PACIFIC JOURNAL OF MATHEMATICS
Vol. 130, No. 2, 1987

APPROXIMATION OF PRIME ELEMENTS IN DIVISION ALGEBRAS OVER LOCAL FIELDS AND UNITARY REPRESENTATIONS OF THE MULTIPLICATIVE GROUP

LAWRENCE CORWIN

Let K be a locally compact, non-Archimedean field of residual characteristic p, and let D be a central division algebra of dimension n^2 over K. In constructing the irreducible unitary representations of $D^\times$, a technical question repeatedly arises. Let $x \in D$, and let x_1 be "close" to x (in the sense that, for the usual absolute value on D, $|x - x_1| < |x|$). Let D_x, D_{x_1} be the subalgebras of elements commuting with x, x_1 respectively. Is it possible to pick a prime element $\eta_1 \in D_{x_1}$ and an element $\eta_0 \in D_x$ that are also close, and how close can η, η_1 be to one another? The first part of this paper analyzes this problem. It turns out that η, η_1 can be chosen close enough to one another so that Clifford-Mackey theory easily permits the construction of $(D^x)^\wedge$ only if $p^2 = n$ or $p^2 | n$. The construction has been given in earlier papers except for the case where $p|n$, $p \neq n$, and $p^2 | n$; the second part of the paper is a construction of $(D^x)^\wedge$ in this remaining case.

We recall some details of the construction of $(D^x)^\wedge$. Let π be a prime of K, and let K_n be an unramified extension of K of degree n, embedded in D. One can choose a prime $\eta \in D$ such that $\eta^n = \pi$ and conjugation by P is an automorphism of K_n generating $\mathrm{Gal}(K_n/K)$. We write $\eta a \eta^{-1} = a^\sigma$, $a \in K_n$; $\sigma \in \mathrm{Gal}(K_n/K)$ is the *Hasse invariant* of D up to isomorphism. (See **[11]** for these and other unreferenced facts about division algebras.) Let R be the group of roots of unity in K_n of order prime to p; let $O = O_D$ be the ring of integers in D, and let $P = \eta O$ be the prime ideal in O. Then one has

$$D^\times = U\langle\eta\rangle, \qquad U = (1 + P)R,$$

where U is the group of units of O, the products are semidirect products, and the first subgroup is normal. The main step in computing the irreducible representations of $D^\times$ is that of computing the irreducibles of $G = 1 + P$, and it is on this step that we concentrate. The general idea is this: a representation π_0 of G will be trivial on some normal subgroup $(1 + P^{m+1})$ of G. Choose m as small as possible; for convenience of exposition, assume m odd, and let $2m' = m + 1$. Then π_0 is a representation of $G/(1 + P^{m+1})$, and $(1 + P^{m'})/(1 + P^{m+1})$ is an Abelian subgroup. Furthermore, $(1 + P^{m'})/(1 + P^{m+1}) \cong P^{m'}/P^{m+1}$ via the map $1 + y \to y$. In this way, $\pi_0|(1 + P^{m'})$ can be regarded as a direct sum of (1-dimensional) characters of $P^{m'}$, all trivial on P^{m+1}. To describe these characters, let χ be an additive character of K that is trivial on $P \cap K$ but nontrivial on $O \cap K$; for $x \in D$, let $\chi_x(y) = \chi(\mathrm{Tr}_{D/K}(xy))$. Then every character on D is of the form χ_x for some $x \in D$; $(P^{m+1})^{\perp} = \{\chi_x: x \in P^{-m}\}$, and restricting to $P^{m'}$ means that x is determined only mod $P^{-m'+1}$. Given π_0, we thus get x (mod $P^{-m'+1}$, and up to conjugation by G) by the condition that χ_x occurs in $\pi_0|_{1+P^{m'}}$. Determining π_0 then becomes a problem in Mackey (or Clifford) theory.

It is here that the relation between n and p becomes important. One needs to determine the $w \in G$ for which $\chi_x(wyw^{-1}) = \chi_x(y)$ for all $y \in 1 + P^{m'}$. It is easy to see that w satisfies this condition iff w commutes with x mod some sufficiently high power of P. In the tamely ramified case and in the case $n = p$, one can arrange to have w and x commute; this simplifies matters. (See **[1]**, **[5]**, **[4]**, **[3]**, and **[10]** for details and further results.) For $n = p^2$, however, matters are less simple. The problem is that one can pick elements $x \in D$ such that $[K(x):K] = p^2$, but such that $K(x)$ contains no extension of degree p over K. However, it may be possible to choose x_1 such that $[K(x_1):K] = p$, and x and x_1 agree modulo some moderately high power of P. Certain elements commuting with x_1 commute with χ_x and not with any element in $K(x) \sim K$. In **[2]**, this problem was handled by showing that in the division algebras D_{x_1}, D_x of elements commuting with x_1, x respectively, one could find prime elements η_1, η_0 respectively that were congruent mod some moderately high power of P; thereafter, the analysis could proceed roughly as before.

It is therefore useful to consider the following type of approximation question: suppose that $x \in P^{j_0} \sim P^{j_0+1}$, and suppose that $x_1 \equiv x \bmod P^{j_1}$, $j_1 > j_0$. Let D_x, D_{x_1} be the division algebras of elements in D commuting with x, x_1 respectively. How close to one another can one choose an element $\eta_0 \in D_x$ and a prime element $\eta_1 \in D_{x_1}$? This is the topic of the first half of this paper. It turns out, unsurprisingly, that the difficulties arise primarily with wild ramification; much of the analysis, therefore, deals with the case of totally wildly ramified extensions, and the notation for this part (§§2, 3) is somewhat different from the notation in other sections.

The results are rather negative; they show that the methods of **[2]** apply directly only to the cases $p^2|n$ and $p^2 = n$. (One needs to replace $P^{s_{j+1}-s_j}$ in Theorem 4.2 by $P^{s_{j+1}-s_1}$ to use these methods, and in general that is impossible.) In the second part of this paper, we construct the irreducible representations for the case $n = pn_0$, with n_0 prime to p. The cases $(n, p) = 1$ and $n = p^2$ have previously been treated, as noted earlier.

PART I. APPROXIMATION THEOREMS

2. Some results on finite fields. Let k be a finite field of characteristic p; for each integer $r \geq 0$, denote by k_r the extension field with $[k_r : k] = p^r$. In particular, $k_0 = k$. Fix $n \geq 1$; we shall be interested primarily in the k_r with $r \leq n$. Let σ generate $\mathrm{Gal}(k_n/k)$. Write $\mathrm{Tr}_{i/j}$ for Tr_{k_i/k_j} (if $i \geq j$), and write Tr_i for $\mathrm{Tr}_{i/0}$; write $N_{i/j}$, N_i for the corresponding norm maps.

PROPOSITION 2.1. (a) *There exist elements* $\alpha_1, \ldots, \alpha_n \in k_n$ *such that*

$$\alpha_r^\sigma - \alpha_r = \prod_{i=1}^{r-1} \alpha_i^{p-1}, \quad 1 \leq r \leq n \ (\textit{and } \alpha_1^\sigma - \alpha_1 = 1); \tag{2.1}$$

moreover, $k_r = k(\alpha_1, \ldots, \alpha_r) = k(\alpha_r)$.

(b) *For* $0 \leq j < p^n$, *define* $\beta_j = \beta(j)$ *by*

$$\beta_j = \prod_{i=1}^{r} \alpha_i^{m_i}, \quad \textit{where } \sum_{i=1}^{r} m_i p^{i-1} = j; \quad \beta_0 = 1.$$

Then the β_j *with* $j < p^r$ *give a vector space basis for* k_r/k.

(c) *Let* V_j *be the vector space spanned by the* β_i *with* $i < j$; *in particular*, $V_0 = \{0\}$. *Then for all* $j \geq 1$, *there is a nonzero* $c \in \mathbf{F}_p$ (*the prime field*) *such that*

$$\beta_j^\sigma - \beta_j - c\beta_{j-1} \in V_{j-1}.$$

(d) $\mathrm{Tr}_m(\beta_{p^m-1}) = (-1)^m$; *for* $j < p^{m-1}$, $\mathrm{Tr}_m(\beta_j) = 0$.

Proof. We proceed by induction on n.

(1) Let $n = 1$. As $\mathrm{Tr}_1 1 = 0$, there exists $\alpha_1 \in k_1$ with $\alpha_1^\sigma - \alpha_1 = 1$. Then $\alpha_1 \notin k_0$, the fixed field of σ, and we must have $k_0(\alpha_1) = k_1$. Now (a) and (b) follow. For (c) and (d), we use the formula

$$\left(\alpha_1^j\right)^\sigma = (\alpha_1 + 1)^j = \alpha_1^j + j\alpha_1^{j-1} + \sum_{i=2}^{j} \binom{j}{i} \alpha_1^{j-i}. \tag{2.2}$$

Now (c) follows immediately; just let $j = 1, 2, \ldots, p-1$. As

$$j\alpha_1^{j-1} \equiv \left(\alpha_1^j\right)^\sigma - \alpha_1^j \bmod V_{j-1}, \qquad j \leq p-1,$$

the second part of (d) is also immediate. Finally, the case $j = p$ gives

$$(\alpha_1^p)^\sigma = (\alpha_1 + 1)^p = \alpha_1^p + 1.$$

Hence σ fixes $\alpha_1^p - \alpha_1$, or $\exists \delta \in k$ with $\alpha_1^p - \alpha_1 - \delta = 0$. The roots of $X^p - X - \delta = 0$ are therefore α_1 and its conjugates. Write the coefficient of X as a symmetric function of the roots:

$$-1 = (-1)^{p-1} \sum_{j=0}^{p-1} N(\alpha_1)/\alpha_1^{\sigma_j} = (-1)^{p-1}\mathrm{Tr}_\nu(N(\alpha_1)/\alpha_1),$$

$$N = \text{norm in } k_1/k.$$

But the conjugates of α_1 are $\alpha_1 + 1, \ldots, \alpha_1 + p - 1$, and thus $N(\alpha_1)/\alpha_1 \equiv \alpha_1^{p-1} \bmod V_{p-1}$. Now the first part of (d) is obvious, since $(-1)^{p-1} \equiv 1 \bmod p$ for all p.

(2) Now assume the result for $n - 1$. Then

$$\mathrm{Tr}_n \beta(p^{n-1} - 1) = p\,\mathrm{Tr}_{n-1}\beta(p^{n-1} - 1) = 0,$$

so that we can find $\alpha_n \in k_n$ with $\alpha_n^\sigma - \alpha_n = \beta(p^{n-1} - 1)$. As $\mathrm{Tr}_{n-1}\beta(p^{n-1} - 1) \neq 0$, $\alpha_n \notin k_{n-1}$. Just as in (1), we get all of (a) and (b) except for the claim that $k_n = k(\alpha_n)$. We do have $k_n = k_{n-1}(\alpha_n)$. Also, $\beta(p^{n-1} - 1) \in k(\alpha_n)$, and (b) and (c), applied to k_{n-1}, show that $k(\beta(p^{n-1} - 1)) = k_{n-1}$. Hence

$$k(\alpha_n) = k(\alpha_n, \beta(p^{n-1} - 1)) = k_{n-1}(\alpha_n) = k_n,$$

as required.

We need to prove (c) only for those $\beta(j)$ with $j > p^{m-1}$. We first prove it for α_n^i, $1 \le i \le p - 1$. For $i = 1$, the result is immediate from the definition of α_n; in general, it follows from the formula

$$(\alpha_n^i)^\sigma = (\alpha_n + \beta(p^n - 1))^i.$$

Now we deal with the remaining case: $\beta_j = \alpha_n^i\beta(l)$, $1 \le l < p^{n-1}$ and $i > 0$. Then

$$\beta_j^\sigma = (\alpha_n^i + \gamma)(\beta(l) + c\beta(l - 1) + \delta),$$

$$\gamma \in V_{ip^{n-1}},\ \delta \in V_{l-1}, \text{ and } c \in \mathbf{F}^p$$

$$= \beta_j = c\alpha_n^i\beta(l - 1) + \alpha_n^i\delta + \varepsilon, \quad \varepsilon \in V_{ip^{n-1}},$$

which is (c) for β_j. The same argument as in (1) now shows that $\mathrm{Tr}_n\beta_j = 0$ if $j < p^{m-1}$.

We need more calculation to get $\mathrm{Tr}_n\,\beta(p^n - 1)$. For $\gamma \in k_n$, define $\gamma\{0\}, \gamma\{1\}, \ldots$ inductively by

$$\gamma\{0\} = \gamma, \qquad \gamma\{j+1\} = \gamma\{j\}^{\sigma} - \gamma\{j\}.$$

An easy induction gives

$$\gamma^{\sigma^s} = \sum_{j=0}^{s} \binom{s}{j} \gamma\{j\}; \tag{2.3}$$

in particular,

$$\gamma^{\sigma^s} = \gamma + \gamma\{s\} \quad \text{if } s \text{ is a power of } p. \tag{2.4}$$

Now let $\gamma = \beta_r$. Applying (c) p times and using (2.4), we get

$$\beta_r^{\sigma^p} - \beta_r \equiv c'\beta_{r-p} \bmod V_{r-p} \quad \text{for some } c' \in \mathbf{F}_p,$$

and an easy induction shows that if s is a power of p, then there is a $c \in \mathbf{F}_p$ such that

$$\beta_r\{s\} = \beta_r^{\sigma^s} - \beta_r \equiv c\beta_{r-s} \bmod V_{r-s}.$$

For $r = p^{n-1}$, we get

$$\alpha_n^{\sigma^{p^{n-1}}} = \alpha_n + c, \qquad c \in \mathbf{F}_p.$$

But $\sigma^{p^{n-1}}$ generates $\mathrm{Gal}(k_n/k_{n-1})$, and $k_n = k_{n-1}(\alpha_n)$. Thus, from (d) in the case $n = 1$,

$$\mathrm{Tr}_{n/n-1}(\alpha_n/c)^{p-1} = -1.$$

As $c^{p-1} = 1$, we have

$$\mathrm{Tr}_{n/n-1}\alpha_n^{p-1} = -1,$$

or

$$\mathrm{Tr}_{n/n-1}\beta(p^n - 1) = -\beta(p^{n-1} - 1).$$

The inductive hypothesis on $\beta(p^{n-1} - 1)$ now gives the first part of (d).

We note two corollaries. The first was essentially proved in the course of the above proof.

COROLLARY 1. *If $s = ap^r$ with $p \nmid a$, then for each j there is a nonzero $c \in \mathbf{F}_p$ with*

$$\beta_j^{\sigma^s} - \beta_j - c\beta_{j-p^r} \in V_{j-p^r}.$$

(If $j - p^r < 0$, then β_{j-p^r} is taken to be 0.)

Proof. This is an easy calculation from (c) and (2.3).

COROLLARY 2. *If* $r < n$, *and if* $\gamma \in k_n$ *satisfies* $\mathrm{Tr}_{n/r}\gamma \in k$, *then* $\gamma \in V_{p^n - p^r + 1}$.

Proof. This follows from (d) (applied to k_n/k_r) and the linearity of the trace.

We now prove a result about "partial traces".

LEMMA 2.2. *Let* $r < s$; *suppose that* $p \nmid a$. *Then* $\exists$ *a nonzero* $c \in \mathbf{F}_p$ *such that*

$$\left(\sum_{i=0}^{ap^{s-r}-1} \beta_j^{\sigma^i p^r} \right) \equiv c\beta(j - p^s + p^{s-r}) \bmod V_{j-p^s+p^{s-r}}.$$

Proof. For $\gamma \in k_n$, define $\gamma\{j\}$ as in the proof of Lemma 2.1. Then (2.4) and induction give

$$\gamma^{\sigma^i p^r} = \sum_{j=0}^{i} \binom{i}{j} \gamma\{jp^r\}.$$

Hence

$$\sum_{i=0}^{ap^{s-r}-1} \gamma^{\sigma^i p^r} = \sum_{i=0}^{ap^{s-r}-1} \sum_{j=0}^{i} \binom{i}{j} \gamma\{jp^r\}$$

$$= \sum_{j=0}^{ap^{s-r}-1} \sum_{i=j}^{ap^{s-r}-1} \binom{i}{j} \gamma\{jp^r\} = \sum_{j=0}^{ap^{s-r}-1} \binom{ap^{s-r}}{j+1} \gamma\{jp^r\}.$$

As

$$\binom{ap^{s-r}}{j+1} \equiv 0 \bmod p \quad \text{if } p^{s-r} \nmid j - 1$$

and

$$\binom{ap^{s-r}}{p^{s-r}} \equiv a \bmod p,$$

we have

$$\sum_{i=0}^{ap^{s-r}-1} \gamma^{\sigma^i p^r} = a\gamma\{p^r(p^{s-r} - 1)\} + \gamma',$$

where γ' is a k-linear combination of terms of the form $\gamma\{p^r(ip^{s-r} - 1)\}$,

$i > 1$. Now let $\gamma = \beta_j$. Since (c) of Proposition 2.1 and an easy induction shows that for each i there is $c \in \mathbf{F}_p$ such that $\gamma\{i\} - c\beta_{j-i} \in V_{j-i}$, the result follows.

Recall that $N_{i/j}$ is the norm map from k_i to k_j ($j < i$). We shall also need the following result about norms and traces in k_m.

PROPOSITION 2.3. *Let $s < m$. Given $\alpha \in k_n^\times$, there exists a nonzero $\lambda \in k_n$ with $N_{n/s}\lambda \in k$ and* $\mathrm{Tr}_n(\alpha\lambda) = 0$.

Proof. We use the following lemma:

LEMMA 2.4. *Let q be a power of p; let $q' = (q^{p^s} - 1)/(q - 1)$. Then for all $n > s$, q' and $(q^{p^n} - 1)/q'$ are relatively prime.*

Proof of the Lemma. Since

$$q^{p^n} - 1 = \left(q^{p^s} - 1\right) \sum_{j=0}^{p^{n-s}-1} q^{jp^s}$$

and $q^{jp^s} \equiv 1 \bmod q^{p^s} - 1$, we see that

$$\left(q^{p^n} - 1\right)/\left(q^{p^s} - 1\right) \equiv p^{n-s} \bmod q^{p^s} - 1.$$

Hence $q'(q - 1)$ and $(q^{p^n} - 1)/q'(q - 1)$ are relatively prime. Similarly, q' and $q - 1$ are relatively prime, and this proves the lemma.

Proof of Proposition 3.3. Let $|k| = q$, $(q^{p^n} - 1)/(q^{p^s} - 1) = r$. As $N_{n/s}\lambda = \lambda^r$, we need to pick λ so that its order divides $(q - 1)r = (q^{p^n} - 1)/q'$. Now let β generate $k_n^\times$ as a cyclic group, and let $\alpha = \beta^t$. From Lemma 3.4, we can find an integer a such that

$$(q - 1)r | t + aq'.$$

Let $\lambda = \beta^{aq'}$. Then the order of λ divides $(q - 1)r$, and $\alpha\lambda$ has order dividing q'. But then $\alpha\lambda \in k_s$, so that $\mathrm{Tr}_n \alpha\lambda = 0$.

3. Prime elements in sub-division algebras: Totally wildly ramified case. We begin by describing the notational conventions in this section, since they are somewhat different from those for other sections. We assume that the index of D over K is p^n (so that $\dim_K D = p^{2n}$); we choose $x \in D$, and assume that $K(x)$ is totally ramified over K. We assume further that x is in general position, or that $|x| \le |x + z|$ for all $z \in K$. Define n by $|x| = p^m$.

We recall some results from [7]. For each $j \in \mathbf{Z}$, let $x_{(j)}$ be an element such that $x_{(j)} \equiv x \bmod P^{-m+j+1}$ and $[K(x_j):K]$ is minimal (subject to the above condition). For $j < 0$, we have $x_{(j)} \in K$, and we take $x_{(j)} = 0$ there; for sufficiently large j, we may take $x_{(j)} = x$. The fields $K(x_{(j)})$ are all totally ramified over K. There are integers $s_0 = 0$, $s_1, \ldots, s_{t-1}$ such that for each r, $[K(x_{(s_r)}):K] > [K(x_{(s_r-1)}):K]$. These integers are the *jump points* of x. Set $s_{-1} = -\infty$, $s_t = \infty$, and define $x_r = x_{(s_r-1)}$, with $x_t = x$. We may (and henceforth do) assume that $x_{(j)} = x_r$ if $s_{r-1} \le j < s_r$. We call the x_j the *approximating elements* for x. Let D_r = algebra of elements commuting with x_r. We shall be interested in how closely we can approximate a prime element in D_r by one in D_{r-1}.

Write K_n for the unramified extension of degree p^n that is normalized by η, and let K_b be the subfield of K_n of degree p^b over K. (Note: K_n is what was called K_{p^n} in the introduction.) We write k_n for the residue class field of K_n (and of D); this corresponds to the notation in §2. The residue class map $\mathcal{O} \to \mathcal{O}/P \cong k_n$ is bijective on $R \cup \{0\}$, and we generally identify $R \cup \{0\}$ with k_n, for notational ease. Thus we write a typical element $y \in D$ as $\sum_{j=j_0}^{\infty} \delta_j \eta^j$, $\delta_j \in k_n$. We use α_j $(1 \le j \le n)$ and $\beta_j = \beta(j)$ $(0 \le j \le p^n - 1)$ for the elements of k_n so denoted in §2, and write

$$x = \sum_{j=-m}^{\infty} \gamma_j \eta^j. \tag{3.1}$$

Let $[K(x_j):K] = p^{a_j} = e_j$, and let $a_j + b_j = n$. Since x is in general position, m is divisible by p^{b_1} but not by p^{b_1+1}. Furthermore, $(\gamma_{-m}\eta^{-m})^{p^{a_1}} \in K$. It follows that we may assume $\gamma_{-m} \in K$, possibly by replacing η with some $\varepsilon\eta$, $\eta \in k_n$. (Every element of the form $c\eta^{-mp^{a_1}}$, $c \in k$, can be written as $(c'\eta^{-m})^{p^{a_1}}$ with $c' \in k$, essentially because $c' \mapsto (c')^p$ is an automorphism of k. See p. 55 of [2].) It also follows from p. 55 of [2] that we may assume (after conjugating x by an element of G) that

$$K_{b_j} \subseteq D_j. \tag{3.2}$$

Note, incidentally, that a_j increases with j, while b_j decreases.

Our first job is to find a "normal form" for the x_j.

PROPOSITION 3.1. *By possibly conjugating x in D, we may assume that*

$$x_i - x_{i-1} = \sum_{j=-m+s_{i-1}}^{\infty} \gamma_{i;j}\eta^j,$$

with $\gamma_{i;-m+s_{i-1}} = c\beta(p^m - p^{b_{i-1}})$ for some nonzero $c \in k$.

Proof. Induction on i. For $i = 1$, this is just our assumption that $\gamma_{-n} \in k$; thus assume $i \geq 2$. Apply Satz 8 of [7] with $v = x_{i-1}$, ω a prime element in D_i such that $\omega \equiv \eta \bmod P^2$, and $\beta = \gamma_{i;-n+s_i}\omega^{-n+s_i}$; if ψ_i is the irreducible polynomial satisfied by x_{i-1}, then

$$\psi_i(x_i) \equiv \left(\mathrm{Tr}_{m/b_{i-1}}\gamma_{i;-m+s_{i-1}}\right)\eta^b \bmod P^{b+1},$$

where b is an integer divisible by p^{b_i} but not by $p^{b_{i-1}}$. Set

$$\delta = \mathrm{Tr}_{n/b_{i-1}}\left(\gamma_{i;-m+s_{i-1}}\right).$$

Then $\delta \in k_{b_{i-1}}$. Furthermore,

$$\left(\delta\eta^b\right)^{p^{a_i}} = \left(N_{n/b_i}\delta\right)\eta^{bp^{a_i}} \in K;$$

otherwise, $N_{n/b_i}\delta \in k_n \setminus k$ and Hensel's lemma implies that $K(x_i)$ is not totally ramified over K. But since $\delta \in k_{b_{i-1}}$,

$$N_{n/b_i}\delta = \left(N_{b_{i-1}/b_i}\delta\right)^{p^{n-b_{i-1}}}.$$

Hence $N_{i/i-1}\delta \in k$, and there exists $d \in k$ with $N_{i/i-1}(d/\delta) = 1$.

Suppose that $\delta \notin k$. Set $h = b_{i-1} - b_i$; let $-m + s_i = g$. Then $p^{b_i}|g$, but $p^{b_{i-1}} \nmid g$. Hence

$$\left(\delta\eta^g\right)^{p^h} = \left(N_{i/i-1}\delta\right)\eta^{gp^h} = \left(d\eta^g\right)^{p^h},$$

and so $\delta\eta^g$ and $d\eta^g$ satisfy the same equation over K. Therefore, they are conjugate in D (and, in fact, by an element of K_m). That is, there is a prime η' such that

$$\delta\eta^g = d(\eta')^g.$$

Hence $(\eta')^{gp^h} = d^{-p^h}(d\eta'^g)^{p^h} = d^{-p^h}(\delta\eta^g)^{p^h} = d^{-p^h}(d\eta^g)^{p_h} = \eta^{gp^h}$, or $(\eta')^{p^{b_{i-1}}} = \eta^{p^{b_{i-1}}}$. But x_{i-1} commutes with $K_{b_{i-1}}$; hence in the expansion of x_{i-1},

$$x_{i-1} = \sum_{j=-m}^{\infty} \gamma_j^{(i-1)}\eta^j, \text{ say},$$

we must have $\gamma_j^{(i-1)} = 0$ unless $p^{b_{i-1}}|j$. It follows that the expansion of x_{i-1} is the same if we use powers of η' instead of powers of η. On the other hand, we have

$$x_i - x_{i-1} = \sum_{j=m-s_{i-1}}^{\infty} \gamma'_{i;j}\eta'^j, \quad \gamma'_{i,-m+s_i} = \gamma_{i,-m+s_i}d/\delta,$$

so that

$$\mathrm{Tr}_{m/b_{i-1}}\gamma'_{i;-m+s_{i-1}} \in k.$$

As η, η' are related by a conjugation in D, this all means that we may assume that $\delta \in k$. Now Corollary 2 of Proposition 2.1 says that

$$\gamma_{l;-n+s_l} \in V_{p^n - p^{b_{l-1}}+1}.$$

To limit $\gamma_{l;-n+s_{l-1}}$ more, we conjugate. Write

$$\gamma_{l;-n+s_{l-1}} = a\beta(p^m - p^{b_{l-1}}) + \gamma, \quad \gamma \in V_{p^n - pb_{l-1}} \quad \text{and} \quad a \in k.$$

The inductive hypothesis gives $\gamma_{l-1,m+s_{l-2}} = c(p^n - p^{b_{l-2}})$, $c \neq 0$. Write $\beta = \beta(p^m - p^{q_{l-2}})$; conjugate x with $1 + \delta\eta^{s_l - s_{l-1}}$, $\delta \in k_{q_{l-2}}$. Then $1 + \delta\eta^{s_l - s_{l-1}}$ commutes with the $\gamma_j\eta^j$ such that $j < -m + s_{i-1}$, and the effect of the conjugation (mod $P^{-m+s_{l-1}}$) is to change γ to

$$\gamma - c\left(\beta\delta^{\sigma^{-m+s_{l-1}}} - \beta^{\sigma^{s_l - s_{l-1}}}\delta\right).$$

Let $\sigma' = \sigma^{s_l - s_{l-1}}$, so that σ' generates $\mathrm{Gal}(k_n/k_{b_l})$; set $\sigma^{-m+s_{l-1}} = \sigma''$. Then $\sigma'' = (\sigma')^{ap^{b_{l-1}-b_l}}$, $p \nmid a$, and, in the notation of §2,

$$\beta\delta^{\sigma''} - \delta\beta^{\sigma'} = \beta\left(\delta + c_0\delta\{p^{b_l}\} + \delta'\right) - \left(\beta + d_0\beta\{p^{b_{l-1}}\} + \beta'\right)\delta$$

(where $c_0, d_0 \in k_0$; $\delta' \in \mathrm{span}(\delta\{p^{b_l}+1\}, \dots, \delta\{p^n\})$; and $\beta' \in \mathrm{span}(\{p^{b_{l-1}}+1\}, \dots, \beta\{p^n\})) = c_0\beta\delta\{p^{b_l}\} - d_0\delta\beta\{p^{b_{l-1}}\} + \beta\delta' - \delta\beta'$. But $\delta \in V_{p^{b_{l-1}}}$, and it is not hard to verify that $\beta\delta\{p^{b_l}\}$ can be any linear combination of $\beta\{p^m - p^{b_{l-2}}\}, \dots, \beta\{p^m - p^{b_{l-1}-1}\}$, while the remaining terms are in $V_{p^m - p^{b_{l-2}}}$. Hence we can choose δ so that

$$\gamma - c\left(\beta\delta^{\sigma''} - \beta^{\sigma'}\delta\right) \in V_{p^m - p^{b_{l-2}}},$$

and we may therefore assume that $\gamma \in V_{p^m - p^{b_{l-2}}}$. Continue inductively; the next step is to conjugate with $1 + \delta\eta^{s_l - s_{l-2}}$, where $\delta \in k_{b_{l-3}}$, and thus to move γ into $V_{p^m - p^{b_{l-3}}}$. Since $b_0 = m$, we eventually move γ to 0. Hence we may assume that

$$\gamma_{l;-m+s_l} = a\beta(p^m - p^{b_{l-1}}), \qquad a \in k.$$

If $a = 0$, then $x_{(s_1)} = x_{(s_{l-1})}$. This is impossible, since s_l is a jump point. Hence $a \neq 0$, and the proof is complete.

LEMMA 3.2. (a) *Let* $\eta_1 = \eta + \delta_2\eta^2 + \delta_3\eta^3 + \cdots$ *be a prime element of* D; *suppose that* $y \in P^t$ *commutes with* $\eta_1 \bmod P^{s+2}$, *with* $s \geq t$ (*i.e.*, $[y, \eta_1] \in P^{s+2}$). *Then there exists* $y_1 \in K(\eta_1)$ *such that* $y_1 \equiv y \bmod P^{s+1}$. (*Note that* $K(\eta_1)$ *is the algebra of elements commuting with* η_1, *since* $[K(\eta_1):K] = p^n$.)

(b) *If* η_1 *is such that* $\delta_j = 0$ *unless* $j \equiv 1 \bmod p^r$ (*where* $r \leq m$), *and if* $y = \sum_{j=t}^{\infty} \varepsilon_j\eta^j$, *with* $\varepsilon_j = 0$ *unless* $p^r | j$, *then one can pick* y_1 *as above so that* $K(y_1)$ *is totally ramified of degree* $\leq p^{n-r}$.

Proof. Let $y = \sum_{j=t}^{\infty} \varepsilon_j \eta^j$; let $\varepsilon_i \eta^i$ be the first nonzero term. If $i \geq s+1$, there is nothing to prove. If $i \leq s$, then $\varepsilon_i \eta^i$ commutes with η (as one sees by computing $[y, \eta_1]$); thus $\varepsilon_i \in k$. Now $y - \varepsilon_i \eta_i^i$ commutes with $\eta_1 \bmod P^{s+2}$, and $|y - \varepsilon_i \eta_1^i| < y$. Proceed inductively to produce y_1.

(b) Continue with the notation of (a). We must have $p^r | i$, and our construction gives $y_1 = \sum_{j=t}^{\infty} \varepsilon'_j \eta_1^j$, where $\varepsilon_j \in k$ and $\varepsilon'_j = 0$ unless $p^r | j$. Hence $K(y_1) \subseteq K(\eta_1^{p^r}) \subseteq K(\eta_1)$, so that $K(y_1)$ is totally ramified. As $\eta_1^{p^r}$ commutes with k_r, the division algebra D_{y_1} of elements commuting with y_1 has index $\geq p^r$ over $K(y_1)$, and this implies that $[K(y_1) : K] \leq p^{n-r}$.

We are now ready for the main result of this section.

THEOREM 3.3. (*Approximation Theorem.*) *Let notation be as mentioned previously.*

(a) *There exist $\eta_1, \ldots, \eta_t$ for $D_1, \ldots, D_t$ respectively, such that for $2 \leq i \leq t$,*

$$\eta_i = \sum_{j=1}^{\infty} \sum_{1 \leq l < i} \delta_{l;j}^{(i)} \eta_l^j,$$

where $\delta_{i-1;l}^{(i)} = 1$; $\delta_{l,j}^{(i)} \in k_{b_{l-1}}$; $\delta_{l;j}^{(i)} = 0$ for $j \leq s_{i-1} - s_{l-1}$ and $l < i-1$; and $\delta_{i-1;j}^{(i)} \in k_{b_{i-1}}$ for $j \leq s_{i-1}$.

(b) *We have $\eta_i = \sum_{j=1}^{\infty} \delta_{i,j}^{\sim} \eta^j$, with $\delta_{i,j}^{\sim} = 0$ unless $p^{b_i} | j - 1$.*

REMARK. From (a), η_i is congruent $\bmod P^{s_{i-1} - s_{i-2} + 2}$ to the prime element $\sum_{j=1}^{p^{s_{i-1} - s_{i-2}}} \delta_{i-1;j}^{(i)} \eta_{i-1}^j$ of D_{i-1}. As the proof will show, we cannot generally do better.

Proof. We use induction on t. For $t = 1$, the first statement is vacuous, while the second simply states that D_1 has a prime η_1 such that conjugation by η_1 generates $\mathrm{Gal}(K_{b_1}/K)$.

Assume the theorem for $t-1$. From Proposition 3.1,

$$x_t - x_{t-1} \equiv c\beta(p^m - p^{b_{t-1}}) \eta^{-m+s_{t-1}} \bmod P^{-m+s_{t-1}+1}, \qquad c \in k \sim \{0\}.$$

We first find a prime η'_t in D such that $\eta_1, \ldots, \eta_{t-1}$ (from the inductive hypothesis), and η'_t satisfy (a), (b) and

(c) η'_t commutes with $x_t \bmod P^{-m+s_{t-1}+2}$,

We do this by following the procedure in the proof of Proposition 3.1. Modulo $P^{-m+s_{t-1}+2}$, we have

$$\begin{aligned} [\eta_{t-1}, x_t] &= [\eta_{t-1}, c\beta_{p^n - p^{b_{t-1}}} \eta^{-m+s_{t-1}}] \\ &= c(\beta_{p^n - p^{b_{t-1}}} - \beta^{\sigma}_{p^n - p^{b_{t-1}}}) \eta^{-m+s_{t-1}+1} \\ &= \gamma \eta^{-m+s_{t-1}+1}, \end{aligned}$$

say, where $\gamma \in V_{p^n - p^{b_{t-1}}}$. Now consider $\eta_{t-1} + \delta\eta_{t-1}^{s_{t-1} - s_{t-2} + 1}$, $\delta \in k_{b_{t-2}}$. Modulo $P^{-m+s_{t-1}+2}$, we have

$$[\delta\eta_{t-1}^{s_{t-1}-s_{t-2}+1}, x_t] = [\delta\eta_{t-1}^{s_{t-1}-s_{t-2}+1}, x_{t-1}]$$

$$(\text{since } x_t - x_{t-1} \in P^{s_{t-1}+1})$$

$$= (\delta x_{t-1} - x_{t-1}\delta)\eta_{t-1}^{s_t - s_{t-1}+1}.$$

But $x_{t-1} = x_{t-2} + \beta_{p^m - p^{b_{t-2}}}$ + higher order terms (which disappear in the commutator once we work mod $P^{-m+s_{t-1}+2}$), and x_{t-2} commutes with δ. Thus (mod $P^{-m+s_{t-1}+2}$)

$$[\delta\eta_{t-1}^{s_{t-1}-s_{t-2}+1}, x_t] = \beta_{p^n - p^{b_{t-2}}}(\delta - \delta^{s_{t-1}-m})\eta.$$

Also, $\delta^{\sigma^{s_{t-1}-n}} = \delta + c'\delta\{p^{b_{t-1}}\}$, $c' \in k \setminus \{0\}$; as δ runs through $k_{b_{t-2}}$, $\delta\{p^{b_{t-1}}\}$ runs through the elements of $V_{p^{b_{t-2}} - p} b_{t-1}$. By choosing δ appropriate, we may arrange to have

$$[\eta_{t-1} + \delta\eta_{t-1}^{s_{t-1}-s_{t-2}}, x_t] = \gamma_{(t-2)}\eta^{-n+s_{t-1}+1} \bmod P^{-n+s_{t-1}+1},$$

$$\gamma_{(t-2)} \in V_{p^n - p^{b_{t-2}}}.$$

Continue inductively; the next step involves adding $\delta'\eta_{t-2}^{s_{t-1}-s_{t-3}}$ to $\eta_{t-1} + \delta\eta_{t-1}^{s_{t-1}-s_{t-2}}$ and showing in the same way that for an appropriate $\delta' \in k_{b_{t-3}}$, we get the commutator to be

$$\gamma_{(t-3)}\eta^{-m+s_{t-1}+1} \pmod{P^{-m+s_{t-1}+2}}, \qquad \gamma_{(t-2)} \in V_{p^n - p^{b_{t-3}}}.$$

After $(t-1)$ steps, we get η'.

Now let x'_t be an element in $K(\eta'_t)$ with $x'_t \equiv x_t \bmod P^{-m+s_{t-1}+2}$ and $[K(x'_t):K] = [K(x_t):K]$; this is possible because of Lemma 3.2 and the minimality of $[K(x_t):K]$. If $x_t = x'_t$, then we are done. If not, then for some $r \geq -m + s_{t-1} + 2$ with p^{b_t}/r and some $\gamma \in V_{p^m - p^{b_t + 1}}$,

$$x_t - x'_t \equiv \gamma\eta^r \bmod P^{r+1}.$$

The argument to prove this is like the one at the start of the proof for Proposition 3.1. Let F be the minimal polynomials satisfied by x_t, and suppose that $x_t - x'_t \in P^{r-1} \setminus P^r$. Then we have $x_t - x'_t \equiv \gamma\eta^r \bmod P^{r+1}$, with $\gamma \neq 0$; moreover, $F(x'_t) = (\mathrm{Tr}_{m/b_t}\gamma)\eta^h \bmod \eta^{h+1}$, where $p^{b_t} \mid h - r$. But $K(F(x_t)) \subseteq K(x_t)$, which shows that $p^{b_t} | h$. Hence $p^{b_t} | r$. Next, $F(x'_t) \cdot (\eta')^{-h}$ commutes with η'; hence $\mathrm{Tr}_{m/b_t}\gamma$ commutes with η', and this shows that $\mathrm{Tr}_{m/b_t}\gamma = 1$. Now apply Corollary 2 of Proposition 2.1.

Modulo P^{r+2} (as the next calculations are also to be understood), we have

$$[\eta'_t, x_t] = \gamma_1\eta^{r+1}, \qquad \gamma_1 \in V_{p^m - p^{b_t}}.$$

We now argue as in the first part of the proof to remove γ_1. For $\delta \in k_{b_{t-1}}$, we get

$$\left[\delta\eta_{t-1}^{r-s_{t-1}+1}, x_t\right] = c\left(\delta\beta\left(p^n - p^{b_{t-1}}\right)^{\sigma^{r-s_t+1}} - \beta\left(p^n - p^{b_{t-1}}\right)\delta^{\sigma^{s_t}}\right)\eta^{r+1},$$
$$0 \neq c \in k.$$

Since

$$\delta\beta\left(p^n - p^{b_{t-1}}\right)^{\sigma^{r-s_t+1}} - \beta\left(p^n - p^{b_{t-1}}\right)\delta^{\sigma^{s_t}} \in -\delta\{p^{b_t}\}\beta\left(p^n - p^{b_{t-1}}\right) + V_{p^n - p^{b_{t-1}}},$$

and since $\delta\{p^{b_t}\}$ can be arbitrary in $V_{p^{b_{t-1}}-p^{b_t}+1}$, we can find δ so that

$$\left[\eta' + \delta\eta_{t-1}^{r-s_{t-1}+1}, x_t\right] = \gamma'_{(t-1)}\pi^{r+1}, \qquad \gamma'_{(t-1)} \in V_{p^m - p^{b_{t-1}}}.$$

Now iterate, next adding $\delta_{(t-2)}\eta_{t-2}^{r-s_{t-2}+1}$ to put the commutator in $(V_{p^m - p^{b_{t-2}}})\eta^{r+1}$. The same inductive argument as earlier shows that we can find η'' commuting with $x_t \bmod P^{t+2}$ and satisfying (a) and (b). The theorem follows by using induction on r and then taking limits.

4. The general approximation theorem. We now remove the restrictions on D that were imposed for §3; thus $[D:K] = n^2$ and n is arbitrary. Let $x \in D$ and write $x = \sum_{j=j_0}^{\infty}\gamma_j\eta^j$; we assume for notational convenience that x is in general position. Let $s_0, \ldots, s_{t-1}$ be the jump points of x, and let $x_1, \ldots, x_t = x$ be approximating elements for x; note that $s_0 = j_0$. Write $L_j = L(x_j)$, and let e_j, f_j be the ramification index and residue class degree respectively of $K(x_j)/K$; let $e'_j = e_j/e_{j-1}$, $f'_j = f_j/f_{j-1}$. (We define $e_0 = f_0 = 1$.) Finally, set D_j = algebra of elements commuting with x_j.

LEMMA 4.1. *Let E_j be the maximal tamely ramified extension in L_j. By conjugating x in G, we may arrange to have $E_1 \subseteq E_2 \subseteq \cdots \subseteq E_t$.*

Proof. Let $x_0 = \alpha_{j_0}\pi^{j_0}$, and let $(n, j_0) = m_0$, $n = n_0 m_0$. Then $K(x_0^{n_0})$ is unramified over K. Let $n_0 = n_1 p^{n_2}$, where $(n_1 p) = 1$, then $E_1 = K(x_0^{p^{n_2}})$ is the maximal tamely ramified extension in $K(x_0)$. Let $y = x_0^{p^{n_2}}$, and let D_y be the algebra of all elements commuting with y.

Let l be the residue class field of D_y, k_n the residue class field of D. Then $[k_n : l] = e(E_0/K)$ is prime to p. Define

$$S_j = \left\{\delta \in k_n : \delta^j \in D_y\right\},$$

$$T_j = \left\{\varepsilon \in k_n : \mathrm{Tr}_{k_n/l}\delta^{-1}\varepsilon = 0 \text{ for all nonzero } \delta \in S_j\right\}.$$

It is easy to verify the following facts (proved as Lemma 2 of [3]):

(a) $S_0 = l_i S_j$ is a vector space over l of dimension 1 if $f(E_0/K) \mid j$ and of dimension 0 otherwise.

(b) T_j is also a vector space over l, and $S_j \oplus T_j = k_n$. (This uses the fact that $[k_n : l]$ is prime to p.)

(c) If $0 \neq \delta \in S_j$, then $S_{j+j'} = \delta S_{j'}^{\sigma^j} = S_{j'}\delta^{\sigma^{j'}}$ and

$$T_{j+j'} = \delta T_{j'}^{\sigma^j} = T_{j'}\delta^{\sigma^{j'}}.$$

Furthermore, $\gamma_{j_0} \in S_{j0}$.

We show first that we can conjugate x by an element of G so that $x \in D_y$. The proof is by induction (plus an easy convergence argument). Suppose that (by conjugating if necessary) every term in the expansion of x through $\gamma_j \eta^j$ commutes with y (i.e., $\gamma_i \in S_i$ if $i \leq j$). For $\varepsilon \in T_{j+1-j_0}$, consider

$$(1 + \varepsilon^{j+1-j_0})x(1 + \varepsilon\eta^{j+1-j_0})^{-1} \equiv x + \left(\varepsilon\gamma_{j_0}^{\sigma^{j+1-j0}} - \gamma_{j_0}\varepsilon^{\sigma^{j0}}\right)\eta^{j+1} \bmod P^{j+2}$$

$$\equiv x + \zeta(\varepsilon)\eta^{j+1}, \quad \text{say.}$$

From (c), $\zeta(\varepsilon) \in T_{j+1}$. On the other hand,

$$\zeta(\varepsilon) = 0 \Leftrightarrow \left[\varepsilon\eta^{j+1-j_0}, \gamma\eta^{j_0}\right] = 0 \Leftrightarrow \left[\varepsilon\eta^{j+1-j_0}, y\right] = 0$$

$$\Rightarrow \varepsilon \in S_{j+1-j_0}.$$

Hence ζ is injective from T_{j+1-j_0} to T_{j+1}, and (a)–(c) imply now that ζ is bijective. Hence we can choose ε so that the coefficient of η^{j+1} after conjugation lies in S_{j+1}, and this is the inductive step.

Thus we have $x \in D_y$. A simple application of Hensel's lemma shows that every $K(x_r)$ contains a tamely ramified extension conjugate to E_1. As this extension is in D_y and the center of y is E_1, the extension must be E_1.

Now the lemma follows by induction on t, since henceforth we can work inside D_y.

THEOREM 4.2. *With notation as above, D_{j+1} has a prime congruent* mod $P^{s_{j+1}-s_j+1}$ *to an element of D_j.*

Proof. We work by induction on j. In view of Lemma 4.1, D_j and D_{j+1} both contain the tamely ramified extension E_j; thus (by passing, if necessary, to the algebra of elements commuting with E_j) we may assume that $K(x_j)$ is totally wildly ramified over K. Similarly, we may assume that $K(x_{j+1})$ is totally ramified over K.

Let E_{j+1} be the tamely ramified piece of the extension for $K(x_{j+1})$, and let L be a totally ramified extension of L_j in D. We may assume further that x_{j+1}, x_j commute with L. Let D_L be the algebra of elements commuting with L, and let, e.g., D_{L,x_j} be the subalgebra of elements in D_L commuting with x_j. From Theorem 3.3, we can choose primes $\tilde{\eta}_j, \eta_{j+1}$ in D_{L,x_j} and $D_{L,x_{j+1}}$ respectively with $\tilde{\eta}_j \equiv \eta_{j+1} \bmod P^{s_{j+1}-s_j-1}$. Then η_{j+1} is also a prime in $D_{x_{j+1}}$, because L is totally ramified over E_{j+1}, while $\tilde{\eta}_j \in D_{x_j}$.

Remark 1. It is natural to ask whether the result of the theorem is best possible. If $(n, p) = 1$, the answer is certainly "no"; in fact, Lemma 4.1 says that in that case, we can find primes η_j for D_{x_j} such that $\eta_j \in D_{x_i}$ for $i < j$. If n is a power of p, the answer (for totally ramified extensions) is "yes" in general; it is easy to construct examples by paralleling the constructions in the proof of Theorem 3.3. In the general case it appears that one cannot do better than Theorem 4.2, but I have not checked an example in detail.

Remark 2. The proof of Theorem 4.2 actually proves a bit more than what is stated. Since the stronger result will be useful in what follows, we state it here as a corollary.

Corollary 4.3. *In the situation of Theorem* 4.2,

(a) *If j_0 is the largest index $< j_1$ such that s_{j_0} is a jump point with wild ramification, then there is a prime in D_{j_1} congruent* mod $P^{s_{j_1}-s_{j_0}+1}$ *to an element of D_{j_0};*

(b) *If there is no index $< j_1$ where wild ramification occurs, then for every $j < j_1$, $D_{j_1} \subset D_j$.*

5. Commutators in division algebras. We shall later need a result about commutators, which we prove now. Let $G_j = 1 + P^j$, $G = G_1$.

Proposition 5.1. *Let $y \in [G, G] \cap G_h$, $h \geq 2$; let r be any integer $> h$. Then we can write y* mod G_r *as a product of commutators,*

$$y = (u_1, v_1)(u_1', v_1') \cdots (u_{r-h}, v_{r-h})(u_{r-h}', v_{r-h}'),$$

where each u_i, u_i' is of the form $1 + \delta_j \eta$ ($\delta_j \in k_n$) and the v_j, v_j' are of the form $1 + \varepsilon_j \eta^{h+j-2}$.

Proof. By an obvious induction argument, it suffices to consider the case $r = h + 1$. In what follows, all calculations on G are performed modulo G_r. Write $y = 1 + \sum_{j=h}^{\infty} \gamma_j \eta^j$. If χ is any character of $D^\times$ that factors through the norm map, then $\chi(y) = 1$. This implies in particular that if $w \in K \cap (1 + P^{1-2h})$, then $\mathrm{Tr}_{D/K} w(y-1) = 0$; see, e.g. Theorem 1 of [2]. Thus $\mathrm{Tr}_{k_n/k}\gamma_h = 0$ if $n \mid h$.

In general, we have

$$(u_1, v_1) = 1 + \left(\delta_1 \varepsilon_1^{\sigma} - \varepsilon_1 \delta_1^{\sigma^{h-1}}\right)\eta^h.$$

If $n \mid h$, let $\delta_1 = 1$. Then

$$(u_1, v_1) = 1 + (\varepsilon_1^{\sigma} - \varepsilon_1)\eta^h;$$

as γ_h has trace 0, we can choose ε_1 so that $(u_1, v_1) \equiv y \bmod G_{r'}$. In this case, we can let $u_1' = v_1' = 1$.

If $n \nmid h$, write

$$\lambda_0 = \delta_1^{\sigma^h} \delta_1^{\sigma^{h+1}} \cdots \delta_1^{\sigma^{h+(t-1)}},$$

where t is the smallest integer such that $n = h + t$; let $\lambda = \delta_1^{\sigma^{h+t}}\lambda_0$. Then

$$\lambda_0\left(\delta_1 \varepsilon_1^{\sigma} - \varepsilon_1 \delta_1^{\sigma^{h-1}}\right) = (\lambda \varepsilon_1)^{\sigma} - \lambda \varepsilon_1.$$

So for fixed δ_1.

$$\gamma_h = \delta_1 \varepsilon_1^{\sigma} - \varepsilon_1 \delta_1^{\sigma^{h-1}} \quad \text{for some } \varepsilon_1 \in k_n \Leftrightarrow \mathrm{Tr}_{k_n/k}(\lambda_0 \gamma_h) = 0.$$

Let $\delta = \delta_1^{\sigma^h}$, so that $\lambda_0 = \delta\delta^{\sigma} \cdots \delta^{\sigma^{t-1}}$. From Hilfsatz 4 of [7], we can (by choosing δ_1 appropriately) make λ_0 any element such that

$$N_{k_n/k_s}\lambda_0 \in k, \qquad s = (h, n).$$

Write $n = p^{n_0}n_1$, where $(p, n_1) = 1$. Suppose first that $p^{n_0} \nmid h$; let $s = p^{s_0}s_1$, and set $s' = p^{s_0}n_1$. From Proposition 2.3, we know that there exists λ_0' such that $N_{k_n/k_{s'}}\lambda_0' \in k_{s'}$, $\mathrm{Tr}_{k_n/k_s'}\lambda_0'\gamma_h = 0$. Since n/s' is a power of p, $N_{k_n/k_s'}$ is an automorphism on $k_{s'}^{\times}$. Thus we can multiply λ_0' by an element of $k_{s'}^{\times}$ to get λ_0 with $N_{k_n/k_s'}\lambda_0 \in k$, $\mathrm{Tr}_{k_n/k_s'}\lambda_0\gamma_h = 0$. Thus we can find δ_1, ε_1 to prove the lemma. Here, too, we have $u_1' = v_1' = 1$.

Therefore we may suppose that $p^{n_0} \mid h$. Restrict attention to elements $\delta_1 \in k_{n_1}$; it is not hard to see that it suffices to consider the case $n_0 = 0$. We are now in the tamely ramified situation. Note that

$$(u_1, v_1)(u_1', v_1') = 1 + \left[\left(\delta_1 \varepsilon_1^{\sigma} - \delta_1^{\sigma^{h-1}}\varepsilon_1\right) + \left(\delta_1' \varepsilon_1'^{\sigma} - \delta_1'^{\sigma^{h-1}}\varepsilon_1'\right)\right]\eta^h.$$

We need to show that the sum in brackets can be made equal to any element of k_n. It suffices, since $\mathrm{Tr}_{k_n/k}$ is faithful on k, to show that then

(a) there exists ε_1, δ_1 such that $\mathrm{Tr}_{k_n/k}(\delta_1 \varepsilon_1^{\sigma} - \varepsilon_1 \delta_1^{\sigma^{h-1}}) = 1$; and

(b) if $\mathrm{Tr}_{k_n/k}\kappa = 0$, then $\exists \delta_1', \varepsilon_1'$ with $\kappa = \delta_1'\varepsilon_1' - \varepsilon_1 \delta_1'^{\sigma^{h-1}}$.

Part (b) is easy; in fact, we can take $\delta_1 = 1$. As for (a), fix δ_1 and suppose that $\mathrm{Tr}_{k_n/k}(\delta_1\varepsilon_1^\sigma - \varepsilon_1\delta_1^{\sigma^{h-1}}) = 0$ for all ε_1. Then $\mathrm{Tr}_{k_n/k}\varepsilon_1^\sigma(\delta_1^{\sigma^h} - \delta_1) = 0$ for all ε_1; hence $\delta_1^{\sigma^h} - \delta_1 = 0$, or σ^h fixes δ_1. We need only choose δ_1 to be outside the fixed field of σ^h to complete the proof.

Part II. Representations of Division Algebras of Index pn_0, $p \nmid n_0$.

6. Some simpler cases. Let D be a division algebra of index pn_0 over its center K, where K has residual characteristic p and $(p, n_0) = 1$. We use the notation of §§1 and 4.

We wish to determine the irreducible unitary representations of $D^\times$. In general, we work by determining those of G. Any such representation has a kernel containing some $(1 + P^{m+1}) = G_{m+1}$ for an $m \geq 0$; choose m to be as small as possible. In this and the next few sections, we assume that m is odd; we remove this assumption in §9. Let $m = 2m' - 1$, and let χ be a character on the Abelian group $G_{m'}/G_{m+1}$ that is nontrivial on G_m. As noted in §1, one can write $\chi = \chi_x$ for some $x \in P^{-m} \setminus P^{-m+1}$. Let $s_0, \ldots, s_{t-1}$ be the jump points for x. We shall assume until §9 that the s_j are all odd.

The construction of the representations of G is done by (mathematical) induction. We assume that the representations of the corresponding group G', and of $D'^\times$, are known if D' is a division algebra whose index over its center K' is a proper divisor of n (of course, K' also has residual characteristic p). We also assume that all irreducibles of G containing $\chi_{x'}$ are known when $x' \in P^{-m+1}$. In this section we deal with some relatively easy cases, leaving the hard work for §§7 and 8.

Case I. x is not in general position. Then there is a central element $x_0 \in P^{-m}$ such that $x - x_0 \in P^{-m+1}$. We may let $x_0 = \gamma_{-m}\eta^{-m}$, in fact. If χ is any character of $K^\times \cap G$ satisfying $\chi(1 + \delta\eta^m) = \psi(\gamma_{-m}\delta)$, then $\chi \circ N_{D/K}$ agrees with χ_{x_0} on P^m. Therefore $\chi_{\tilde{x}} = \chi_x(\chi \circ N_{D/K})^{-1}$ is a character on P^{m-1}. Moreover, π_0 contains $\chi_{\tilde{x}} \Leftrightarrow \pi_0 \otimes (\chi \circ N_{D/K})$ contains χ_x, and the representations containing $\chi_{\tilde{x}}$ are assumed known.

Henceforth we assume that x is in general position. Write

$$x = \sum_{j=-m}^{\infty} \gamma_j\eta^j, \qquad x_0 = \gamma_{-m}\eta^{-m} \notin K.$$

Case II. $t = 1$. Then $K(x_0)$ and $K(x)$ have the same ramification index and residue class degree. The following argument is like that in [**6**]; indeed, it applies whenever $t = 1$ (regardless of n). It is also not strictly necessary for the construction in our case, but I think that it may be useful to have the following result stated explicitly.

THEOREM 6.1. *Suppose that x is in general position and that $t = 1$. Then $y \in D^x$ commutes with χ_x on $G_{m'} \Leftrightarrow y \in G_{m'} \cdot D_x$. Moreover, χ_x extends to a character on $G_{m'}D_x$. Let χ be any such extension. For each representation π_0 of $D_x^\times$ trivial on $G_{m'} \cap D_{x'}^\times$ extend π_0 to $G_m \cdot D_x$ by making it trivial on $G_{m'}$. Then $\chi \otimes \pi_0$ induces to an irreducible representation of $D^\times$; moreover, every representation of $D^\times$ containing χ_x is obtained in this way.*

Proof. Satz 2 of [**8**] gives the result about elements commuting with χ_x. If $w \in G_{m'}$ and $y \in G_{m'} \cdot D_x$, then $\chi_x((w, y)) = 1$ because y commutes with χ_x, while χ_x is 1 on $(D_x^\times, D_x^\times \cap G_{m'})$ because χ_x factors through the norm map on $D_x^\times$. Hence χ_x extends to $G_{m'} \cdot D_x^\times$. $\chi_x \otimes \pi_0$ is a multiple of χ_x on $G_{m'}$, and Theorem 6 of [**9**] implies that it induces to an irreducible representation of $D_x^\times$. Finally, we show that we obtain all representations of $D^\times$ containing χ_x which are trivial on η^n. We may assume that $\chi(\eta^n) = 1$. The set $S = \{\pi_0 \in (D_x^\times)^\wedge : \pi_0$ is trivial on $G_{m'} \cap D_x^\times$ and on $\eta^n\}$ satisfies

$$\sum_{\pi_0 \in S} (\operatorname{Dim} \pi_0)^2 = [G \cap D_x : G_{m'} \cap D_x] \cdot e_1(q^{f_1} - 1),$$

since the left-hand side is $[D_x^\times : \langle \eta^n \rangle \cdot G_{m'}]$. On the other hand,

$$[D^\times : D_x^\times \cdot G_{m'}] = [G : G_{m'}]([G \cap D_x : G_{m'} \cap D_x])^{-1}(n/e_1)\frac{(q^n - 1)}{(q^{f_1} - 1)}.$$

Since χ_x appears in $\pi = \operatorname{Ind}_{G_{m'}D_x^x \to D^x}(\chi_x \otimes \pi_0)$ exactly $(\dim \pi_0)$ times, we see that the π-primary subspace in $\operatorname{Ind}_{G'_m\langle\eta^n\rangle \to D^x}\chi_x$ has dimension $(\dim \pi_0)^2[D^\times : D_x^\times \cdot G_m]$. Hence, by Frobenius reciprocity, these subspaces account for a subspace of dimension

$$\sum_{\pi_0 \in S} (\dim \pi_0^2)[D^x : D_x^x \cdot G_m] = [G : G_{m'}]n(q^n - 1) = [D^x : G_{m'}\langle \eta^n \rangle],$$

or for all of $\operatorname{Ind}_{G'_m\langle\eta^n\rangle \to D^x}\chi_x$. This proves the theorem.

REMARKS. 1. We have dealt with $D^\times$ rather than G in this theorem; obviously, there is a similar theorem for G. When we come to deal with the case m even, the representation χ_x will not extend to $G_{m'}D_x^\times$, and we need to use a Weil representation; see, e.g., [**4**].

2. In general, $K(x)$ is not determined up to conjugacy by $x \bmod p^{-m'}$; this is most evident in the case where $K(x)$ is inseparable. This fact makes it more difficult to arrange for a good parametrization of $(D^{\times})^{\wedge}$.

Case III. The first jump point involves only tame ramification. From Lemma 4.1, we may assume that the $\gamma_j \eta^j$ all commute. Now the construction of [**5**] (or [**1**]) yields all irreducible representations of $D^{\times}$ that agree with χ_x on G_m, and hence all such representations agreeing with χ_x on $G_{m'}$. The proofs are exactly as in [**1**]; we omit details.

Case IV. The first jump point is not totally wildly ramified. Let K_0 be the largest tamely ramified field in $K(x_0)$; we may assume again (Lemma 4.1) that every $\gamma_j \eta^j$ commutes with K_0. Let D_0 be the algebra of elements commuting with K_0, and construct all representations σ of $D_0^{\times} \cap G$ containing $\chi_x | D_0^{\times} \cap G_{m'}$. The same construction as in [**1**] shows that there is a subgroup N_0 of $G_{m'}$ on which χ_x is trivial, which is normalized by $D_0^{\times} \cap G$, and which satisfies

$$N_0\left(D_0^{\times} \cap G\right) = G_{m'}\left(D_0^{\times} \cap G\right), \qquad N_0 \cap \left(D_0^{\times} \cap G\right) \subseteq G_{m+1}.$$

Then we can extend σ to $G_{m'}(D_0^{\times} \cap G)$ by making it trivial on N_0. Induce σ to G to get an irreducible π containing χ_x. That π is irreducible and that every π containing χ_x is obtained in this way can be proved essentially as in Case III, by following the corresponding proofs in [**1**].

7. Extending χ_x. We henceforth assume that

(a) the element x is in general position;

(b) the first jump point of x, $s_0 = -m$, is totally wildly ramified.

Let $-s_j = 2s_j' - 1$ (recall that we are assuming that the s_j are all odd), and define $H = H_x$ to be the group

$$G_{s_0'}\left(G_{s_1'} \cap D_{x_1}\right) \cdots \left(G_{s_{t-1}'} \cap D_{x_{t-1}}\right)\left(G \cap D_{x_t}\right).$$

We wish to show that χ_x extends to a character of H. This is equivalent to:

THEOREM 7.1. *If $y \in [H, H] \cap G_{s_0'}$, then $\chi_x(y) = 1$.*

Proof. This follows the lines of the proof of Lemma 8 of [**2**]. We write y as a product of commutators. We note that

$$(7.1) \qquad \left(v_1 v_2, w\right) = \left(v_1 v_2 v_1^{-1},\ v_1 w v_1^{-1}\right)\left(v_1, w\right)$$

and

$$(v, w_1w_2) = (v, w_1)\left(w_1vw_1^{-1}, w_1w_2w_1^{-1}\right). \tag{7.2}$$

In this way, we can let y be a product of commutators of the form (v, w), where $v = 1 + \gamma\eta_0^r$, $w = 1 + \gamma\eta^s$, and η_0, η are specified primes, while $\gamma, \delta \in k_n$. Similarly, we can commute commutators by using

$$u_2u_1 = u_1\left(u_1^{-1}u_2u_1\right),$$

where, if $u_2 = (v, w)$, then $u_1^{-1}u_2u_1 = (u_1^{-1}vu_1, u_1^{-1}wu_1)$.

We proceed by a lengthy sequence of steps.

(a) If $v, w \in H$ and $vwv^{-1} \in G_{s_0'} = G_{m'}$, then $w \in G_{m'}$ (since $G_{m'}$ is normal in G), and $\chi_x(vwv^{-1}) = \chi_x(w)$, since $\chi_x(vwv^{-1}) = \chi_{v^{-1}xv}(w)$ and elements of H preserve χ_x. In particular, $\chi_x((v, w)) = 1$.

(b) The following computation will arise repeatedly in the proof: if x and v commute, then

$$\operatorname{Tr}_{D/K} x\left(vuv^{r-1} - uv^r\right) = \operatorname{Tr}_{D/K}\left(vxuv^{r-1} - xuv^r\right) = 0,$$

since $\operatorname{Tr}_{D/K}(ab) = \operatorname{Tr}_{D/K}(ba)$.

(c) Let $H' = H_x' = G_{s_0'}(G_{s_1'} \cap D_{x_1}) \cdots (G_{s_{t-1}'} \cap D_{x_{t-1}})$. If $u, v \in H'$, then $(u, v) \in G_{s_0'}$ and $\chi_x((u, v)) = 1$. To prove this, it suffices to consider the case where $u \in G_{s_i'} \cap D_{x_i}$ and $v \in G_{s_j} \cap D_{x_j}$, as repeated use of (7.1) and (7.2) shows. Assume $i \leq j$, for definiteness; write $u = 1 + u_0$, $v = 1 + v_0$. Then modulo P^{m+1},

$$\begin{aligned}(u, v) &= 1 + (u_0v_0 - v_0u_0) + \left(v_0u_0^2 - u_0v_0u_0\right) + \left(v_0u_0v_0 - u_0v_0^2\right)\\ &= (1 + u_0v_0 - v_0u_0)\left(1 + v_0u_0^2 - u_0v_0u_0\right)\left(1 + v_0u_0v_0 - u_0v_0^2\right).\end{aligned}$$

Since $u_0v_0 - v_0u_0 \in P^{m - s_j + 1}$, we have

$$\chi_x(1 + u_0v_0 - v_0u_0) = \chi_{x_j}(1 + u_0v_0 - v_0u_0) = 1,$$

from (b) (note that x_j and v_0 commute). The other terms are taken care of similarly.

(d) Now (a) and (c) reduce us to considering

$$w = (u_1, v_1) \cdots (u_r, v_r), \tag{7.3}$$

where one of each u_j, v_j is in $G \cap D_x$. We shall assume for notational convenience that $u_j \in G \cap D_x$ for all j; this will not affect the proof. We may also assume that $u_j = 1 + u_{j,0}$, $v_j = 1 + v_{j,0}$, where the $u_{j,0}, v_{j,0}$ are "monomials":

$$u_{j,0} = \delta\eta_t^a; \quad v_{j,0} = \varepsilon\eta_l^b, \qquad a = a_j,\ b = b_j \tag{7.4}$$

where $\delta,\ \varepsilon \in k_n,\ v_{j_0} \in G_{s'_s} \cap D_{x_l}$, and η_l, η_t are primes for D_{x_l}, D_{x_t} respectively. We fix these primes so that η_l is congruent mod $P^{s_l - s_1}$ to an element η'_l of D_{x_1}.

We may assume also that for the first r_0 commutators, and only for these, we have $v_j \in D_{x_t}$. Then the product of these commutators is in $G_{s'_{t-1}+1}$ (since every other commutator is in $G_{s'_{t-1}+1}$), and we may, therefore, assume from Proposition 5.1 that $a_j + b_j \geq s'_{t-1}$ for all j. (In fact, we can have $a_j =$ the order of a prime in D_x, for $j \leq r_0$).

Write $u'_j = 1 + u'_{j,0}$, $v'_j = 1 + v'_{j,0}$, where

(7.5) $\quad u'_{j,0} = \delta(\eta'_t)^a,\ v'_{j,0} = \varepsilon \eta'^b_l \quad \big(\eta'_t, \eta'_l$ primes in D_{x_1} related to η_t, η_1

as in Theorem 4.2 and Corollary 4.3$\big)$;

the a, b, δ, ε in (7.5) agree with those in (7.4). Let

$$y' = (u'_1, v'_1) \cdots (u'_r, v'_r).$$

Then y' is a commutator in D_{x_1}, so that

$$\chi_{x_1}(y') = 1.$$

The proof consists of showing that $\chi_x(y) = \chi_{x_1}(y)$.

(e) Write $(u_j, v_j) = 1 + w_{j,0} + w_{j,1}$, where

$$w_{j,0} = \sum_{i=1}^{\infty} (-1)^i \big(v_j u_j^i - u_j v_j u_j^{i-1}\big);$$

write $y = 1 + w_0 + w_1$, where $w_0 = \sum_{j=1}^r w_{j,0}$. If one multiplies out all the commutators, w_1 consists of all terms of degree ≥ 2 in the v_j. For instance, we have (mod G_{m+1})

$$w_{j,1} = \sum_{i=1}^{\infty} (-1)^i \big(u_{j,0} v_{j,0} u_{j,0}^{i-1} v_{j,0} - v_{j,0} u_{j,0}^i v_{j,0} + v_{j,0} u_{j,0}^i v_{j,0}^2 - u_{j,0} v_{j,0} u_{j,0}^{i-1} v_{j,0}^2\big),$$

and w_1 is a sum of the $w_{j,1}$, plus products of the $w_{j,0}$ and $w_{j,1}$. Thus $w_1 \in P^{m'}$; as $w \in G_{m'}$, we have $w_0 \in P^{m'}$, and

$$\chi_x(1 + w_0 + w_1) = \chi_x(1 + w_0)\chi_x(1 + w_1).$$

Similarly, we write $(u'_j, v'_j) = 1 + w'_{j,0} + w'_{j,1}$ and

$$y' = (u'_1, v'_1) \cdots (u'_r, v'_r) = 1 + w'_0 + w'_1;$$

the same argument as above shows that

$$\chi_{x_1}(y') = \chi_{x_1}(1 + w'_0)\chi_{x_1}(1 + w'_1), \quad w'_0 \text{ and } w'_1 \in P^{m'}.$$

(f) We have

$$\chi_x(1 + w_0) = 1$$

from (b), since each u_j commutes with x. Similarly, $\chi_{x_1}(1 + w_0') = 1$.

Moreover, w_1 and w_1' are congruent mod $P^{-s_1} + 1$. The reason is that each term of w_1 is at least quadratic in the $u_{j,0}$; moreover, if $u_{j,0}$ appears, so does $v_{j,0}$. By Corollary 4.3 (applied to $j_0 = 0$), $u_{j,0}v_{j,0} \equiv u'_{j,0}v'_{j,0}$ mod $P^{s_r - s_0 + c - 1}$, where $v_j \in D_{x_r}$ and c is the order of $u'_{j,0}v'_{j,0}$. This order is at least $P^{s'_r + 1}$. Thus $u_{j,0}v_{j,0} \equiv u'_{j,0}v'_{j,0}$ mod $P^{m - s'_r}$. Now suppose that the term of w_1 contains a product $u_{j,0}v_{j,0}u_{i,0}v_{i,0}$, where $v_{i,0} \in D_{x_{r'},j,0}$ and $r' \geq r$. Then $u_{i,0}v_{i,0}$ and $u'_{i,0}v'_{i,0}$ are in $P^{s'_{r'} + 1}$ (and congruent mod $P^{m - s'_{r'}}$); hence the products are congruent mod $P^{m - s'_r + s'_{r'} + 1}$, and therefore congruent mod P^{m+1}. The remaining terms in $w_{j,1}$ are of the form $u_{j,0}v_{j,0}^2$ or $v_{j,0}u_{j,0}v_{j,0}$. But, e.g., $u_{j,0}v_{j,0}^2 \equiv u'_{j,0}v'^2_{j,0}$ mod $P^{m - s'_r + s'_r} = P^m$, and χ_x, χ_{x_1} agree on G_m.

(g) It follows that $\chi_{x_1}(1 + w_1') = \chi_x(1 + w_1)$; (e) and (f) give

$$\chi_x(w) = \chi_{x_1}\big((1 + w_0')(1 + w_1')\big) = 1,$$

since $(1 + w_0')(1 + w_1')$ is congruent mod P^{m+1} to a commutant in D_{x_1}. This proves the theorem.

8. Construction of the representations. In this section, we construct "enough" representations of $H = H_x$ so that inducing to G produces all the desired irreducibles. This is not too difficult, but takes some time.

Recall that $K(x_1)$ is totally wildly ramified. Hence we may (and do) assume that $\gamma_{-m} \in k$. This implies that the residue class fields of $D_1, \ldots, D_t$ are all extensions of degree prime to p. The residue class field of D_{x_1} is k_{n_0}; let l_j be the residue class field of D_j, and let $k(x_j)$ have ramification index and residue class degree e_j, f_j respectively.

We know that H_x is generated by $G_{m'}$ and elements $1 + \delta\eta_j^r$, where η_j is a prime of D_j that is close to an element of D_1, $\delta \in l_j$, and $r \geq s_j'$. Moreover, $\eta_j \equiv \delta_j'\eta^{f_j}$ mod $P^{f_j + 1}$ for some $\delta' \in k_{n_0}$. For each integer r, let

$$S_r(j) = \left\{\delta \in k_{n_0};\ \text{there is an element of } D_j \text{ congruent mod } P^{r+1} \text{ to } \delta\eta^r\right\},$$

$$T_r(j) = \left\{\varepsilon \in l_1 : \mathrm{Tr}_{k_{n_0}/l_j}(\varepsilon\delta^{-1}) = 0 \text{ for all nonzero } \delta \in S_r(j)\right\}.$$

We sometimes write S_r, T_r for $S_r(t)$, $T_r(t)$.

LEMMA 8.1. (a) $S_0(j) = l_j$,

(b) *If* $\delta \in S_r(j)$ *is nonzero, then*

$$S_{r+r'}(j) = \delta^{\sigma^{r'}}S_{r'}(j) = \delta S_{r'}(j)^{\sigma^r}$$

and

$$T_{r+r'}(j) = \delta^{\sigma^{r'}} T_{r'}(j) = \delta T_{r'}(j)^{\sigma^r}.$$

(c) *Both* $S_r(j)$ *and* $T_r(j)$ *are vector spaces over* l_j, *and* $S_r(j) \oplus T_r(j) = k_{n_0}$.

(d) $\mathrm{Dim}_{l_j'} S_r(j) = 1$ *if* $F_j \mid r$ *and* 0 *otherwise.*

Proof. This is essentially done as Lemma 2 of [**3**].

Now let N be the subgroup of H generated by $G_{m'}$ and the elements $1 + \varepsilon\eta_j^r \in D_j \cap H$ with $\varepsilon \in T_r$.

LEMMA 8.2. (a) *N is normal in H,*

(b) $H/N \cong (G \cap D_x)/(G_{m'} \cap D_x)$.

Proof. (a) Since

$$\left(1 + \varepsilon_1\eta_{j_1}^{r_1}\right)\left(1 + \varepsilon_2\eta_{j_2}^{r_2}\right) \equiv 1 + \varepsilon_1\eta_{j_1}^{r_1} + \varepsilon_2\eta_{j_2}^{r_2} \bmod P^{m'}$$

if the $1 + \varepsilon_i\eta_{j_i}^{r_i}$ are generators of N, and since $\eta_{j_i}^{r_i}$ is congruent to an element of D_x modulo $P^{m'}$, it is not hard to see that N is composed entirely of elements of the form

$$w = 1 + \sum_{j=j_0}^{\infty} \varepsilon_j \eta_t^j, \qquad 2j_0 \geq m' \text{ and } \varepsilon_j \in T_j \text{ if } j < m', \tag{8.1}$$

while H is composed of elements of the form

$$y = 1 + \sum_{j=1}^{\infty} \delta_j \eta_t^j, \qquad \delta_j \in S_j \text{ if } 2j < m'. \tag{8.2}$$

To prove (a), it suffices to show that every element of the form (8.2) normalizes the elements of the form (8.1).

Write

$$w = w_{j_0} w_{j_0+1} \cdots w_{m'}, \quad w_j = 1 + \varepsilon_j \eta_t^j \quad \text{for } j < m';$$

then $w_{m'} \in 1 + P^{m'}$. Similarly, one can write

$$y = y_1 \cdots y_{s'} \quad y_j = 1 + \delta_j' n_t^j \quad \text{with } \delta_j' \in S_j \text{ for } 2j < m';$$

$$y_s \in 1 + P^s \text{ and } 2s \geq m'.$$

Then y_s commutes with $w \bmod 1 + P^{m'}$ and $w_{m'}$ commutes with each $y_j \bmod 1 + P^{m'}$. It thus suffices to show that $y_i w_j y_i^{-1} \in N$ for $i < s$ and

$j < m'$. This is a straightforward calculation:

$$y_i w_j y_i^{-1} = 1 + \varepsilon_j \eta_t^j + \left(\delta_i' \varepsilon_j^{\sigma^i} - \varepsilon_j \delta_i'^{\sigma^j}\right)\eta_t^{i+j} + \left(\delta_i' \varepsilon_j^{\sigma^i} \delta_i'^{\sigma^{i+j}} - \varepsilon_j \delta_i'^{\sigma^j} \delta_i'^{\sigma^{i+j}}\right)\eta_t^{2i+j} + \cdots,$$

and repeated application of Lemma 8.1 shows that $y_i w_j y_i^{-1} \in N$.

As for (b), $G \cap D_x$ injects into H and hence maps into H/N; from the form of elements in N and H given in (8.1) and (8.2), it is easy to verify that the map is surjective and has $G_{m'} \cap D_x$ as kernel.

Any representation of $G \cap D_x$ that is trivial on $G_{m'} \cap D_x$ can thus be as a representation of H trivial on N. Take an extension of χ_x to G (guaranteed by Theorem 7.1); call the extension χ_x as well. Then $\chi_x \otimes \sigma$ is also a representation of H, and is a multiple of χ_x on N.

Let $H' = G_{m'}(G_{-s_1} \cap G_{x_1}) \cdots (G_{-s_{t-1}} \cap D_{x_{t-1}})(G \cap D_x)$. The key results we need about H' and H are contained in the following proposition:

PROPOSITION 8.3. (a) $[H' : H] = [H : G_{m'}(G \cap D_x)]$.

(b) *If $y \in G$ is such that $\chi_x(ywy^{-1}) = \chi_x(w)$ for all $w \in G_{m'}$, then $y \in H'$; conversely, $y \in H' \Rightarrow \chi_x(ywy^{-1}) = \chi_x(w)$ for all $w \in G_{m'}$.*

(c) *If $y \in H'$ is such that $\chi_x(ywy^{-1}) = \chi_x(w)$ for all $w \in H$ such that $ywy^{-1} \in H$, then $y \in H$; conversely, $\chi_x(ywy^{-1}) = \chi_x(w)$ if $w, y \in H$.*

Before proving Proposition 8.3, we show how it solves the problem of constructing representations of G containing χ_x. For each $\sigma \in (D_x \cap G/D_x \cap G_{m'})^\wedge$, let $\pi_\sigma = \mathrm{Ind}_H^G(\sigma \otimes \chi_x)$.

THEOREM 8.4. *The π_σ are distinct irreducibles of G, and* $\mathrm{Ind}_{G_m}^G \chi_x \cong \bigoplus \sigma[H' : H](\mathrm{Dim}\,\sigma)\pi_\sigma$.

Proof. The irreducibility follows from Proposition 8.3 (c) and Theorem 6 of [**9**]. Proposition 8.3 (c) and Theorem 7 of [**9**] also imply that the π_σ are distinct.

Frobenius reciprocity says that π_σ appears in $\mathrm{Ind}_{G_m}^G \chi_x$ with a multiplicity equal to the multiplicity of χ_x in $\pi_\sigma|_{G_m}$. But

$$\pi_\sigma|_{G_m} \cong \bigoplus_{y \in G/H} \sigma \otimes \chi_{yxy^{-1}}|_{G_m} \cong (\dim\sigma) \bigoplus_{y \in G/H} \chi_{yxy^{-1}},$$

and $\chi_{yxy^{-1}} = \chi_x \Leftrightarrow y \in H'/H$ from Proposition 8.3(b). Thus the multiplicity of π_σ is $[H' : H]\dim\sigma$. Finally,

$$\begin{aligned}[H' : H](\dim\sigma)\dim\pi_\sigma &= [H' : H][G : H]\dim^2\sigma \\ &= [G : G_{m'}(D_x \cap G)] \cdot \dim^2\sigma\end{aligned}$$

by Proposition 8.3(a). Summing over σ shows that we have accounted for a subrepresentation of $\mathrm{Ind}_{G_m}^G \chi_x$ of dimension

$$\sum_\sigma [G:G_{m'}(d_x \cap G)]\dim^2\sigma = [G:G_{m'}(D_x \cap G)][D_x \cap G : D_x \cap G_{m'}]$$
$$= [G:G'_m],$$

and hence for all of $\mathrm{Ind}_{G_m}^G \chi_x$.

We still need to prove Proposition 8.3. For part (b), write $w = 1 + w_0$, $w_0 \in P^{m'}$. Then

$$\chi_x(ywy^{-1}) = \psi \circ \mathrm{Tr}_{D/K}(xyw_0y^{-1}) = \psi \circ \mathrm{Tr}_{D/K}(y^{-'}xyw_0).$$

This is equal to $\psi \circ \mathrm{Tr}_{D/K}(xw_0)$ for all $w_0 \in P^{m'}$ iff

$$x - y^{-1}xy \in P^{-m'+1},$$

and this congruence holds iff $y \in H'$ by Satz 2 of [**8**]. Half of part (c) is easy; $y, w \in H \Rightarrow \chi_x(ywy^{-1}) = \chi_x(w)$ from Theorem 7.1. For the rest of (c) and for (a), we need to do some more work.

The field k_n is the compositum of k_p and k_{n_0}. Define $\alpha_1, \beta_1 \in k_p$ as in §2.

Lemma 8.5. *Let $x - x_j \equiv \zeta_j\eta^{s_j} \bmod P^{s_j+1}$, with $j \geq 1$. Then by conjugating, we may assume that $\zeta_j = \beta_1\zeta'_j$, $\zeta'_j \in S_{s_j}(j)$ and $\zeta'_j \neq 0$.*

Proof. We have seen that $x - x_0 \equiv \gamma_{-m}\eta^{-m} \bmod P^{-m+1}$, with $\gamma_{-m} \in k$ and $\gamma_{-m} \neq 0$. Since x_0 is totally wildly ramified, we have $(m, p) = m_0$.

Assume the lemma for $j - 1$. We certainly have

$$\zeta_j = \sum_{i=0}^{p-1} \alpha_1^i \zeta_{i,j} \quad \text{for appropriate } \zeta_{i,j} \in k_{n_0}.$$

We describe anything of the form

$$\sum_{i=0}^{p-2} \alpha_1^i \gamma_j, \qquad \gamma_j \in k_{n_0},$$

as "small". Note that if we conjugate x with $1 + \gamma\eta^{s_j+m}$, we get $x + \gamma_0(\gamma^{\sigma^{-m}} - \gamma)\eta^{s_j} (\bmod P^{s_j+1})$, and $\gamma_0(\gamma^{\sigma^{-m}} - \gamma)$ can be any element α of k_n such that $\mathrm{Tr}_{k_n/k_{n_0}} \alpha = 0$; that is, we can always get rid of any small element by conjugating.

Thus we have $\xi_j = \alpha_1^{p-1}\zeta'_1$, $\zeta'_j \in k_{n_0} = S_{s_j}(1)$. Now conjugate x with $1 + \gamma\eta_1^{s_j - s_1}$, where $\gamma \in k_{n_0} = l_1$; we get (mod P^{s_j+1}, as all future computations are made)

$$x + \left(\delta\zeta_{1'}^{\sigma^{s-s_1}} - \delta^{\sigma^{s_1}}\zeta_1\right)\eta^{s_j}, \quad \delta \text{ any element of } k_{n_0}.$$

Now $\zeta_1 = \alpha_1^{p-r}\zeta_1'$ with $\zeta_1' \in k_{n_0}$. Moreover, Satz 8 of [7] shows that the largest tamely ramified extension of $K(\zeta_1\eta^{s_j})$ is (conjugate to) the largest tamely ramified extension of x_1; this implies that $\delta \in S_r(2) \Leftrightarrow \delta\eta^r$ and $\zeta_1'\eta^{s_1}$ commute. So set

$$F_1(\delta) = \delta\zeta_1'^{\sigma^{s_j-s_1}} - \delta^{\sigma^{s_1}}\zeta', \qquad \delta \in k_{n_0} = S_{s_j-s_1}(1).$$

This is a k-linear map, and it is $0 \Leftrightarrow \delta \in S_{s_j-s_1}(2)$. For $\delta \in T_{s_j-s_1}(2)$, Lemma 8.1 implies that $F_1(\delta) \in T_{s_j}(2)$. Hence counting implies that F_1 maps k_{n_0} into $T_{s_j}(2)$. Moreover,

$$\delta\zeta_1^{\sigma^{s_j-s_1}} - \delta^{\sigma^{s_1}}\zeta_1 = \alpha^{p-1}F_1(\delta) + \text{a small term}.$$

Thus we can change γ_{s_j} (the coefficient of η^{s_j} in the expansion of x) by any element of the form $\alpha^{p-1}\varepsilon$, $\varepsilon \in T^{s_j}(2)$. It follows that we can arrange to have $\zeta_j = \alpha_1^{p-1}\zeta_{j;2}$, with $\zeta_{j;2} \in S_j(2)$.

We continue inductively. Conjugate x with $1 + \gamma\eta_2^{s_j-s_1}$, $\gamma \in l_2$; we get $x + (\delta\zeta_2^{\sigma^{s_j-s_2}} - \delta^{\sigma^{s_2}}\zeta_2)\eta^{s_j}$, where δ is any element of $S_{s_j-s_1}(2)$. The same argument as before shows that $\delta \in S_r(2) \Leftrightarrow \delta\eta^r$ commutes with $\zeta_1'\eta^{s_1}$ and $\zeta_2'\eta^{s_2}$. So define $F_2(\delta) = \delta\zeta_2'^{\sigma^{s_j-s_2}} - \delta^{\sigma^{s_2}}\zeta_2'$, $\delta \in S^{s_j-s_2}(2)$; by the same argument as above, F_2 maps $S^{s_j-s_2}(2)$ onto $T^{s_j}(3)$. It follows that we can have $\zeta_j = \alpha_1^{p-1}\zeta_{j;3}$, $\zeta_{j;3} \in T_j(3)$, and the same inductive procedure gives the result.

Corollary. *In the above setup, let L_j be the maximal tamely ramified extension in $K(x_j)$. Choose $\zeta_{j,0}$ such that $\zeta_{j,0}\eta_j^{s_j/f_j} \equiv \zeta_j'\eta^{s_j} \bmod \eta^{s_j+1}$. Then L_{j+1} is conjugate* (mod *an element of G*) *to the maximal tamely ramified extension in $L_j(\zeta_{j,0}\eta_j^{s_j/f_j})$.* (This follows from Satz 8 of [7], as we observed in the course of the proof of Lemma 8.5.)

Remark 1. The corollary shows the following useful fact: suppose that $\delta \in S_r(j)$. Then $\delta \in S_r(j+1)$ if $\delta\eta^r$ and $\zeta_j'\eta^{s_j}$ commute, or, equivalently, if $\delta\eta^r$ and $\zeta_{0;j}^{s_j/f_j}$ commute mod P^{s_j+r+1}. The reason is that $\delta\eta^r$ then commutes with the largest tamely ramified extension in $K(\zeta_j'\eta^{s_j})$. Another way of stating the condition is that $\delta'\eta_j^r$ is congruent mod P^{rf_j+1} to an element of $K(x_{j+1})$ if $\delta' \in S_0(j)$ and $[\delta\eta_j^r, \zeta_{0;j}\eta^{s_j/f_j}] = 0$.

Remark 2. While the process of Lemma 8.5 may mean that the tamely ramified extensions in $K(x_j)$ are no longer contained in the algebra generated by k_{n_0} and η^p, conjugating by elements of G does not change the membership of the $S_r(j)$ or $T_r(j)$.

We now return to the proof of Proposition 8.3. Part (a) is a matter of counting. For each $r \le m'$, we compute card$(H \cap G_r)/(H \cap G_{r+1})$. If $s'_j \le r < s'_{j-1}$, this number is 1 if $f_j \nmid r$ and card $l_j = q^{n/e_j}$ if f_j/r. Since $f_j \mid (2s'_j - 1, 2s'_{j-1} - 1)$, we see that for $j \le t$,

$$\prod_{s'_j \le r < s'_{j-1}} [H \cap G_r : H \cap G_{r+1}] = q^{l(n/e_j f_j)(s'_{j-1} - s'_j)}.$$

A similar calculation for $r < s'_t$ gives

$$\prod_{1 \le r < s'_t} [H \cap G_r : H \cap G_{r+1}] = q^{(n/e_t f_t)(s'_{t-1} - s'_t)}, \qquad 2s'_t - 1 = -s_t = f_t.$$

So

$$\log_q[H : H \cap G_{m'}] = \sum_{j=1}^{t} \frac{n}{n_j}\left(s'_{j-1} - s'_j\right), \qquad n_j = \left[K_j : K\right].$$

Similarly,

$$\log_q[G \cap D_x : G_{m'} \cap D_x] = \frac{n}{n_t}\left(s'_0 - s'_t\right)$$

and

$$\log_q\left[H' : H' \cap G_{m'}\right] = \frac{n}{n_t}\left(s'_0 - (s_{t-1} - s_0) - s_{t'}\right) + \sum_{j=1}^{t-1} \frac{n}{n_j}(-s_j + s_{j-1}).$$

It is now easy to verify that

$$2\log_q[H : H \cap G_{m'}] = \log_q[G \cap D_x : G_{m'} \cap D^x] + \log_q\left[H' : H' \cap G_{m'}\right],$$

which proves (a).

As for the second half of (c), assume that $\chi_x(ywy^{-1}) = \chi_x(w)$ for all $w \in H$ such that $ywy^{-1} \in H$, but that $y \notin H$. Since $y \in H'$ (from (b)), we may assume (after multiplying by an element of H) that

$$y = 1 + \varepsilon\eta_j^r + \text{higher order terms},$$

where $1 + \varepsilon\eta_j^r c G_{-s_j} \cap D_j$ and $1 + \varepsilon\eta_j^r \notin G_{s'_j} \cap D_j$. We have $\varepsilon\eta_j^r \in P^{rf_j} \sim P^{rf_j - 1}$, and we may assume that $\varepsilon\eta_j^r$ is not congruent mod P^{rf_j+1} to an element of D_{j-1}.

Let $w_\delta = 1 + \delta_0\delta\eta_j^{r'}$, where $(r + r')f_j = -s_j$, $\delta \in l_j$ is fixed, and δ runs over l_j. The w_δ are all in D_j, and (with $w_\delta = 1 + w_{\delta,0}$ and $y = 1 + y_0$)

$$yw_\delta y^{-1}w_\delta^{-1} \equiv 1 + \sum_{i=1}^{\infty} (-1)^{i-1}\left(y_0 w_{\delta,0} y_0^{i-1} - w_{\delta,0} y_0^i\right)$$
$$+ \sum_{i=2}^{\infty} (-1)^{i-1}\left(w_{\delta,0} y_0 w_{\delta,0}^{i-1} - y_0 w_{\delta,0}^i\right) \bmod P^{m+1}.$$

Each term in the sums is in $P^{m'}$, and each term with an index $i \geq 2$ is in P^{-s_j+1}. We have

$$\chi_x\left(yw_\delta y^{-1}w_\delta^{-1}\right) = \chi_x(1 + y_0w_{\delta,0} - w_{\delta,0}y_0)$$
$$\cdot \prod_{i=2}^{\infty} \chi_x\left(1 + (-1)^{i-1}\left(y_0w_{\delta,0}y_0^{i-1} - w_{\delta,0}y_0^i\right)\right)$$
$$\cdot \prod_{i=2}^{\infty} \chi_x\left(1 + (-1)^{i-1}\left(w_{\delta,0}y_0w_{\delta,0}^{i-1} - y_0w_{\delta,0}^i\right)\right),$$

and (b) of the proof of Theorem 7.1 shows that every term in the two infinite products is 1, since we can replace χ_x with χ_{x_j}. Similarly, $\chi_{x_j}(1 + y_0w_{\delta,0} - w_{\delta,0}y_0) = 1$, since x_j commutes with y and w_δ. This implies that

$$\zeta_0\varepsilon^{\sigma^{s_j}} - \zeta^{\sigma^{rF_j}}\varepsilon = 0,$$

since ζ_0, $\varepsilon \in l_j$ and $[k_{n_0}:l_j]$ is prime to p. But then $\zeta_0\eta_j^{-(r+r')}$ and $\varepsilon\eta_j^r$ commute. As noted in the Remark after Lemma 8.5, this means that $\varepsilon\eta_j^r$ is congruent (mod P^{rF_r+1}) to an element of D_{j+1}, which contradicts our assumption on y. This finishes the proof of Proposition 8.3 and of Theorem 8.4.

9. Extending to $D^\times$; removing hypotheses. In this section, we deal with two issues: extending the representations of G to representations of $D^\times$, and removing the assumption that the s_j are odd. The procedures are essentially those of [**1**], [**2**] and [**6**], and our discussion will be brief.

The elements of $k_n^\times$ that commute with π_0 (obtained from χ_x) are those in D_x. Indeed, if $\delta \in k_n^\times \cap D_x$, then δ commutes with χ_x, and we could simply extend χ_x to $H(k_n^\times \cap D_x)$. If $\delta \in k_n^\times$ but $\delta \in D_x$, then the argument of Theorem 8.4 is easily adapted to show that π and the representation π^δ defined by $\pi^\delta(y) = \pi(\delta y\delta^{-1})$ are disjoint. Hence we can extend π_0 to $G(k_n \cap D_x)$ and induce to get the irreducibles of $G \cdot k_n^\times$.

The situation for extending to $D^\times$ is similar; arguing as in §5 of [**2**], one can show that the elements of $D^\times$ commuting with a representation π_σ of $G \cdot k_n^\times$ that comes from χ_x are precisely the elements of $D_x^\times(G \cap k_n^\times)$. Thus we need to extend π_1 to $G \cdot k_n^\times \cdot \langle\eta_t\rangle$ (since η_t is a prime in D_x). Since $\langle\eta_t\rangle \cong Z$, there is no Mackey obstruction; however, we have no good way of describing $\pi_1(\eta_t)$. Inducing to $D^\times$ then gives an irreducible π.

To see how to deal with even s_j, consider the case where m is even. Define m' by $2m' = -m$. Now χ_x is initially defined on $G_{m'+1}$. We want to define an extension of χ_x to

$$H: G_{m'}\left(D_{x_1} \cap G_{s_1'}\right) \cdots \left(D_{x_{t-1}} \cap G_{s_{t-1}'}\right)\left(D_{x_t} \cap G\right).$$

The argument of §7 lets us extend χ_x to $G_{m'+1}(D_{x_1} \cap G_{s'_1}) \cdots (D_{x_t} \cap G) = H_0$. To go to H from H_0, note that a set of coset representatives for $G_{m'}/G_{m'+1}$ consists of the elements $y_\delta = 1 + \delta\eta^{m'}$, $\delta \in k_n$. The map μ: $(\delta_1, \delta_2) \mapsto \chi_x(y_{\delta_1} y_{\delta_2} y_{\delta_1}^{-1} y_{\delta_2}^{-1})$ is an antisymmetric bilinear form on k_n, and δ is in the radial of $\mu \Leftrightarrow y_\delta \in H$. There is a unique irreducible projective representation of H/H_0 with the above form μ as multiplier; when p is odd, it corresponds to a Heisenberg-type representation on the k-vector space $k_n/\text{Rad}\,\mu$. Tensor this representation with χ_x to get a representation which might be called χ'_x; on H_0, χ'_x is a multiple of χ_x. The reasoning in §8 applies to show that $\text{Ind}_H^G \chi'_x \otimes \sigma$ is irreducible and that these irreducibles exhaust $\text{Ind}_{G_m}^G \chi_x$; even the changes in the counting arguments are not difficult. If some other s_j is even, we again get a Heisenberg-type representation on $G_{s'_j}$, where $2s'_j = -s_j$; the details are similar to those sketched above. Extending to those elements of $k_n^\times$ that commute with x now involves the Weil representation; see, e.g., [**4**].

REFERENCES

[1] L. Corwin, *Representations of division algebras over local fields*, Adv. in Math., **10** (1973), 387–394.

[2] ______, *Representation of division algebras over local fields*: II, Pacific J. Math., **101** (1982), 49–70.

[3] L. Corwin and P. J. Salley, *Characters of division algebras and of* Gl_n *for p-adic fields on the elliptic set (tamely ramified case)*, in preparation.

[4] L. Corwin and R. Howe, *Computing characters of tamely ramified p-adic division algebras*, Pacific J. Math., **73** (1977), 461–477.

[5] R. Howe, *Representation theory for division algebras over local fields (tamely ramified case)*, Bull. Amer. Math. Soc., **17** (1971), 1063–1066.

[6] ______, *Kirillov theory for compact p-adic groups*, Pacific J. Math., **73** (1977), 365–381.

[7] H. Koch, *Eisensteinsche Polynomfolgen und Arithmetik in Divisionsalgebren über lokalen Körpern*, Math. Nachr., **104** (1981), 229–251.

[8] ______, *Zur Arithmetik in Divisionsalgebren über lokalen Körpern*, Math. Nachr., **104** (1981), 9–19.

[9] G. W. Mackey, *On induced representations of groups*, Amer. J. Math., **73** (1951), 576–592.

[10] A. Moy, *Local constants and the tame Langlands correspondence*, Amer. J. Math.

[11] A. Weil, *Basic Number Theory*, New York, Springer, 1967.

Received June 20, 1986 and in revised form August 17, 1986.

RUTGERS UNIVERSITY
NEW BRUNSWICK, NJ 08903