^{B \cap B^w}\right) \\ &= \sum_{\gamma \in \Gamma} i_{C(\gamma)}\left(\psi'^{\gamma}_{A_t,\lambda}, \psi'^{w}_{A_t,\lambda}\right), \end{aligned}$$

where

$$\Gamma = B \cap [U(A_t)B_{t+l-1}]^w \backslash B \cap B^w / [B^w \cap U(A_t)B_{t+l-1}]$$

and

$$C(\gamma) = [B \cap [U(A_t)B_{t+l-1}]^w] \cap [B^w \cap U(A_t)B_{t+l-1}]^\gamma,$$

for each $\gamma \in \Gamma$. Since for each γ, $C(\gamma) \supset K_{(t+l)_2}$ and since $\psi'_{A_t,\lambda} \mid_{K_{(t+l)_2}} = \psi'_{A_t}$, we see that the equations above show that

$$i_{B \cap B^w}\left(\psi'^{\,B}_{A_t,\lambda_1}, \left(\psi'^{w}_{A_t,\lambda_2}\right)^{B^w}\right) \leq \sum_{\gamma \in \Gamma} i_{K_{(t+l)_2}}\left(\psi'^{\gamma}_{A_t}, \psi'^{w}_{A_t}\right).$$

But as in the proof of Lemma 8, we can take each index γ to be of the form $\left(\begin{smallmatrix} 1 & 0 \\ \omega c & d \end{smallmatrix}\right)$, for some c and d in $\mathcal{O}$, and it is then clear that $\gamma A_t \gamma^{-1} - wA_t w^{-1}$ does not lie in $M_{(t+l)_1}$, so that

$$i_{K_{(t+l)_2}}\left(\psi'^{\gamma}_{A_t}, \psi'^{w}_{A_t}\right) = i_{K_{(t+l)_2}}\left(\psi'_{\gamma A_t \gamma^{-1}}, \psi'_{wA_t w^{-1}}\right) = 0,$$

for each $\gamma \in \Gamma$.

We have now proved the following lemma:

LEMMA 14. $\psi'^{K}_{A_t}$ *is isomorphic to the direct sum* $\sum_{\lambda \in \Lambda} \psi'^{\,K}_{A_t,\lambda}$, *where*

$$\Lambda = \left[U(A_t)/\left[U(A_t) \cap K_{(t+l)_2}\right]\right]^\wedge \quad \textit{if } t + l \textit{ is even},$$

and

$$\Lambda = [U(A_t)/[U(A_t) \cap B_{t+l-1}]]^\wedge \quad \text{if } t + l \text{ is odd.}$$

In either case, the summands are irreducible and nonisomorphic.

By the remarks following Lemma 12, $\psi_{A_t}'^K$ and $(\rho^{\eta_t}|_{K \cap H^{\eta_t}})^K$ intertwine. The lemma above gives a decomposition of $\psi_{A_t}'^K$ into irreducible subrepresentations, and it follows that one of these summands must be a subrepresentation of $(\rho^{\eta_t}|_{K \cap H^{\eta_t}})^K$. Here we recall that the representation ρ of H is in both the unramified and ramified cases determined by a representation λ of U_E which factors through $U_E/U_E^{l_2}$ in the first case and through U_E/U_E^{l-1} in the latter. Because $U(A_t) \subset U_E^{\eta_t}$, if we define λ_t by $\lambda_t = \lambda^{\eta_t}|_{U(A_t)}$, then λ_t defines a representation which factors through $U(A_t)/[U(A_t) \cap B_{t+l-1}]$ and so also through $U(A_t)/[U(A_t) \cap K_{(t+l)_2}]$, so that $\psi_{A_t,\lambda_t}'^K$ is an irreducible component of $\psi_{A_t}'^K$ in both the unramified and ramified cases. We claim that $\psi_{A_t,\lambda_t}'^K$ is also a subrepresentation of $(\rho^{\eta_t}|_{K \cap H^{\eta_t}})^K$. The following lemma, which corresponds to Lemma 11 and is proved similarly, is needed to prove this fact.

LEMMA 15. *In the case that π is ramified, so $H = U_E B_{l-1}$, $B_{t+l-1} \cap H^{\eta_t} = B_{t+l-1} \cap B_{l-1}^{\eta_t}$. If π is unramified, so $H = U_E K_{l_1}$, then $B_{t+l-1} \cap H^{\eta_t} = B_{t+l-1} \cap K_{l_1}^{\eta_t}$; furthermore, $B_{t+l-1} \cap K_{l_1}^{\eta_t} = B_{t+l-1} \cap (K_{l_2} N_{l_1}^w)^{\eta_t}$ in this case.*

The lemma above allows us to prove:

LEMMA 16. *In all cases, $\psi_{A_t,\lambda_t}'^K$ is a subrepresentation of $(\rho^{\eta_t}|_{K \cap H^{\eta_t}})^K$.*

Proof. Because $\psi_{A_t,\lambda_t}'^K$ is irreducible, it suffices to show that $i_K(\psi_{A_t,\lambda_t}'^K, (\rho^{\eta_t}|_{K \cap H^{\eta_t}})^K) > 0$. By Frobenius reciprocity, we know that

$$i_K\left(\psi_{A_t,\lambda_t}'^K, (\rho^{\eta_t}|_{K \cap H^{\eta_t}})^K\right) = \sum_{\gamma \in H_t \backslash K / K \cap H^{\eta_t}} i_{H_t^\gamma \cap P_0 K_t}\left(\psi_{A_t,\lambda_t}'^\gamma, \rho^{\eta_t}\right),$$

where $H_t = U(A_t) K_{(t+l)/2}$ if $t + l$ is even, and $H_t = U(A_t) B_{t+l-1}$ if $t + l$ is odd. Thus

$$i_K\left(\psi_{A_t,\lambda_t}'^K, (\rho^{\eta_t}|_{K \cap H^{\eta_t}})^K\right) \geq i_{H_t \cap (K \cap H^{\eta_t})}\left(\psi_{A_t,\lambda_t}', \rho^{\eta_t}\right) = i_{H_t \cap H^{\eta_t}}\left(\psi_{A_t,\lambda_t}', \rho^{\eta_t}\right),$$

where as before $H = U_E K_{l_1}$ if E is unramified over F and $H = U_E B_{l-1}$ if E is ramified over F. In the cases that E is ramified or E is unramified and l is even, we have seen that $\rho = \psi_{A,\lambda}$ on H. The proof of the result is similar in these cases; we prove it in the case that $H_t = U(A_t) K_{(t+l)_2}$ and

E is ramified so that $H = U_E B_{l-1}$. In this case, because $U(A_t) \subset U_E^{\eta_t}$ and by the lemma above,

$$H_t \cap H^{\eta_t} = U(A_t)\left[K_{(t+l)_2} \cap H^{\eta_t}\right] = U(A_t)\left[K_{(t+l)_2} \cap B_{l-1}^{\eta_t}\right].$$

Because we have defined $\psi_{A_t}'^{\sim} = (\psi_A^{\sim})^{\eta_t}|_{U(A_t)}$, $\lambda_t = \lambda^{\eta_t}|_{U(A_t)}$, and because $\psi'_{A_t} = \psi_A^{\eta_t}$ on $K_{(t+l)_2} \cap B_{l-1}^{\eta_t}$ (as seen in the proof of Lemma 12), we see that $i_{H_t \cap H^{\eta_t}}(\psi'_{A_t,\lambda_t}, \rho^{\eta_t}) = i_{H_t \cap H^{\eta_t}}(\psi'_{A_t,\lambda_t}, \psi_A^{\eta_t}) = 1$ in these cases.

In the case that E is unramified over F and l is odd, $\rho = \xi_{A,\lambda}$ on H and restricts to $\zeta_{A,\lambda'}$ on $U_F U_E^1 K_{l_1}$, where $\zeta_{A,\lambda'} = (\psi_{A,\lambda',1})^{U_F U_E^1 K_{l_1}}$. There are two cases, depending on the definition of H_t; their proofs are similar and we assume that $H_t = U(A_t) B_{t+l-1}$ in what follows. Thus

$$\begin{aligned}
&i_{H_t \cap H^{\eta_t}}\left(\psi'_{A_t,\lambda_t}, \rho^{\eta_t}\right)\\
&\quad = i_{H_t \cap H^{\eta_t}}\left(\psi'_{A_t,\lambda_t}, \left[(\psi_{A,\lambda',1})^{U_F U_E^1 K_{l_1}}\right]^{\eta_t}\right)\\
&\quad = i_{H_t \cap H^{\eta_t}}\left(\psi'_{A_t,\lambda_t}, \left[(\psi_{A,\lambda',1})^{\eta_t}\right]^{(U_F U_E^1 K_{l_1})^{\eta_t}}\right)\\
&\quad = i_{H_t \cap H'^{\eta_t}}\left(\psi'_{A_t,\lambda_t}, (\psi_{A,\lambda',1})^{\eta_t}\right), \quad \text{where } H' = U_F U_E^1 K_{l_1}.
\end{aligned}$$

$U(A_t) \subset (U_F U_E^1)^{\eta_t}$, so applying the lemma above we find that

$$\begin{aligned}
H_t \cap H'^{\eta_t} &= U(A_t)\left[B_{t+l-1} \cap K_{l_1}\right]\\
&= U(A_t)\left[B_{t+l-1} \cap \left(K_{l_2} N_{l_1}^w\right)\right] = U(A_t)\left[B_{t+l-1} \cap B_l^w\right].
\end{aligned}$$

As above, we see that we have constructed ψ'_{A_t,λ_t} in such a way that it is equal to $(\psi_{A,\lambda',1})^{\eta_t}$ on $H_t \cap H'^{\eta_t}$, so again $i_{H_t \cap H^{\eta_t}}(\psi'_{A_t,\lambda_t}, \rho^{\eta_t}) = 1$. □

Finally, we shall prove that $\psi'^K_{A_t,\lambda_t} = (\rho^{\eta_t}|_{K \cap H^{\eta_t}})^K$ by showing that their degrees are equal. The following lemmas imply this fact.

LEMMA 17. *$\psi'^K_{A_t,\lambda_t}$ has degree $(q-1)q^{t+l-2}(q+1)$.*

Proof. If $t+l$ is even, then ψ'_{A_t,λ_t} is a representation of $U(A_t)K_{(t+l)_2}$ and $(t+l)_2 \geq 1$. Hence since ψ'_{A_t,λ_t} has degree one, $\psi'^K_{A_t,\lambda_t}$ has degree:

$$\begin{aligned}
&\left[K : U(A_t)K_{(t+l)_2}\right]\\
&\quad = \left[K : K_{(t+l)_2}\right]\left[U(A_t)K_{(t+l)_2} : K_{(t+l)_2}\right]^{-1}\\
&\quad = \left[K : K_{(t+l)_2}\right]\left[U(A_t) : U(A_t) \cap K_{(t+l)_2}\right]^{-1}\\
&\quad = \left[K : K_{(t+l)_2}\right]\left[U : U^{(t+l)_2}\right]^{-1}\left[\mathfrak{p} : \mathfrak{p}^{(t+l)_2}\right]^{-1}\\
&\quad = \left[(q-1)^2 q^{4(t+l)_2-3}(q+1)\right]\left[(q-1)q^{(t+l)_2-1}\right]^{-1}\left[q^{(t+l)_2}\right]^{-1}\\
&\quad = (q-1)q^{t+l-2}(q+1).
\end{aligned}$$

If $t + l$ is odd, then ψ'_{A_t,λ_t} is a representation of $U(A_t)B_{t+l-1}$, and again ψ'_{A_t,λ_t} has degree one. Thus in this case, $\psi'^{K}_{A_t,\lambda_t}$ has degree:

$$\begin{aligned}
&[K:B][B:U(A_t)B_{t+l-1}] \\
&\quad = [K:B][B:B_{t+l-1}][U(A_t)B_{t+l-1}:B_{t+l-1}]^{-1} \\
&\quad = [K:B][B:B_{t+l-1}][U(A_t):U(A_t)\cap B_{t+l-1}]^{-1} \\
&\quad = [K:B][B:B_{t+l-1}][U:U^{((t+l-1)_2}]^{-1}[\mathfrak{p}:\mathfrak{p}^{((t+l-1)_1}]^{-1} \\
&\quad = [q+1][(q-1)^2q^{2(t+l-2)}][(q-1)q^{((t+l-1)_2-1}]^{-1}[q^{((t+l-1)_1}]^{-1} \\
&\quad = (q-1)q^{t+l-2}(q+1). \qquad \square
\end{aligned}$$

To compute the degree of $(\rho^{\eta_t}|_{K\cap H^{\eta_t}})^K$, we use the following lemma, whose proof is similar to that of Lemma 11:

LEMMA 18.

$$K\cap(U_EK_{l_1})^{\eta_t} = U(A_t)[K\cap K_{l_1}^{\eta_t}].$$
$$K\cap(U_EB_{l-1})^{\eta_t} = U(A_t)[K\cap B_{l-1}^{\eta_t}].$$

LEMMA 19. *In both the ramified and unramified cases, and for any* $t\geq 0$, $(\rho^{\eta_t}|_{K\cap H^{\eta_t}})^K$ *has degree* $(q-1)q^{t+l-2}(q+1)$.

Proof. In the case that E is unramified over F, we have $H = U_EK_{l_1}$, and applying the lemma above, we find that $K\cap H^{\eta_t} = U(A_t)[K\cap K_{l_1}^{\eta_t}]$. Note that

$$K\cap K_{l_1}^{\eta_t} = \begin{pmatrix} U^{l_1} & \mathfrak{p}^M \\ \mathfrak{p}^{t+l_1} & U^{l_1}\end{pmatrix} \supset K_{t+l_1},$$

where $M = M(t,l) = \max(0, l_1 - t)$, and it follows that $U(A_t)\cap K_{l_1}^{\eta_t} = U^{l_1}I + \mathfrak{p}^M A$. Thus in this case, $(\rho^{\eta_t}|_{K\cap H^{\eta_t}})^K$ has degree:

$$\begin{aligned}
&[K:K\cap H^{\eta_t}]\deg(\rho) \\
&\quad = [K:K_{t+l_1}][K\cap H^{\eta_t}:K_{t+l_1}]^{-1}\deg(\rho) \\
&\quad = [K:K_{t+l_1}][K\cap H^{\eta_t}:K\cap K_{l_1}^{\eta_t}]^{-1}[K\cap K_{l_1}^{\eta_t}:K_{t+l_1}]^{-1}\deg(\rho) \\
&\quad = [K:K_{t+l_1}][U(A_t)(K\cap K_{l_1}^{\eta_t}):K\cap K_{l_1}^{\eta_t}]^{-1} \\
&\qquad \times[K\cap K_{l_1}^{\eta_t}:K_{t+l_1}]^{-1}\deg(\rho)
\end{aligned}$$

(continues)

(continued)

$$= \left[K : K_{t+l_1}\right]\left[U(A_t) : U(A_t) \cap K_{l_1}^{\eta_t}\right]^{-1}\left[K \cap K_{l_1}^{\eta_t} : K_{t+l_1}\right]^{-1}\deg(\rho)$$
$$= \left[(q-1)^2 q^{4(t+l_1)-3}(q+1)\right]\left[(q-1)q^{l_1-1+M}\right]^{-1}$$
$$\times\left[q^{3t+l_1-M}\right]^{-1}\deg(\rho)$$
$$= (q-1)q^{t+2l_1-2}(q+1)\deg(\rho).$$

Recall that when l is odd, $\rho = \xi_{A,\lambda}$ and has degree q, while when l is even, $\rho = \psi_{A,\lambda}$ and has degree 1. Thus $\deg(\rho) = q^{l_2 - l_1}$ in either case. Since $l = l_2 + l_1$, we see that $(\rho^{\eta_t}|_{K \cap H^{\eta_t}})^K$ has degree

$$(q-1)q^{t+l-2}(q+1),$$

as claimed.

In the case that E is ramified over F, we have $H = U_E B_{l-1}$, and applying the lemma above, we find that $K \cap H^{\eta_t} = U(A_t)[K \cap B_{l-1}^{\eta_t}]$. Note that

$$K \cap B_{l-1}^{\eta_t} = \begin{pmatrix} U^{l_1} & \not{p}^N \\ \not{p}^{t+l_2} & U^{l_1} \end{pmatrix} \supset K_{t+l_2},$$

where $N = N(t,l) = \max(0, (l-1)_1 - t)$, so it follows that $U(A_t) \cap B_{l-1}^{\eta_t} = U^{l_1}I + \not{p}^N A$. Thus in this case, $(\rho^{\eta_t}|_{K \cap H^{\eta_t}})^K$ has degree:

$$[K : K \cap H^{\eta_t}]\deg(\rho)$$
$$= \left[K : K_{t+l_2}\right]\left[K \cap H^{\eta_t} : K_{t+l_2}\right]^{-1}\deg(\rho)$$
$$= \left[K : K_{t+l_2}\right]\left[K \cap H^{\eta_t} : K \cap B_{l-1}^{\eta_t}\right]^{-1}\left[K \cap B_{l-1}^{\eta_t} : K_{t+l_2}\right]^{-1}\deg(\rho)$$
$$= \left[K : K_{t+l_2}\right]\left[U(A_t)(K \cap B_{l-1}^{\eta_t}) : K \cap B_{l-1}^{\eta_t}\right]^{-1}$$
$$\times\left[K \cap B_{l-1}^{\eta_t} : K_{t+l_2}\right]^{-1}\deg(\rho)$$
$$= \left[K : K_{t+l_2}\right]\left[U(A_t) : U(A_t) \cap B_{l-1}^{\eta_t}\right]^{-1}\left[K \cap B_{l-1}^{\eta_t} : K_{t+l_2}\right]^{-1}\deg(\rho)$$
$$= \left[(q-1)^2 q^{4(t+l_2)-3}(q+1)\right]\left[(q-1)q^{l_1-1+N}\right]^{-1}$$
$$\times\left[q^{3(t+l_2)-2l_1-N}\right]^{-1}[1]$$
$$= (q-1)q^{t+l-2}(q+1). \qquad \square$$

We have thus found our desired decomposition of π into irreducible K-types. We recall that $(\sigma|_{P_0 K_t})^K \cong (\rho|_{K \cap H^{\eta_t}})^K$ and can state our final theorem:

THEOREM 3. *If π has level $l > 1$, then the sum $\sigma^K \oplus \sum_{t\geq 1}(\sigma^{\eta_t}|_{P_0K_t})^K$ is a complete decomposition of $\pi|_K$ into irreducible K-types. The summands are of degree $(q-1)q^{t+l-2}(q+1)$, $t = 0, 1, 2, \ldots,$ respectively.*

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