PACIFIC JOURNAL OF MATHEMATICS
Vol. 132, No. 2, 1988

LOEWY SERIES AND SIMPLE PROJECTIVE MODULES IN THE CATEGORY $\mathcal{O}_S$

RONALD S. IRVING AND BRAD SHELTON

Results are obtained on the Loewy length and Loewy series of generalized Verma modules and projective modules in certain categories $\mathcal{O}_S$ of modules over a complex, semisimple Lie algebra. The results obtained rely on a study of the behavior of Loewy series under translation functors and on the existence of simple projective modules in suitable blocks of $\mathcal{O}_S$. An example is given of two generalized Verma modules such that the space of $\mathcal{O}_S$-homomorphisms from the first to the second is two-dimensional.

1. Introduction.

1.1. In this paper we study the Loewy series of generalized Verma modules and self-dual projectives in the category $\mathcal{O}_S$ associated to a complex semisimple Lie algebra $\mathfrak{g}$ and a parabolic subalgebra $\mathfrak{p}_S$. The principal theme is the translation of data from a block of $\mathcal{O}_S$ associated to a non-regular weight to the blocks arising from other, possibly regular, weights, especially in case the first block contains a simple, projective module. In particular, we find that $\mathcal{O}_S$ contains simple projectives for any choice of parabolic subalgebra if $\mathfrak{g}$ is of type A_n, and thereby obtain precise formulas for Loewy length in $\mathcal{O}_S$. For other Lie algebras, the presence of a simple projective depends on the choice of $\mathfrak{p}_S$. We consider some cases where simple projectives do exist, and study in detail the smallest example in which a simple projective fails to exist: $\mathfrak{g}$ is of type D_4 and $\mathfrak{p}_S$ is a minimal parabolic. For this example we find a generalized Verma module whose socle is a direct sum of two isomorphic copies of a simple module. Thus, we obtain a pair of generalized Verma modules whose space of homomorphisms is two-dimensional.

1.2. To describe our results in more detail, recalling earlier related work along the way, we need to introduce some notation to be used throughout the paper. Any unexplained terminology can be found in [8], [9]. However, some of the notation below differs from that in [8], [9] because of the convention used here for highest weights of Verma modules. We fix a complex simple Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$ and Borel subalgebra $\mathfrak{b}$ containing $\mathfrak{h}$; corresponding to these choices are a root system R, with positive roots R^+ containing a base of simple roots B, and Weyl group $\mathscr{W}$. The lattice of integral weights is denoted by $P(R)$. We denote by $\check{\alpha}$ the co-root $2\alpha/(\alpha, \alpha)$, and $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha$. The Verma module of highest weight $\lambda - \rho$ is denoted $M(\lambda)$, with simple top $L(\lambda)$ and projective cover $P(\lambda)$ in the category $\mathscr{O}$.

For any $\lambda \in \mathfrak{h}^*$, we associate the relative root system R_λ with base B_λ and Weyl group $\mathscr{W}_\lambda$ as in [8], and denote by w_λ the longest element of $\mathscr{W}_\lambda$. To any $w \in \mathscr{W}_\lambda$ we associate the τ-invariant $\tau_\lambda(w) = \{\alpha \in B_\lambda | ws_\alpha < w\}$, where $<$ denotes the Bruhat order of $\mathscr{W}_\lambda$.

Let S be a subset of B, and let $\mathfrak{p}_S$ be the associated parabolic subalgebra, with semisimple part $\mathfrak{g}_S$, nilradical $\mathfrak{m}_S$, and Weyl group $\mathscr{W}_S$. (For more details, see [8, 2.0]). Let $P_S^{++} = \{\lambda \in \mathfrak{h}^* | (\lambda, \check{\alpha}) \in \mathbf{N}^+$ for $\alpha \in S\}$. For $\lambda \in P_S^{++}$, we form the coset space ${}^S\mathscr{W}_\lambda = \mathscr{W}_S \backslash \mathscr{W}_\lambda$, and identify it with the set of minimal length coset representatives in $\mathscr{W}_\lambda$. In this way ${}^S\mathscr{W}_\lambda$ inherits a Bruhat order and length function, with longest element ${}^S w_\lambda$. The generalized Verma module of highest weight $\lambda - \rho$, for $\lambda \in P_S^{++}$, is denoted $M_S(\lambda)$ and its projective cover in $\mathscr{O}_S$ is $P_S(\lambda)$. We denote by $\mathscr{O}_S^\lambda$ the full subcategory of $\mathscr{O}_S$ consisting of modules whose composition factors lie in the set $\{L(w\lambda) | w \in {}^S\mathscr{W}_\lambda\}$. For λ regular, $\mathscr{O}_S^\lambda$ is a block of $\mathscr{O}_S$, but typically $\mathscr{O}_S^\lambda$ may split into more than one block. Let Λ be the coset $\lambda + P(R)$ of $\mathfrak{h}^*/P(R)$. Then $\mathscr{O}_S^\Lambda = \bigoplus_{\mu \in \Lambda} \mathscr{O}_S^\mu$; in other words, $\mathscr{O}_S^\Lambda$ consists of the part of the category $\mathscr{O}_S$ accessible from $\mathscr{O}_S^\lambda$ via translation functors.

Given λ and Λ as above and $\mu \in \Lambda$, the Jantzen translation functors $T_\lambda^\mu : \mathscr{O}_S^\lambda \to \mathscr{O}_S^\mu$ and $T_\mu^\lambda : \mathscr{O}_S^\mu \to \mathscr{O}_S^\lambda$ are defined [12]. They are exact and adjoint to each other. We will denote by θ_λ^μ the functor $T_\mu^\lambda \circ T_\lambda^\mu$. Let $B_\mu^0 = \{\alpha \in B_\mu | (\mu, \alpha) = 0\}$, with $\mathscr{W}_\mu^0 = \{w \in \mathscr{W} | w\mu = \mu\}$, and let w_μ^0 be the longest element of $\mathscr{W}_\mu^0$, which is itself the Coxeter group generated by $\{s_\alpha | \alpha \in B_\mu^0\}$. In case λ is regular and $B_\mu^0 = \{\alpha\}$, we write θ_α for θ_λ^μ.

Assume $\lambda \in P_S^{++}$ is regular. The set

$$\{w \in {}^S\mathscr{W}_\lambda | L(w\lambda) \text{ is a summand of } \operatorname{soc} M_S(y\lambda) \text{ for some } y \in {}^S\mathscr{W}_\lambda\}$$

is denoted ${}^S X_\lambda$. It is proved in [8] that the corresponding socular simples $\{L(w\lambda) | w \in {}^S X_\lambda\}$ have self-dual projective covers $P_S(w\lambda)$, with respect to the contravariant duality functor D defined on $\mathscr{O}$. Moreover, for each $w \in {}^S X_\lambda$ there is a unique element $\check{w} \in {}^S\mathscr{W}_\lambda$ maximal in the set $\{y \in {}^S\mathscr{W}_\lambda | (M_S(y\lambda) : L(w\lambda)) \neq 0\}$. We also let $\overline{w}$ denote the longest element of the Coxeter group generated by $\{s_\alpha | \alpha \in \tau_\lambda(w)\}$.

1.3. We can now review the results on Loewy length from **[9]**. Let $\ell\ell(M)$ denoe the Loewy length of a module M. Let λ be a dominant regular weight, that is, a regular weight with $(\lambda, \check{\alpha}) \geq 0$ for $\alpha \in R_\lambda^+$, and let μ be a dominant weight in $\lambda + P(R)$. One of the main results of **[9]**, an extension of Vogan's conjecture, is that $\ell\ell\theta_\alpha M \leq \ell\ell M + 2$ for $M \in \mathscr{O}^\lambda$. This allows one to prove that $\ell\ell M(w\lambda) = \ell(w_\lambda w) + 1$, so $\ell\ell M(\lambda) = \ell(w_\lambda) + 1$, and $\ell\ell P(w_\lambda\lambda) = 2\ell(w_\lambda) + 1$.

For M in $\mathscr{O}^\mu$, it is also proved that $\ell\ell T_\mu^\lambda M \geq \ell\ell M + 2\ell(w_\mu^0)$. In 2.1 we will prove that for M simple, equality holds. This is an easy consequence of a theorem of Bernstein and Gelfand **[1]** and the θ_α result above. With this in mind, let us note for λ integral how the Loewy length results in $\mathscr{O}^\lambda$ could be easily derived. As shown in **[9]**, it suffices to prove $\ell\ell P(w_\lambda\lambda) = 2\ell(w_\lambda) + 1$. Let $\mu = 0$. Then $\mathscr{O}^\mu$ consists only of the simple, projective module $L(\mu)$. Therefore $\ell\ell T_\mu^\lambda L(\mu) = 1 + 2\ell(w_\mu^0) = 1 + 2\ell(w_\lambda)$. But $T_\mu^\lambda L(\mu)$ inherits projectivity and self-duality from $L(\mu)$, and $P(w_\lambda\lambda)$ is the only possible self-dual projective in $\mathscr{O}^\lambda$. We simultaneously obtain the self-duality of $P(w_\lambda\lambda)$ and its Loewy length.

This observation can be taken as the starting point of the present paper. For arbitrary λ, there may be no simple projective in $\mathscr{O}^\Lambda$, hence no T_μ^λ or θ_μ^λ to use as above. Instead, we must work with the θ_α's, which always exist even for non-integral λ; this is the reason longer proofs of self-duality and Loewy length were required in **[8]**, **[9]**. When we pass to $\mathscr{O}_S^\Lambda$ for $\lambda \in P_S^{++}$, even for integral λ, there may be no simple projective in $\mathscr{O}_S^\Lambda$ at all. But if there is one, we can derive information in $\mathscr{O}_S^\Lambda$. We should note that the idea of using a simple projective in $\mathscr{O}$ and translation functors to obtain information on other projectives can be found in a paper of J. Humphreys **[6]**. As Humphreys observed, this is a standard technique in the representation theory of Chevalley groups, with Steinberg modules playing the role of simple projectives.

1.4. Let us recall what is known. As noted already, it is proved in **[8]** that $P_S(w\lambda)$ is self-dual for $w \in {}^S X_\lambda$, and these are the only possible self-dual projectives. If $\mathscr{O}_S^\mu$ contains a simple projective $L(w\mu)$ for $\mu \in \Lambda$, then $T_\mu^\lambda L(w\mu)$ must be a self-dual projective. This yields the existence of some self-dual projective in $\mathscr{O}_S^\lambda$, but an argument in **[8]** shows that self-duality of $P_S(w\lambda)$ for all $w \in {}^S X_\lambda$ follows. (The proof of this in **[8]** seemed to require the validity of the Jantzen conjecture, which we prefer to avoid, so that the general proof of self-duality was preferred. On the other hand, this dependence on Jantzen's conjecture can be eliminated, **[10]**.)

Regarding Loewy length, suppose for some $x \in {}^S X_\lambda$ that $x\bar{x} = \check{x}$. Then it is proved in **[9]** that $\ell\ell M_S(\check{x}\lambda) = \ell(\bar{x}) + 1$ and $\ell\ell P_S(x\lambda) = 2\ell(\bar{x})+1$. Moreover, by an argument analogous to that in **[8**, §9] (with a similar dependence on Jantzen's conjecture, which can be eliminated by **[10]**) we obtain $\ell\ell M_S(\check{w}\lambda) = \ell(\bar{x}) + 1$ and $\ell\ell P_S(w\lambda) = 2\ell(\bar{x}) + 1$ for all $w \in {}^S X_\lambda$. We prove in §2 that the existence of x with $x\bar{x} = \check{x}$ is closely related to the presence of a simple projective in $\mathscr{O}_S^\Lambda$. In fact, we have:

PROPOSITION. *There is a simple projective in $\mathscr{O}_S^\Lambda$ if and only if for some $x \in {}^S X_\lambda$ satisfying $x\bar{x} = \check{x}$ there is a weight $\mu \in \Lambda$ with $B_\mu^0 = \tau_\lambda(x)$. In this case, $P_S(x\lambda) = T_\mu^\lambda L(y\mu)$ for $L(y\mu)$ the simple projective.*

Thus, the presence of a simple projective yields, as in 1.3, the desired Loewy length conclusions. Although in general this is a stronger hypothesis than $x\bar{x} = \check{x}$, for λ integral the existence of the desired μ is automatic. The advantage of this result is that it is not obvious in general how to find $x \in {}^S X_\lambda$ with $x\bar{x} = \check{x}$, but as we will see in §3, finding simple projectives can be easy. In particular, for $\mathfrak{g}$ of type A_n and S arbitrary, there is always a simple projective in $\mathscr{O}_S^{P(R)}$, as well as for several families of parabolics in the other classical cases, and all maximal parabolics. This allows us to extend the Loewy length results of **[9]** considerably.

Observe in addition that the existence of $x \in {}^S X_\lambda$ with $x\bar{x} = \check{x}$ is a condition on the Coxeter group $\mathscr{W}_\lambda$ and the choice of parabolic subgroup $\mathscr{W}_S$. This can be seen since all the data involved is encoded in Kazhdan-Lusztig polynomials, and the validity of the Kazhdan-Lusztig conjecture for arbitrary λ shows that the data depends only on $\mathscr{W}_\lambda$ and S. Thus, if we find a simple projective in $\mathscr{O}_S^{P(R)}$ for a particular S and Weyl group $\mathscr{W}$, the existence of a suitable x as well as the resulting Loewy length information carries over to ${}^{S'}\mathscr{W}_\lambda$ and $\mathscr{O}_{S'}^\lambda$, for λ a weight of any semisimple Lie algebra such that $(\mathscr{W}_\lambda, B_\lambda) \cong (\mathscr{W}, B)$ and S' corresponds to S under the isomorphism.

1.5. A natural question raised by these considerations is what we can say about Loewy series of generalized Verma modules and self-dual projectives in $\mathscr{O}_S^\nu$ for ν dominant but not necessarily regular. Even if $\mathscr{O}_S^\nu$ contains no simple projective, it may split into a sum of small blocks which can be analyzed. If so, and if we can control the increase of Loewy length under T_ν^λ for λ regular in $\nu + P(R)$, we can again obtain information for $\mathscr{O}_S^\lambda$. In particular, we should expect self-dual projectives in $\mathscr{O}_S^\lambda$ to have Loewy length $2\ell(w_\nu^0)$ more than in $\mathscr{O}_S^\nu$, and generalized Verma modules to have maximal Loewy length $\ell(w_\nu^0)$ more. In turn, passing to other non-regular μ's in $\nu + P(R)$, we can obtain information for $\mathscr{O}_S^\mu$.

We carry out this process for certain examples in §§4 and 5, with an indication in §2 of how one may be able to control T_ν^λ even when $\mathscr{O}_S^\nu$ contains no simple projective. The example in §4 is for $\mathfrak{g}$ of type A_n and $\mathfrak{p}_S$ a particular minimal parabolic. We choose a ν for which $\mathscr{O}_S^\nu$ contains a simple projective and pass through $\mathscr{O}_S^\lambda$ to $\mathscr{O}_S^\mu$ for a particular μ of interest and analyze $\mathscr{O}_S^\mu$. In §5 we consider $\mathfrak{g}$ of type D_4 and $\mathfrak{p}_S$ the minimal parabolic associated to the simple root at the center of the Coxeter graph. This is the smallest example for which $\mathscr{O}_S^{P(R)}$ contains no simple projective. We are able to choose a $\nu \in P_S^{++}$ for which $\mathscr{O}_S^\nu$ can be analyzed and T_ν^λ controlled, so that for μ taken to the highest root, we obtain Loewy length information in $\mathscr{O}_S^\mu$.

1.6. It is possible in some cases to use the knowledge of $\ell\ell P_S(w\mu)$ for $P_S(w\mu)$ a self-dual projective to obtain the layers of the socle and radical filtrations on $P_S(w\mu)$. A guess was formulated in **[9]** for μ regular which can be extended in general. Let μ be a dominant weight with $w\mu \in P_S^{++}$ and $P_S(w\mu)$ self-dual. The proposal for layers of the radical filtration for $P_S(w\mu)$ is as follows:

$$(*)\quad \operatorname{rad}_r P_S(w\mu) = \bigoplus_{z\in\mathscr{W}_\mu/\mathscr{W}_\mu^0}\ \bigoplus_{i<\ell\ell M_S(z\mu)} (\operatorname{rad}_i M_S(z\mu) : L(w\mu))\operatorname{rad}_{r-i} M_S(z\mu).$$

Here z runs over coset representatives of $\mathscr{W}_\mu$ modulo $\mathscr{W}_\mu^0$, and if $z\mu \notin P_S^{++}$ then of course $M_S(z\mu) = 0$ so there is no harm including such z's in the sum. The rationale for this guess is given in **[9]**. As is also pointed out, we may be able to confirm the guess in certain cases in a manner summarized by the following Lemma:

LEMMA. *Suppose* (i) *the layers of radical filtrations for all $M_S(z\mu)$ are known,* (ii) *the Loewy length $2t+1$ of $P_S(w\mu)$ is known and coincides with the number of non-zero layers of* $(*)$, *and* (iii) *for each $i < t$ the guesses for layers i and $2t - i$ in* $(*)$ *coincide. Then* $(*)$ *correctly depicts the layers in both the radical and socle filtrations of $P_S(w\mu)$.*

The proof of this is essentially given in [9], and it is used in many examples in [9].

In our examples of §§4 and 5 it is possible to determine the Loewy layers of all the generalized Verma modules by direct calculation. As indicated in 1.5, we also know the Loewy length of the self-dual projectives. Thus, we are in a position to apply the Lemma, and hypothesis (iii) is satisfied as well. This is particularly of interest for the D_4 example of §5. For, as observed at the end of [9], all the previous examples for which $(*)$ had been confirmed have generalized Verma modules occuring once, if at all, in a Verma flag for $P_S(x\mu)$. But in the D_4 example there is a multiplicity 2. This yields more substantive evidence for the validity of $(*)$ in general: for all parabolics and singular as well as regular weights.

Another interesting feature of the D_4 example is that for appropriate y and w, the socle of $M_S(w\mu)$ is $L(y\mu) \oplus L(y\mu)$. Thus, we find that $\mathrm{Hom}_{\mathscr{O}_S}(M_S(y\mu), M_S(w\mu))$ has dimension 2. Moreover, for λ a dominant regular weight, we are able to preserve enough information under T_μ^λ to show that $M_S(w\lambda)$ has socle $L(y\lambda) \oplus L(y\lambda)$, where w is the shortest element in the coset $w\mathscr{W}_\mu^0$ and y is the longest element in $y\mathscr{W}_\mu^0$. This answers negatively the question raised in several places (for λ general or λ regular) of whether Hom spaces between generalized Verma modules must be 0 or 1 dimensional.

1.7. Some of the results in this paper were obtained while the first author was a visitor at U.C.S.D. in the winter of 1985; he thanks the mathematics department for its hospitality. Both authors were partially supported during the work by the N.S.F., the second via an N.S.F. postdoctoral fellowship.

2. Loewy length under translation.

2.1. In this section the Loewy length results discussed in the introduction will be proved. We will fix throughout a dominant regular weight λ, with $\Lambda = \lambda + P(R)$, and another dominant weight μ in Λ.

PROPOSITION. *Let M be a module in $\mathscr{O}^\lambda$. Then $\ell\ell T_\mu^\lambda T_\lambda^\mu M \leq \ell\ell(M) + 2\ell(w_\mu^0)$.*

REMARK. In case $B_\mu^0 = \{\alpha\}$ for some $\alpha \in B_\lambda$, the functor $T_\mu^\lambda T_\lambda^\mu$ is θ_α and the formula is $\ell\ell\theta_\alpha M \leq \ell\ell(M) + 2$. As noted in 1.3, this is the extension of Vogan's conjecture proved in [9]. The proposition may be viewed as a generalization, and the proof depends on this special case.

Proof. Suppose $\ell(w_\mu^0) = r$ and w_μ^0 factors as $s_{\alpha_1} \cdots s_{\alpha_r}$ with $\alpha_i \in B_\lambda$. Let $\theta = \theta_{\alpha_r} \circ \cdots \circ \theta_{\alpha_1}$. By induction, we obtain the inequaltiy $\ell\ell\theta M \leq \ell\ell M + 2r$. Thus it suffices to prove that θ_λ^μ is a summand of θ; that is, $\theta_\lambda^\mu M$ is a summand of θM for all M in $\mathcal{O}^\lambda$. By the fundamental theorem of Bernstein-Gelfand on projective functors [**1**; 3.3, 3.5], this will follow if $\theta_\lambda^\mu M(\lambda)$ is a summand of $\theta M(\lambda)$.

The module $\theta_\lambda^\mu M(\lambda)$ is $T_\mu^\lambda M(\mu)$. By results of [**12**] this is a projective module with a Verma flag whose constituents are $\{M(y\lambda) | y \in \mathscr{W}_\mu^0\}$, from which we see that $T_\mu^\lambda M(\mu) = P(w_\mu^0 \lambda)$. On the other hand, $\theta M(\lambda)$ is also projective, and applying the θ_{α_i}'s in turn we find that it has a Verma flag with $M(w_\mu^0\lambda)$ as uppermost quotient. Thus $P(w_\mu^0\lambda)$ is a summand, as desired. □

Corollary 1. *Let w be an element of $\mathscr{W}_\lambda$ of maximal length in the coset $w\mathscr{W}_\mu^0$. Then $T_\mu^\lambda T_\lambda^\mu L(w\lambda)$ has Loewy length $1 + 2\ell(w_\mu^0)$.*

Proof. The hypothesis on w is equivalent to $T_\lambda^\mu L(w\lambda)$ being nonzero, in which case it equals $L(w\mu)$. Thus $T_\mu^\lambda T_\lambda^\mu L(w\lambda) = T_\mu^\lambda L(w\mu)$, and by [**9**], the Loewy length is at least $1 + 2\ell(w_\mu^0)$. The preceding Proposition yeilds equality. □

Corollary 2. *Suppose $L(w\mu)$ is projective in $\mathcal{O}_S^\mu$ for some $w \in \mathscr{W}_\lambda$. Then $T_\mu^\lambda L(w\mu)$ is a self-dual projective in $\mathcal{O}_S^\lambda$ of Loewy length $1+2\ell(w_\mu^0)$.*

Proof. The Loewy length formula follows from Corollary 1 once we note that $T_\lambda^\mu L(w'\lambda) = L(w\mu)$ for w' the longest element in wW_μ^0. □

2.2. We obtain in this subsection the result discussed in 1.4.

Proof of Proposition 1.4. Suppose x exists in ${}^S X_\lambda$ with $x\bar{x} = \check{x}$ and $\mu \in \Lambda$ satisfies $B_\mu^0 = \tau_\lambda(x)$. Then $\bar{x}$ is exactly w_μ^0. The proof in [**9**; 4.3] shows that $P_S(x\lambda)$ has a Verma flag with constituents $\{M_S(xz\lambda) | z \in \mathscr{W}_\mu^0\}$, each occurring once. Therefore $T_\lambda^\mu P_S(x\lambda)$ is a projective module whose Verma flag constituents are all copies of $M_S(x\mu)$. This means each $M_S(x\mu)$ is itself a self-dual projective, forcing $M_S(x\mu)$ to be $L(x\mu)$. The same argument used in 2.1 shows that $T_\mu^\lambda L(x\mu) = P_S(x\lambda)$.

Conversely, suppose $L(x\mu)$ is a simple projective for some $\mu \in \Lambda$, and x is the longest element in $x\mathscr{W}_\mu^0$. Then $T_\mu^\lambda L(x\mu) = P_S(x\lambda)$ and the Verma flag constituents are $\{M_S(xz\lambda) | z \in \mathscr{W}_\mu^0\}$. Thus $\check{x}$ is the shortest element of $x\mathscr{W}_\mu^0$, equalling $x\bar{x}$. □

As noted in 1.4, in the integral case the hypothesis on the existence of μ is automatic, so we obtain:

COROLLARY. *Assume λ is integral. Then $\mathscr{O}_S^\Lambda$ contains a simple projective if and only if there is an $x \in {}^S X$ with $x\bar{x} = \check{x}$. In this case, $P_S(x\lambda)$ has Loewy length $2\ell(\bar{x}) + 1$ and $M_S(\check{x}\lambda)$ has Loewy length $\ell(\bar{x}) + 1$.*

2.3. As discussed in 1.5, one might expect for μ and λ as in 2.1 that $\ell\ell T_\mu^\lambda M = \ell\ell M + 2\ell(w_\mu^0)$. One collection of modules for which this is known to hold is the self-dual projectives in $\mathscr{O}^\Lambda$, for which we have $T_\mu^\lambda P(w_\lambda \mu) = P(w_\lambda \lambda)$, with $\ell\ell P(w_\lambda\mu) = 2(\ell(w_\lambda) - \ell(w_\mu^0)) + 1$ and $\ell\ell P(w_\lambda\lambda) = 2\ell(w_\lambda) + 1$ [**9**]. It is also known for certain categories $\mathscr{O}_S^\Lambda$ in case $B_\mu^0 = \{\alpha\}$ for some $\alpha \in B_\lambda$ [**2**]. The result for $\mathscr{O}^\Lambda$ can sometimes be transferred to $\mathscr{O}_S^\Lambda$.

LEMMA. *Let M and N be modules in $\mathscr{O}^\mu$ with $\ell\ell M < \ell\ell N$. Then $\ell\ell T_\mu^\lambda M < \ell\ell T_\mu^\lambda N$.*

Proof. There is no harm in assuming that N has simple top $L(x\mu)$, with $M \subseteq \operatorname{rad} N$. Then $T_\mu^\lambda N$ is an extension of $T_\mu^\lambda \operatorname{rad} N$ by $T_\mu^\lambda L(x\mu)$. In particular, we have a sequence of epimorphisms $T_\mu^\lambda N \to T_\mu^\lambda L(x\mu) \to L(x\lambda)$, assuming x is chosen of maximal length in $x\mathscr{W}_\mu^0$. Therefore, if $T_\mu^\lambda N$ has simple top, the desired conclusion holds. But by the adjointness of T_μ^λ and T_λ^μ, we obtain

$$\operatorname{Hom}_{\mathscr{O}}(T_\mu^\lambda N, L(y\lambda)) = \operatorname{Hom}_{\mathscr{O}}(N, T_\lambda^\mu L(y\lambda)),$$

so $T_\mu^\lambda N$ does have simple top. □

COROLLARY. *Let M be a subquotient of $\operatorname{soc}^{r+s} P(w_\lambda\mu)/\operatorname{soc}^r P(w_\lambda\mu)$ in $\mathscr{O}^\mu$, with $\ell\ell M = s$. Then $\ell\ell T_\mu^\lambda M = s + 2\ell(w_\mu^0)$.*

Proof. We know $\ell\ell T_\mu^\lambda M \geq s + 2\ell(w_\mu^0)$, so it suffices to prove

$$\ell\ell T_\mu^\lambda(\operatorname{soc}^{r+s} P(w_\lambda\mu)/\operatorname{soc}^r P(w_\lambda\mu)) = s + 2\ell(w_\mu^0).$$

But $\ell\ell T_\mu^\lambda P(w_\lambda\mu) = \ell\ell P(w_\lambda\mu) + 2\ell(w_\mu^0)$, so repeated application of the lemma and its dual yields the result. □

We will apply the corollary in 5.1 to the self-dual projective $P_S(w\mu)$ in place of M.

3. Simple projectives. In this section we will prove that simple projectives exist in $\mathcal{O}_S^\Lambda$ for certain choices of $\mathfrak{g}$ and $\mathfrak{p}_S$. As noted in 1.4, we may as well restrict to integral weights, so throughout this section Λ is $P(R)$, the lattice of integral weights.

3.1. *Type A_n.* Let $\mathfrak{g} = s\ell_{n+1}$ and let $B = \{\alpha_1 \dots, \alpha_n\}$, with $(\alpha_i, \check{\alpha}_{i+1}) = -1$. Let $\{\omega_1, \dots, \omega_n\}$ be the fundamental dominant weights. An integral weight ν can be represented by an element $(a_1, \dots, a_{n+1})$ of $\mathbb{C}^{n+1}$, with $(\nu, \check{\alpha}_i) = a_i - a_{i+1} \in \mathbb{Z}$, so $\nu = \sum_{i=1}^n (a_i - a_{i+1})\omega_i$. We may let $\{\varepsilon_1, \dots, \varepsilon_{n+1}\}$ denote the standard orthonormal basis of $\mathbb{C}^{n+1}$, with $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. The Weyl group $\mathscr{W}$ acts on $\mathbb{C}^{n+1}$ and $\mathfrak{h}^*$ by permutation of the ε_i's. Let S be a subset of B, given by $S = \{\alpha_i | i \in I_S\}$, with

$$I_S = \{i_1 + 1, i_1 + 2, \dots, i_1 + r_1, i_2 + 1, \dots, i_2 + r_2, \dots, i_s + 1, \dots, i_s + r_s\}$$

and $i_j > i_{j-1} + r_{j-1}$ for all j with $2 \le j \le s$. Thus $\nu \in P_S^{++}$ if and only if $a_i - a_{i+1} \in \mathbf{N}^+$ for $i \in I_S$.

Definition. Let ν_S be the weight $(b_1, \dots, b_{n+1})$ with $b_{i_j+k} = r_j - k + 1$ and $b_i = 0$ for $i \notin I_S$.

Proposition. *The only weight in $\mathscr{W}\nu_S \cap P_S^{++}$ is ν_S.*

Proof. Let us introduce additional notation. Let $(\lambda_1, \dots, \lambda_s)$ be a permutation of $(r_1, \dots, r_s)$ so that $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_s$, and let $(\mu_1, \dots, \mu_t)$ be the dual partition. Thus $t = \max\{r_i\}$ and $\mu_i = |\{j | \lambda_j \ge i\}|$. Let μ_S be the dominant weight $(t, \dots, t, t-1, \dots, t-1, \dots, 1, \dots, 1, 0, \dots, 0)$, where the integer i is repeated μ_i times for $i > 0$ and 0 occurs q times with $q = n + 1 - \sum_{i=1}^t \mu_i$. Notice that μ_S is the unique dominant weight in $\mathscr{W}\nu_S$.

Let $(a_1, \dots, a_{n+1})$ lie in $\mathscr{W}\mu_S \cap P_S^{++}$. We must have $a_i - a_{i+1} > 0$ for $i \in I_S$. Thus for each j between 1 and s one obtains $a_{i_j+1} > a_{i_j+2} > \cdots > a_{i_j+r_j} > a_{i_j+r_j+1} \ge 0$. Therefore, $a_{i_j+k} \ge r_j - k + 1 = b_{i_j+k}$ for $1 \le k \le r_j$. For those j with $r_j = t$ we obtain $a_{i_j+k} = b_{i_j+k}$. Proceeding inductively through j's with $r_j = t - 1$, etc., we obtain $a_i = b_i$ for all i. □

Corollary. *Let S be a subset of B as above and let λ be a dominant, integral regular weight. Let μ_S, ν_S, and $(\mu_1, \dots, \mu_t)$ be as above. Let*

$$m_S = \binom{\mu_1}{2} + \cdots + \binom{\mu_t}{2} + \binom{q}{2}.$$

(i) *The category* $\mathcal{O}_S^{\mu_S}$ *contains a unique non-zero module* $L(\nu_S)$, *which is projective.* (ii) *Given* $w \in {}^S X$, *the Loewy length of* $P_S(w\lambda)$ *is* $2m_S + 1$ *and the Loewy length of* $M_S(\check{w}\lambda)$ *is* $m_S + 1$.

Proof. Part (i) is an immediate consequence of the Lemma. For part (ii), by 2.2 and the discussion of 1.4, we need to show that $\ell(w_{\mu_S}^0) = m_S$. The Coxeter group $\mathscr{W}_{\mu_S}^0$ is of type $A_{\mu_1 - 1} \times \cdots \times A_{\mu_{t-1}} \times A_{q-1}$, and the length of the longest element in a Coxeter group of type A_ℓ is $\ell(\ell+1)/2$, or $\binom{\ell+1}{2}$. The result follows. □

3.2. *Type* D_n. Let $\mathfrak{g}$ be of type D_n with $B = \{\alpha_1, \dots, \alpha_n\}$, where $\{\alpha_1, \dots, \alpha_{n-1}\}$ form a root system of type A_{n-1} as in 3.1 and $(\alpha_n, \check{\alpha}_{n-2}) = -1$, so $\alpha_n = \varepsilon_{n-1} + \varepsilon_n$. The Weyl group $\mathscr{W}$ acts on $\mathbb{C}^n$ via all permutations of $\{\varepsilon_i\}$ and all changes of an even number of signs. Let S be a subset $\{\alpha_i | i \in I_S\}$ of $B \backslash \{\alpha_n\}$ with I_S as in 3.1. Let $\nu_S = (b_1, \dots, b_n)$ with

$$\begin{aligned} b_{i_j+1} &= \left[\frac{r_j+1}{2}\right], \\ b_{i_j+k} &= b_{i_j+1} - k + 1 \quad \text{for } 1 \le k \le r_j + 1, \\ b_i &= 0 \quad \text{for } i \text{ and } i-1 \text{ not in } I_S. \end{aligned}$$

Notice in particular that $b_{i_j+r_j+1} = -[r_j/2]$. Let $m_i = |\{j | r_j = i\}|$, with $t = \max\{r_j\}$ and $t' = 2[(t+1)/2] - 1$.

Proposition. *The set* $\mathscr{W}\nu_S \cap P_S^{++}$ *has cardinality* $2^{m_1+m_3+\cdots+m_{t'}}$, *and* ν_S *is its unique maximal element.*

Proof. An argument like that in the proof of Proposition 3.1 shows that $(a_1, \dots, a_n)$ lies in $\mathscr{W}\nu_S \cap P_S^{++}$ if and only if $a_i = 0$ for $i, i+1 \notin I_S$, while $a_{i_j+k} = b_{i_j+k}$ for $k \le r_j + 1$ if r_j is even and either $a_{i_j+k} = b_{i_j+k}$ for all $k \le r_j + 1$ or $a_{i_j+k} = b_{i_j+k} - 1$ for all $k \le r_j + 1$ if r_j is odd. This yields the claimed size for $\mathscr{W}\nu_S \cap P_S^{++}$. We also see for $\bar{a} \in \mathscr{W}\nu_S \cap P_S^{++}$ that $\nu_S - \bar{a}$ is either 0 or a sum of roots of the form $\varepsilon_i + \varepsilon_{i+1}$. □

The proposition makes clear that the parity of the r_j's exerts a strong influence on the category $\mathcal{O}_S^\Lambda$. The most manageable choice of S yields the following.

Corollary. *Let* S *be a subset of* $B \backslash \{\alpha_n\}$ *such that all* r_j*'s are even. Then* $M_S(\nu_S)$ *is a simple projective.*

Remarks. (1) In the setting of the corollary, one obtains Loewy length results for $\mathcal{O}_S^\lambda$ analogous to those in 3.1.

(2) Results similar to those above can be formulated for subsets S with $\alpha_n \in S$.

(3) In §5, we will consider in detail the first case not handled by the corollary, with $S = \{\alpha_2\}$ and $n = 4$. In this case, we will find that $\mathcal{O}_S^\Lambda$ contains no simple projective.

3.3. *Type B_n*. In types B_n and C_n one can obtain results similar to those of 3.2, and in fact one can do a little better. But the improvement in results requires Jantzen's criterion for the simplicity of a generalized Verma module, which we will review at the appropriate point.

We will work with B_n only, the C_n analysis being essentially identical. The positive roots are $\{\varepsilon_i - \varepsilon_j | 1 \leq i < j \leq n\} \cup \{\varepsilon_i | 1 \leq i \leq n\}$, with $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i \leq n-1$ and $\alpha_n = \varepsilon_n$. The Weyl group $\mathscr{W}$ acts via permutations and arbitrary sign changes of $\{\varepsilon_i\}$. Given $S \subseteq B \backslash \{\alpha_n\}$, let ν_S be defined exactly as in 3.2, and let I_S, i_j, r_j, m_i be as before.

PROPOSITION. *The set $\mathscr{W}\nu_S \cap P_S^{++}$ has cardinality $2^{m_1+m_3+\cdots+m_{l'}}$, and ν_S is its unique maximal element.*

We omit the proof, which is identical to that of Proposition 3.2. In 3.2 we deduced the simplicity of $M_S(\nu_S)$ if no r_j is odd. Here we may allow odd r_j's provided no odd value occurs more than once.

COROLLARY. *Let S be a subset of $B\backslash\{\alpha_n\}$ such that for j odd either $m_j = 0$ or $m_j = 1$. Then $M_S(\nu_S)$ is a simple projective.*

Proof. (1) It suffices to prove that $M_S(\nu_S)$ is simple, for the maximality of ν_S and BGG reciprocity **[13]** imply that $P_S(\nu_S) = M_S(\nu_S)$. To do this we use Jantzen's criterion **[11]**, see also **[5]**, which we may summarize as follows. Let $A = \{\alpha \in R^+ \backslash R_S^+ | (\nu_S, \alpha) > 0\}$, where we recall that R_S^+ is the set of positive roots associated to $\mathfrak{g}_S$, and $R^+\backslash R_S^+$ is the set of roots of the nilradical $\mathfrak{m}_S$ of $\mathfrak{p}_S$. Let $A_1 = \{\alpha \in A |$ there is a $\beta \in R_S^+$ such that $(s_\alpha \nu_S, \beta) = 0\}$. Then $M_S(\nu_S)$ is simple if and only if one of three conditions holds:

(i) A is empty.

(ii) $A = A_1$.

(iii) For each $\alpha \in A\backslash A_1$ there is an $\alpha' \in A\backslash A_1$ so that $s_{\alpha'}\nu_S = w s_\alpha \nu_S$ for some $w \in \mathscr{W}_S$ of odd length.

(2) We will consider each type of root in $R^+\backslash R_S^+$ systematically in order to verify that one of the three conditions above is satisfied. We begin with the short positive roots $\{\varepsilon_l\}$, none of which lies in R_S^+. Let

$\alpha = \varepsilon_l$. Then α lies in A if and only if $b_l > 0$, in which case $s_\alpha \nu_S$ differs only by having $-b_l$ as its lth coordinate. Choose j and k so $l = i_j + k$. If $k > 1$, or $k = 1$ and r_j is even, then $b_{l+2b_l} = -b_l$ and $s_\alpha \nu_S$ is orthogonal to $\varepsilon_l - \varepsilon_{l+2b_l}$, so $\alpha \in A_1$.

Thus, α is in $A \backslash A_1$ only if $l = i_j + 1$ for some j for which r_j is odd. In this case consider the root $\alpha' = \varepsilon_l + \varepsilon_{l+b_l}$. The weight $s_{\alpha'} \nu_S$ differs from ν_S via a 0 in entry l and $-b_l$ in entry $l + b_l$. Thus α' also lies in $A \backslash A_1$, and one can pass from $s_\alpha \nu_S$ to $s_{\alpha'} \nu_S$ via a reflection about $\varepsilon_l - \varepsilon_{l+b_l}$.

(3) We have shown that $A \neq A_1$ only if some r_j is odd, in which case condition (iii) is satisfied for the short roots in A. If all the long roots of A other than those which we have just paired with the short roots of $A \backslash A_1$, lie in A_1, then condition (iii) will be satisfied.

Consider $\alpha = \varepsilon_p + \varepsilon_q$. If α lies in A then $b_p + b_q > 0$. We may assume that $b_p \geq b_q$ without loss of generality, so $b_p > 0$ and $p \in I_S$. Assume $b_p > b_q$. Then $s_\alpha \nu_S$ has $-b_q$ as its coordinate in entry p and in entry $p + b_p + b_q$, so $s_\alpha \nu_S$ is orthogonal to $\varepsilon_p - \varepsilon_{(p+b_p+b_q)}$ and $\alpha \in A_1$.

Alternatively, assume $b_p = b_q$, and let $p = i_j + k$ and $q = i_{j'} + k'$. If one of k or k' is > 1, say k, then $s_\alpha \nu_S$ has coordinate $-b_q$ in positions q and $q + 2b_q$, so $s_\alpha \nu_S$ is orthogonal to $\varepsilon_q - \varepsilon_{q+2b_q}$ and $\alpha \in A_1$. If instead $k = k' = 1$, then $b_p = b_q$ implies $r_j = r_{j'}$. The hypothesis of the theorem is used here, insuring that r_j is even. Therefore $b_{p+r_j} = -b_p$ and $s_\alpha \nu_S$ is orthogonal to $\varepsilon_p - \varepsilon_{p+r_j}$, so $\alpha \in A_1$.

(4) Finally we must examine roots α of the form $\varepsilon_p - \varepsilon_q$ with $p < q$ and $b_p > b_q$. If $b_p = 0$ or $b_q = 0$, it is clear that $\alpha \in A_1$. Otherwise, let $p = i_j + k$ and $q = i_{j'} + k'$. If $j = j'$, then $\alpha \in R_S^+$, so we may assume $j \neq j'$. But then $s_\alpha \nu_S$ is orthogonal to $\varepsilon_p - \varepsilon_{(p+b_p-b_q)}$, so $\alpha \in A_1$. This completes the proof. □

3.4. *Maximal parabolics.* In this subsection we prove for $\mathfrak{g}$ classical that $\mathcal{O}_S^\Lambda$ contains a simple projective if $\mathfrak{p}_S$ is a maximal parabolic subalgebra of $\mathfrak{g}$. For type A_n, this follows from 3.1. Let us consider the D_n case in detail, using the notation of 3.2.

DEFINITION. Let $\mathfrak{g}$ be of type D_n and $S = B \backslash \{\alpha_r\}$. Define a dominant integral weight ν_S as follows.

(i) If $r = 1$ let $\nu_S = (0, n-2, n-1, \ldots, 1, 0)$.

(ii) If $1 < r \leq n-2$ and $r = 2p$, let $q = n - r$ and $\nu_S = (p, p-1, \ldots, 1, -1, -2, \ldots, -p, q, \ldots, 1)$.

(iii) If $1 < r \leq n-2$ and $r = 2p+1$, let $q = n - r - 1$ and $\nu_S = (p, p-1, \ldots, 1, 0, -1, \ldots, -p, q, \ldots, 0)$.

(iv) If $r = n - 1$ let $\nu_S = (n-1, n-2, \ldots, 2, 1, 1)$

(v) If $r = n$ and $r = 2p$ or $2p + 1$, let ν_S be $(p, \ldots, 1, -1, \ldots, p)$ or $(p, \ldots, 1, 0, -1, \ldots, p)$.

PROPOSITION. *Let $\mathfrak{g}$ be of type D_n and $S = B \backslash \{\alpha_r\}$. For ν_S defined as above, $\mathscr{W}\nu_S \cap P_S^{++} = \{\nu_S\}$ and $L(\nu_S)$ is projective in $\mathscr{O}_S^\Lambda$.*

REMARK. For λ dominant integral, this provides Loewy length formulas in $\mathscr{O}_S^\lambda$.

Proof. The proposition is easily proved case-by-case. Let us consider case (ii) as a typical example, with $(a_1, \ldots, a_n) \in \mathscr{W}\nu_S \cap P_S^{++}$. Then $a_i - a_{i+1} \in \mathbf{N}^+$ for $i \neq r$ and $a_{n-1} > |a_n|$. Suppose first that $p \leq q$, so $|a_i| \leq q$ for all i. If $|a_n| > 1$, then $a_{r+1} \geq |a_n| + q - 1 > q$, which is impossible. If $a_n = -1$, then $a_{n-1} \geq 2$ and $a_{n-k} \geq k+1$ for $1 \leq k \leq q-1$, forcing $a_{n-k} = k+1$. Thus $\bar{a}$ differs from ν_S in the last q entries by a single change of sign. Since $\mathscr{W}$ can only perform an even number of sign changes, some number must appear twice among $a_1, \ldots, a_r$, contrary to hypothesis. Thus $a_n = 1$, from which we obtain $(a_1, \ldots, a_n) = \nu_S$. If instead $p > q$, then the hypothesis that $a_{r+1} > q$ or $a_n = -1$ lead to a repetition among $a_1, \ldots, a_r$, by similar arguments, so again $(a_1, \ldots, a_n) = \nu_S$. □

For $\mathfrak{g}$ of type B_n or C_n a similar construction works. We will list the relevant data, using the notation of 3.3, but omit any proofs.

DEFINITION. Let $\mathfrak{g}$ be of type B_n or C_n and let $S = B \backslash \{\alpha_r\}$.

(i) If $r = 1$ let $\nu_S = (0, n-1, \ldots, 1)$.

(ii) If $r > 1$ and $r = 2p$, let $q = n - r$ and $\nu_S = (p, \ldots, 1, -1, \ldots, -p, q, \ldots, 1)$.

(iii) If $r > 1$ and $r = 2p+1$, let $q = n - r$ and $\nu_S = (p, \ldots, 1, 0, -1, \ldots, p, q, \ldots, 1)$.

PROPOSITION. *Let $\mathfrak{g}$ be of type B_n or C_n and $S = B \backslash \{\alpha_r\}$. Then $\mathscr{W}\nu_S \cap P_S^{++} = \{\nu_S\}$ and $L(\nu_S)$ is projective.*

REMARK. If $\mathfrak{p}_S$ is a maximal parabolic of Hermitian symmetric type, and λ is a dominant, regular integral weight, then $\mathscr{O}_S^\lambda$ has been studied in detail in [2] and [4]. In particular, the Leowy filtrations for generalized Verma modules can be completely described, and the proposed description in 1.6 of self-dual projectives is correct [2].

4. An example of type A_n. Let $\mathfrak{g}$ be of type A_n, with $\{\alpha_1, \ldots, \alpha_n\}$ the simple roots as in 3.1. Let μ be the highest root $\alpha_1 + \cdots + \alpha_n$ and $S = \{\alpha_1\}$. In this section we study the category $\mathscr{O}_S^\mu$. The orbit $\mathscr{W}\mu$ is the set R of all roots, and $\mathscr{W}\mu \cap P_S^{++}$ is the set

$$\{\alpha_1 + \cdots + \alpha_i | 1 \leq i \leq n\} \cup \{-(\alpha_2 + \cdots + \alpha_i) | 2 \leq i \leq n\}.$$

For brevity, let $\mu_i = \alpha_1 + \cdots + \alpha_{i+1}$ and $\mu_{-i} = -(\alpha_2 + \cdots + \alpha_{i+1})$ for $1 \leq i < n$, with $\mu_0 = \alpha_1$. Notice that the weights in $\mathscr{W}\mu \cap P_S^{++}$ are totally ordered with respect to the usual order on weights.

The structure of generalized Verma modules in $\mathscr{O}_S^\mu$ is easily calculated, using the known structure of Verma modules in $\mathscr{O}^\mu$ [7]. (Although this is unpublished, a variant of the required calculations is done in 5.1.) One obtains the following result.

Proposition. (i) *For $0 \leq i \leq n-1$ the module $M_S(\mu_{-i})$ is uniserial of Loewy length $n-i$ with socle filtration layers as below.*

$$\begin{array}{c} L(\mu_{-i}) \\ \hline L(\mu_{-i-1}) \\ \hline \vdots \\ \hline L(\mu_{-n+1}) \end{array}$$

(ii) *The module $M_S(\mu_1)$ is a non-trivial extention of $L(\mu_0)$ by $L(\mu_1)$.*

(iii) *For $1 < i \leq n-1$, the module $M_S(\mu_i)$ has coincident socle and radical filtrations, of Loewy length $i+1$, with layers as below.*

$$\begin{array}{c} L(\mu_i) \\ \hline L(\mu_{i-1}) \oplus L(\mu_{-i+1}) \\ \hline \vdots \\ \hline L(\mu_1) \oplus L(\mu_{-1}) \\ \hline L(\mu_0) \end{array}$$

In particular, the socular simples of $\mathscr{O}_S^\mu$ are $L(\mu_0)$ and $L(\mu_{-n+1})$. We can determine the structure of the projective covers by the procedure suggested in 1.6.

Corollary. *The self-dual projectives $P_S(\mu_0)$ and $P_S(\mu_{-n+1})$ in $\mathscr{O}_S^\mu$ have Loewy length $2n-1$, their radical and socle filtrations coincide, and the layers are given by the formula of* 1.6(*).

Proof. Let $\nu = (1, 0, \ldots, 0)$ and let λ be a regular weight of $P(R)^+$. It is obvious that $\mathscr{W}\nu \cap P_S^{++} = \{\nu\}$, so $L(\nu)$ is projective. By 2.1 or 3.1, self-dual projectives in $\mathcal{O}_S^\lambda$ have Loewy length $1+2\ell(w_\nu^0) = 1+n(n-1)$. Thus, by 1.4, self-dual projectives in $\mathcal{O}_S^\mu$ have Loewy length at most $1 + n(n-1) - 2\ell(w_\mu^0) = 2n - 1$. The generalized Verma module of highest weight occurring in a Verma flag for $P(\mu_{-n+1})$ (or $P(\mu_0)$), by BGG reciprocity and the proposition, is $M_S(\mu_0)$ (or $M_S(\mu_{n-1})$). Both these modules have Loewy length n, so the usual argument using self-duality (see **[9]**) forces the Loewy length of the self-dual projectives to be at least $2n - 1$. Thus the Loewy length is $2n - 1$, and we can easily verify that the hypotheses of Lemma 1.6 are satisfied, from which the corollary follows. □

REMARKS. (1) This example provides evidence that some general statements proposed in **[9]** for categories $\mathcal{O}_S^\lambda$ with λ regular, and proved in certain cases, may carry over to $\mathcal{O}_S^\mu$ for μ non-regular. The proposal for Loewy filtration layers of self-dual projectives makes sense, as noted in 1.6, and is correct here. In addition, it is the case that the generalized Verma modules of maximum Loewy length are precisely those occurring in Verma flags of self-dual projectives with maximal highest weight (or those of maximal highest weight containing a particular socular simple as composition factor), this length being half of (1+ the Loewy length of self-dual projectives).

(2) Another noteworthy feature of this example is the location of the socular weights μ_0 and μ_{-n+1} in the ordering of highest weights of simples in $\mathcal{O}_S^\mu$. For λ regular, the socular weights form a right cell **[8]**, so every weight obtained from a socular weight by going down in the ordering via a sequence of simple reflections is still socular and the resulting set ${}^S X_\lambda$ is a pathwise connected subset of ${}^S\mathscr{W}_\lambda$ in an obvious sense. This connectivity property is not the case in the example, provided $n > 2$, suggesting for a general non-regular μ that a good description of the socular simples in $\mathcal{O}_S^\mu$ may be difficult.

5. An example of type D_4.

5.1. Throughout this section $\mathfrak{g}$ is a Lie algebra of type D_4. We depart from the notation of 3.2, letting $B = \{\eta, \alpha_1, \alpha_2, \alpha_3\}$ with $(\eta, \check{\alpha}_i) = -1$ and $(\alpha_i, \check{\alpha}_j) = 0$ for all i and j. Here $\alpha_1 = \varepsilon_1 - \varepsilon_2$, $\alpha_2 = \varepsilon_3 - \varepsilon_4$ and $\alpha_3 = \varepsilon_3 + \varepsilon_4$. We shall fix the parabolic subalgebra $\mathfrak{p}_S$ with $S = \{\eta\}$. Our main goal is an analysis of $\mathcal{O}_S^\mu$ with μ the highest root, but we postpone this until 5.2. Here we will consider $\mathcal{O}_S^\nu$ with $\nu = (1, 0, 0, 0)$. In order to apply Corollary 2.3, we actually need to understand $\mathcal{O}^\nu$.

Below are the eight weights of $\mathscr{W}\nu$ listed with respect to the Bruhat order, and to the right a re-labelling of the weights which we will use for convenience.

$$
\begin{tikzcd}[column sep=tiny]
 & \nu \arrow[d, dash] & \\
 & \nu - \alpha_1 \arrow[d, dash] & \\
 & \nu - \alpha_1 - \eta \arrow[dl, dash] \arrow[dr, dash] & \\
\nu - \alpha_1 - \eta - \alpha_2 \arrow[dr, dash] & & \nu - \alpha_1 - \eta - \alpha_3 \arrow[dl, dash] \\
 & \nu - \alpha_1 - \eta - \alpha_2 - \alpha_3 \arrow[d, dash] & \\
 & \nu - \alpha_1 - 2\eta - \alpha_2 - \alpha_3 \arrow[d, dash] & \\
 & \nu - 2\alpha_1 - 2\eta - \alpha_2 - \alpha_3 &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}[column sep=tiny]
 & \nu_1 \arrow[d, dash] & \\
 & \nu_2 \arrow[d, dash] & \\
 & \nu_3 \arrow[dl, dash] \arrow[dr, dash] & \\
\nu_4 \arrow[dr, dash] & & \nu_5 \arrow[dl, dash] \\
 & \nu_6 \arrow[d, dash] & \\
 & \nu_7 \arrow[d, dash] & \\
 & \nu_8 &
\end{tikzcd}
$$

Observe that $\mathscr{W}\nu \cap P_S^{++} = \{\nu_2, \nu_6\}$.

Proposition. (i) *The socle and radical filtrations of Verma modules in $\mathscr{O}^\nu$ coincide, with layers depicted in the diagram below.*

$$
\begin{array}{c}
L(\nu_1) \\ \hline
L(\nu_2) \oplus L(\nu_7) \\ \hline
L(\nu_3) \oplus L(\nu_6) \\ \hline
L(\nu_4) \oplus L(\nu_5) \\ \hline
L(\nu_6) \\ \hline
L(\nu_7) \\ \hline
L(\nu_8)
\end{array}
\qquad
\begin{array}{c}
L(\nu_2) \\ \hline
L(\nu_3) \oplus L(\nu_6) \\ \hline
L(\nu_4) \oplus L(\nu_5) \\ \hline
L(\nu_6) \\ \hline
L(\nu_7) \\ \hline
L(\nu_8)
\end{array}
\qquad
\begin{array}{c}
L(\nu_3) \\ \hline
L(\nu_4) \oplus L(\nu_5) \\ \hline
L(\nu_6) \\ \hline
L(\nu_7) \\ \hline
L(\nu_8)
\end{array}
$$

$$
\begin{array}{c}
L(\nu_4) \\ \hline
L(\nu_6) \\ \hline
L(\nu_7) \\ \hline
L(\nu_8)
\end{array}
\qquad
\begin{array}{c}
L(\nu_5) \\ \hline
L(\nu_6) \\ \hline
L(\nu_7) \\ \hline
L(\nu_8)
\end{array}
\qquad
\begin{array}{c}
L(\nu_6) \\ \hline
L(\nu_7) \\ \hline
L(\nu_8)
\end{array}
\qquad
\begin{array}{c}
L(\nu_7) \\ \hline
L(\nu_8)
\end{array}
\qquad
L(\nu_8)
$$

(ii) *The self-dual projective $P(\nu_8)$ has coincident socle and radical filtrations, with layers given by* 1.6(*).

Proof. By **[9]**, the Loewy length of $P(\nu_8)$ is $2\times 7-1$ or 13, since the Bruhat order has 7 levels, so part (ii) follows from (i) via Lemma 1.6. To prove part (i), we use Jantzen's character formulas for his filtrations on a Verma module $M(\xi)$ and a quotient $M(\xi)/M(s_\gamma\xi)$, in **[12**; 5.3, 5.16], and the resulting multiplicity 1 criterion of **[12**; §5]. The only Verma module multiplicities which are not 0 or 1 are the multiplicities of $L(\nu_6)$ in $M(\nu_1)$ and $M(\nu_2)$, and $L(\nu_7)$ in $M(\nu_1)$. The character formulas yield $(M(\nu_2)/M(\nu_3):L(\nu_6))=1=(M(\nu_1)/M(\nu_2):L(\nu_7))$, while $(M(\nu_1)/M(\nu_2):L(\nu_6))=0$. Thus the multiplicities indicated in the proposition are correct, as is the structure of $M(\nu_i)$ for $3\le i\le 8$.

Since $\ell\ell P(\nu_8)=13$, we obtain as in **[9]** that $\ell\ell M(\nu_1)=7$ and $\ell\ell M(\nu_2)=6$. It follows that the diagrams for $M(\nu_1)$ and $M(\nu_2)$ correctly depict the layers of their radical filtrations. (For instance, since all composition factors of $M(\nu_2)$ but the copy of $L(\nu_6)$ not in $M(\nu_3)$ are standard factors, the only possible alternative to that shown would place $L(\nu_6)$ in a layer above $L(\nu_3)$, but then the Loewy length would be too large.)

That the diagrams for $M(\nu_1)$ and $M(\nu_2)$ also depict the socle filtrations can be checked directly by calculating in appropriate weight spaces. Alternatively, we can already apply Lemma 1.6, obtaining the Loewy filtrations of $P(\nu_8)$, from which we can read off the socle filtrations of $M(\nu_1)$ and $M(\nu_2)$ as submodules of $P(\nu_8)$, proving the proposition. □

REMARK. One can use exactly the same sort of arguments to prove the proposition of §4.

COROLLARY. (i) *The category $\mathscr{O}_S^\nu$ has $L(\nu_2)$ and $L(\nu_6)$ as its simples; $M_S(\nu_2)$ is an extension of $L(\nu_6)$ by $L(\nu_2)$ and $P_S(\nu_6)$ is an extension of $M_S(\nu_2)$ by $L(\nu_6)$.*

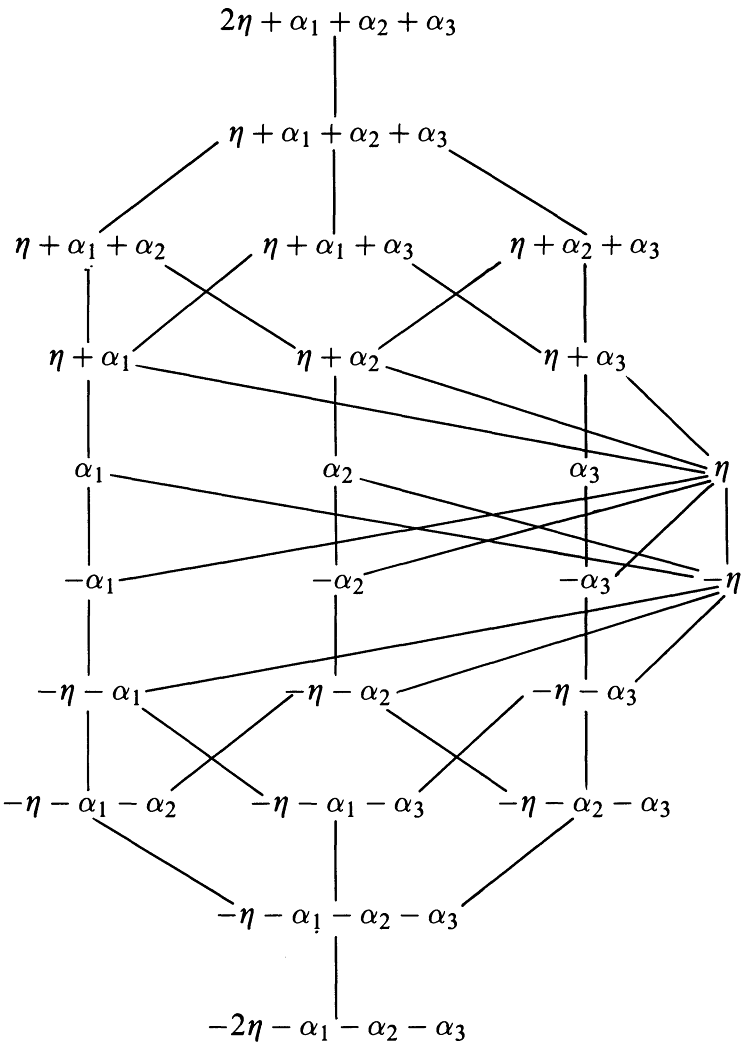
(ii) *The module $P_S(\nu_6)$ is a subquotient of* $\operatorname{soc}^7 P(\nu_8)/\operatorname{soc}^4 P(\nu_8)$ *in the category $\mathscr{O}^\nu$.*

(iii) *Let λ be a dominant integral regular weight. Self-dual projective modules in $\mathscr{O}_S^\lambda$ have Loewy length* 15.

(iv) *There is no simple projective in $\mathscr{O}_S^{P(R)}$.*

Proof. Parts (i) and (ii) follow from the Proposition by inspection, and (iii) follows from (ii) by Corollary 2.3, since $\ell(w_\nu^0)=6$. For part (iv), if there were a simple projective in $\mathscr{O}_S^\mu$ for some integral μ, then by 2.1 the length of w_μ^0 would be 7. But no parabolic subgroup of $\mathscr{W}$ has a longest element of length 7. □

5.2. Let μ be the highest root $2\eta + \alpha_1 + \alpha_2 + \alpha_3$ of R. In analyzing $\mathcal{O}_S^{\mu}$, we will need to consider some Verma modules in $\mathcal{O}^{\mu}$, so let us recall the root system $\mathcal{W}\mu$ with its Bruhat order:



For brevity, let $\mu^- = -\eta - \alpha_1 - \alpha_2 - \alpha_3$. Then the weights of $\mathscr{W}\mu \cap P_S^{++}$ are

$$\begin{array}{ccc} & \mu & \\ \eta+\alpha_1 & \eta+\alpha_2 & \eta+\alpha_3 \\ & \eta & \\ -\alpha_1 & -\alpha_2 & -\alpha_3 \\ & \mu^- & \end{array}$$

The roles of α_1, α_2, α_3 are completely symmetric, so any statement proved for one of them has a parallel statement for the other two, an observation we will use freely. Our main result is the following.

Proposition. (i) *The socle and radical filtrations of generalized Verma modules in $\mathcal{O}_S^\mu$ coincide. The layers of some of these modules are depicted in the diagram below. Those not depicted are given by symmetry.*

$$M_S(\mu) : \begin{array}{c} L(\mu) \\ \hline \bigoplus_i L(\eta+\alpha_i) \\ \hline L(\eta)\oplus L(\eta) \\ \hline \bigoplus_i L(-\alpha_i) \\ \hline L(\mu^-) \end{array} \qquad M_S(\eta+\alpha_1) : \begin{array}{c} L(\eta+\alpha_1) \\ \hline L(\eta) \\ \hline L(-\alpha_2)\oplus L(-\alpha_3) \\ \hline L(\mu^-) \end{array}$$

$$M_S(\eta) : \begin{array}{c} L(\eta) \\ \hline \bigoplus_i L(-\alpha_i) \\ \hline L(\mu^-)\oplus L(\mu^-) \end{array} \qquad M_S(-\alpha_1) : \begin{array}{c} L(-\alpha_1) \\ \hline L(\mu^-) \end{array} \qquad M_S(\mu^-) : L(\mu^-)$$

(ii) *The self-dual projective $P_S(\mu^-)$ has coincident socle and radical filtrations, with layers given by* 1.6(*).

Remarks. (1) This example is of interest because it has a socular simple, $L(\mu^-)$, with multiplicity > 1 in a generalized Verma module, the two appearances in fact occurring in the socle. This has not been known to occur in any previous examples. Because $(M_S(\eta) : L(\mu^-)) = 2$, there are two appearances of $M_S(\eta)$ in a Verma flag for $P_S(\mu^-)$, so it is qualitatively different from the other self-dual projectives for which 1.6(*) is known to hold, yet 1.6(*) is still valid. This provides good evidence that the validity of 1.6(*) in general is plausible.

(2) The two appearances of $L(\eta)$ is $M_S(\mu)$ were observed in [3], in which Conze and Dixmier provided the first example of a Verma module, $M(\mu)$, with infinitely many submodules. These infinitely many submodules already occur in $M_S(\mu)$.

(3) The two appearances of $L(\mu^-)$ in $M_S(\eta)$ can be understood to be mandated by the self-duality of $P_S(\mu^-)$ and the two appearances of $L(\eta)$ in $M_S(\mu)$. For the two copies of $L(\eta)$ in $\operatorname{soc}^3 M_S(\mu)$ or $\operatorname{soc}^3 P_S(\mu^-)$ must be matched by two copies of $L(\eta)$ in

$$P_S(\mu^-)/\operatorname{rad}^3 P_S(\mu^-).$$

These two $L(\eta)$'s correspond to tops of copies of $M_S(\eta)$ in a Verma flag for $P_S(\mu^-)$. Thus $2 = [P_S(\mu^-) : M_S(\eta)] = (M_S(\eta) : L(\mu^-))$.

Proof. (a) Part (ii) follows from (i) by Lemma 2.3, provided $\ell\ell P_S(\mu^-) = 9$. By Corollary 5.1 and 1.3, we have $\ell\ell P_S(\mu^-) \leq 15 - 2\ell(w_\mu^0) = 9$. But the validity of (i) yields $\ell\ell M_S(\mu) = 5$ and $\ell\ell P_S(\mu^-) \geq 9$.

(b) To calculate the generalized Verma modules, we will use the Jantzen character formulas, as in 5.1, and direct calculation of weight vectors. This will require additional notation for $\mathfrak{g}$ and its universal enveloping algebra $U(\mathfrak{g})$. Let $\{x_\varepsilon, y_\varepsilon | \varepsilon \in R^+\} \cup \{h_\varepsilon | \varepsilon \in B\}$ be a Chevalley basis of $\mathfrak{g}$, with $\mathfrak{b}$ spanned by the x_ε's and h_ε's.

Among other relations, we have

$$\begin{aligned} &[x_\varepsilon, y_\varepsilon] = h_\varepsilon \quad \text{for } \varepsilon \in B \\ &[x_{\alpha_1}, y_{\eta+\alpha_1}] = y_\eta, \qquad [x_\eta, y_{\eta+\alpha_1}] = -y_{\alpha_1} \\ &[x_{\alpha_2}, y_{\eta+\alpha_2}] = -y_\eta, \qquad [x_\eta, y_{\eta+\alpha_2}] = y_{\alpha_2} \\ &[x_{\alpha_3}, y_{\eta+\alpha_3}] = -y_\eta, \qquad [x_\eta, y_{\eta+\alpha_3}] = y_{\alpha_3} \end{aligned}$$

Let v be a highest weight vector of the Verma module $M(\mu)$, of weight $\mu - \rho$.

(c) We begin with an analysis of $M_S(\mu)$. This is $M(\mu)/M(s_\eta\mu)$, which has a Jantzen filtration by [12; 5.16]. Working in $M(\mu)$, we find that the vector $w_1 = (y_{\alpha_2}y_{\eta+\alpha_3} - y_{\alpha_3}y_{\eta+\alpha_2})v$ is not in $M(s_\eta\mu)$ and has as image in $M_S(\mu)$ a highest weight vector. Analogously we obtain vectors $w_2 = (y_{\alpha_1}y_{\eta+\alpha_3} + y_{\alpha_3}y_{\eta+\alpha_1})v$ and $w_3 = (y_{\alpha_1}y_{\eta+\alpha_2} + y_{\alpha_2}y_{\eta+\alpha_1})v$ of weights

$\eta+\alpha_2-\rho$ and $\eta+\alpha_3-\rho$ which have highest weight vectors as images in $M_S(\mu)$. The Jantzen character formula on the filtration $M_S(\mu)^i$ yields $(\bigoplus_{i>0} M_S(\mu)^i : L(\eta+\alpha_j)) = 1$, so $(M_S(\mu) : L(\eta+\alpha_j)) = 1$ and we have accounted for the multiplicity. The images of $y_{\alpha_j} w_j$ are also highest weight vectors in $M_S(\mu)$, of weight $\eta-\rho$, as is any linear combination, and $y_{\alpha_1} w_1 - y_{\alpha_2} w_2 + y_{\alpha_3} w_3 = 0$.

These are essentially the vectors listed by Conze and Dixmier **[3]**, and they account for two appearances of $L(\eta)$ in $M_S(\mu)$. To show that $(M_S(\mu) : L(\eta)) = 2$, one can calculate in the $\eta-\rho$ weight space of $M_S(\mu)$, finding that all vectors not in the span of $\{y_{\alpha_j} w_j | 1 \le j \le 3\}$ are cyclic. The Jantzen character formula yields $(\bigoplus_{i>0} M_S(\mu)^i : L(\eta)) = 4$, so both copies of $L(\eta)$ lie in $M_S(\mu)^2/M_S(\mu)^3$.

The Jantzen multiplicity criterion yields $(M(s_\eta \mu) : L(-\alpha_j)) = 1$, which allows one to compute via the character formula that $(\bigoplus_{i>0} M_S(\mu)^i : L(-\alpha_j)) = 3$. One copy of $L(-\alpha_j)$ can be accounted for, since $(y_{\alpha_i} y_\eta \pm y_{\eta+\alpha_i}) y_{\alpha_j} w_j$ for $i \neq j$ has a highest weight vector as image in $M_S(\mu)$. Thus, this copy lies below a copy of $L(\eta)$, and must be in $M_S(\mu)^2$. If it is in $M_S(\mu)^3$, then it accounts for the 3 in the character formula and $(M_S(\mu) : L(-\alpha_j)) = 1$. But otherwise, by self-duality of $M_S(\mu)^2/M_S(\mu)^3$, there must be at least two appearances of $L(-\alpha_j)$ in this quotient, producing 4 in the character formula.

We have determined all composition factors of $M_S(\mu)$ except $L(\mu^-)$, and associated highest weight vectors to each factor, allowing one to verify that their relative location is as claimed in the proposition. Since the socle of $M_S(\mu)$ is simple **[8]**, a contradiction would result if $L(\mu^-)$ is not a composition factor. Thus $L(\mu^-)$ is the socle. But the maximality of μ forces the socle to have multiplicity 1 **[8]**, so the structure of $M_S(\mu)$ is determined.

(d) The Jantzen character formula yields $(M_S(-\alpha_i) : L(\mu^-)) = 1$, so the structure of $M_S(-\alpha_i)$ and $M_S(\mu^-)$ are as claimed.

(e) Turning to $M_S(\eta)$, we have

$$(M_S(\eta) : L(-\alpha_i)) = 1 \quad \text{and} \quad \left(\bigoplus_{i>0} M_S(\eta)^i : L(\mu^-)\right) = 4.$$

The structure of $M_S(\mu)$ yields $\dim \operatorname{Hom}_{\mathcal{O}}(M_S(\eta), M_S(\mu)) = 2$. Taking two independent homomorphisms, we obtain two non-isomorphic submodules of $M_S(\mu)$ with top $L(\eta)$ and socle $L(\mu^-)$, so $(M_S(\eta) : L(\mu^-)) \ge 2$. If the multiplicity is > 2, then $(M_S(\eta)^2 : L(\mu^-)) \le 1$ and $(M_S(\eta)^1/M_S(\eta)^2 : L(\mu^-)) \ge 3$. The self-duality of $M_S(\eta)^1/M_S(\eta)^2$ implies that it must have a copy of $L(\mu^-)$ in its top, as well as its socle, with Loewy length 3. But then $M_S(\mu)$ has this extra copy of $L(\mu^-)$ as well, a contradiction. Thus $M_S(\eta)$ has the claimed structure.

(f) Each $M_S(\eta+\alpha_j)$ maps onto a submodule of $M_S(\mu)$ which has the structure claimed for $M_S(\eta+\alpha_j)$. Thus we need only check that the composition factors all occur with multiplicity 1. The only multiplicity which isn't obvious is $(M_S(\eta+\alpha_j) : L(\mu^-))$, and the Jantzen character formula yields $(\bigoplus_{i>0} M_S(\eta+\alpha_j)^i : L(\mu^-)) = 3$. An argument like that at the end of (c) shows that $(M_S(\eta+\alpha_j)^3 : L(\mu^-)) = 1$, so $(M_S(\eta+\alpha_j) : L(\mu^-)) = 1$, completing the proof. □

5.3. Let λ be a dominant, integral regular weight; choose y of longest length in $\mathscr{W}$ with $y\mu = \mu^-$ and w of shortest length with $w\mu = \eta$. Notice that y equals ${}^S w_\lambda$ (cf. 1.2). Thus $y\lambda$ is the lowest weight in ${}^S\mathscr{W}_\lambda\lambda$.

PROPOSITION. *The socle of* $M_S(w\lambda)$ *is* $L(y\lambda) \oplus L(y\lambda)$, *and* $\mathrm{Hom}_{\mathscr{O}}(M_S(y\lambda), M_S(w\lambda))$ *is two-dimensional.*

Proof. (1) The module $T_\mu^\lambda P_S(\mu^-)$, which is $T_\mu^\lambda P_S(y\mu)$, is a self-dual projective with $L(y\lambda)$ in its top, and the argument of Lemma 2.3 shows that the top is simple, so $T_\mu^\lambda P_S(\mu^-) = P_S(y\lambda)$. By Corollary 5.1, its Loewy length is 15. The analogue of Corollary 2.3 with $P_S(y\mu)$ in place of $P(w_\lambda\mu)$ can be proved in the same manner. Combining this with Proposition 5.2(ii), which completely describes the socle filtration of $P_S(\mu^-)$, we may conclude that $\ell\ell T_\mu^\lambda M = \ell\ell M + 6$ for M any of the simples or generalized Verma modules in $\mathscr{O}_S^\mu$.

(2) The simples of $\mathscr{O}_S^\lambda$ which are not annihilated by T_λ^μ are precisely those $L(x\lambda)$ for which x is of longest length in the coset $x\mathscr{W}_\mu^0$; for such an x and for z in $\mathscr{W}$ we have $(M_S(z\lambda) : M_S(x\lambda)) = (M_S(z\mu) : M_S(x\mu))$ [12]. It follows from the adjointness of T_μ^λ and T_λ^μ, and the previous fact, that the socle of $T_\mu^\lambda M_S(w\mu)$ is $L(y\lambda) \oplus L(y\lambda)$. Also $T_\mu^\lambda M_S(w\mu)$ has a Verma flag with constituents $\{M_S(wz\lambda) | z \in \mathscr{W}_\mu^0\}$. The generalized Verma module with maximal highest weight among these, $M_S(w\lambda)$, must be a submodule, leaving two possibilities: either the socle of $M_S(w\lambda)$ is as claimed, or it is just a single copy of $L(y\lambda)$.

(3) By part (1), the Loewy length of $T_\mu^\lambda M_S(w\mu)$ is 9 and that of $T_\mu^\lambda L(w\lambda)$ is 7, the second module being a homomorphic image of the first. Also, the Verma flag of $T_\mu^\lambda M_S(w\mu)$ allows us to deduce that $L(w\lambda)$ has multiplicity one in it, and therefore multiplicity one in $T_\mu^\lambda L(w\mu)$ as well. As $L(w\lambda)$ is the maximal weight composition factor occurring, it corresponds to the top of a submodule N which is the unique

homomorphic image of $M_S(w\lambda)$ in $T_\mu^\lambda L(w\mu)$. The module $T_\mu^\lambda L(w\mu)$ is self-dual, and the usual argument shows that it has simple top and socle $L(ww_\mu^0\lambda)$. Moreover, this simple has multiplicity 1 in $M_S(w\lambda)$, being a standard composition factor, so that N can be described as the smallest homomorphic image of $M_S(w\lambda)$ in which this factor survives. We may deduce from this that N must also contain the standard composition factors $L(wz\lambda)$ for all $z \in \mathscr{W}_\mu^0$. This forces N to have Loewy length at least 4, taking into account that $\mathscr{W}_\mu^0$ is of type $A_1 \times A_1 \times A_1$. Since $T_\mu^\lambda L(w\mu)$ is self-dual of Loewy length 7, we see that the copy of $L(w\lambda)$ must occur in the middle layer both in the socle and radical filtrations.

Pulling this information back to $T_\mu^\lambda M_S(w\mu)$, we find that the unique copy of $L(w\lambda)$ is in $\mathrm{rad}_3\, T_\mu^\lambda M_S(w\mu)$, and that the inverse image K of the socle of $T_\mu^\lambda L(w\mu)$ lies in $\mathrm{rad}^6\, T_\mu^\lambda M_S(w\mu)$. Thus K has Loewy length 3. However, K is a submodule of $M_S(w\lambda)$ with simple top $L(ww_\mu^0\lambda)$, so $T_\lambda^\mu K$ is a submodule of $T_\lambda^\mu M_S(w\lambda)$, or $M_S(w\mu)$, with $L(w\mu)$ in its top. Thus, $T_\lambda^\mu K = T_\lambda^\mu M_S(w\lambda) = M_S(w\mu)$. This implies that $(K : L(y\lambda)) = 2$. The socle of K, by the conclusion of (2), is either $L(y\lambda)$ or $L(y\lambda) \oplus L(y\lambda)$. But $L(y\lambda)$ cannot be in the top of K and cannot extend itself nontrivially, so the only place for the two copies of $L(y\lambda)$ in K is in the socle. This completes the proof. □

REFERENCES

[1] J. N. Bernstein and S. I. Gelfand, *Tensor products of finite and infinite dimensional representations of semisimple Lie algebras*, Comp. Math., **41** (1980), 245–285.

[2] D. H. Collingwood, R. S. Irving and B. Shelton, *Filtrations on generalized Verma modules for Hermitian symmetric pairs*, J. Reine Angew. Math., to appear.

[3] N. Conze and J. Dixmier, *Idéaux primitifs dans l'algèbre enveloppante d'une algèbre de Lie semi-simple*, Bull. Soc. Math., **96** (1972), 339–351.

[4] T. J. Enright and B. Shelton, *Categories of highest weight modules: applications to classical Hermitian symmetric pairs*, Memoirs Amer. Math. Soc., **367** (1987).

[5] T. J. Enright and J. A. Wolf, *Continuation of unitary derived functor modules out of the canonical chamber*, Mém. Soc. Math. France (N.S.), **15** (1984), 139–156.

[6] J. E. Humphreys, *A construction of projective modules in the category $\mathscr{O}$ of Bernstein-Gelfand-Gelfand*, Indag. Math., **39** (1977), 301–303.

[7] R. S. Irving, *Projective modules in the category $\mathscr{O}$*, typescript, 1982.

[8] ———, *Projective modules in the category $\mathscr{O}_S$: self-duality*, Trans. Amer. Math. Soc., **291** (1985), 701–732.

[9] ———, *Projective modules in the category $\mathscr{O}_S$: Loewy series*, Trans. Amer. Math. Soc., **291** (1985), 733–754.

[10] ———, *The socle filtration of a Verma module*, preprint, 1986.
[11] J. C. Jantzen, "*Kontravariante Formen auf induzierten Darstellungen halbeinfacher Lie-algebren*, Math. Ann., **226** (1977), 53–65.
[12] J. C. Jantzen, *Moduln mit Einem Höchsten Gewicht*, Lecture Notes in Mathematics #750, Springer-Verlag, Berlin, Heidelberg and New York, 1979.
[13] A Rocha-Caridi, *Splitting criteria for g-modules induced from a parabolic and the Bernstein-Gelfand-Gelfand resolution of a finite-dimensional, irreducible g-module*, Trans. Amer. Math. Soc., **262** (1980), 335–366.

Received October 14, 1986.

UNIVERSITY OF WASHINGTON
SEATTLE, WA 98195

AND

UNIVERSITY OF OREGON
EUGENE, OR 97403