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PSEUDOCONVEX DOMAINS WITH PEAK FUNCTIONS AT EACH POINT OF THE BOUNDARY

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Under certain conditions, each point of the boundary of a smoothly bounded weakly pseudoconvex domain D in $\mathbf{C}^n$ is a peak point of $A^\infty(D)$.

1. Introduction. Let D be a bounded pseudoconvex domain with C^∞ boundary. We denote by $A^\infty(D)$ the set of holomorphic functions in D which have a C^∞ extension to $\overline{D}$. A compact subset E of ∂D is a peak set for $A^\infty(D)$ if there exists $f \in A^\infty(D)$ such that $f = 0$ on E and $\operatorname{Re} f > 0$ on $\overline{D} \setminus E$. Such a function will be called a strong support function for E. If $E = \{p\}$, p is a peak point for $A^\infty(D)$.

In **[6]**, **[18]** it is proved that each point of a strictly pseudoconvex domain is a peak point for $A^\infty(D)$ with a strong support function holomorphic in the neighborhood of $\overline{D}$ and in [7], **[17]** it is proved that each strongly pseudoconvex point of a weakly pseudoconvex domain with C^∞ boundary is a peak point for $A^\infty(D)$. These results fail in the case of weakly pseudoconvex domains **[4]**, **[13]**. Other results about smoothly varying peaking functions in pseudoconvex domains may be found in **[1]**, **[5]**, **[14]**.

If D is strictly pseudoconvex, Chaumat and Chollet proved in [3] that each closed subset of a peak set for $A^\infty(D)$ is a peak set for $A^\infty(D)$. The assertion is also true for bounded pseudoconvex domains in $\mathbf{C}^2$ of finite type **[15]** and for bounded pseudoconvex domains in $\mathbf{C}^2$ with isolated degeneracies **[11]** or with (NP) property **[12]**.

In **[16]** is given an example of convex domain in $\mathbf{C}^2$ not of finite type whose weakly pseudoconvex boundary points form a line segment which is a peak set for $A^\infty(D)$, but there is a point which is not a peak point for $A^\infty(D)$.

Here we prove that, under certain assumptions, each point of the boundary of a weakly pseudoconvex domain is a peak point for $A^\infty(D)$.

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2. A Morse lemma for non-negative strictly q-pseudoconvex functions.

LEMMA 1. *Let φ be a real-valued non-negative function defined in a neighborhood of $0 \in \mathbf{C}^n$ such that $\varphi(0) = 0$. We suppose that the complex Hessian of φ at 0 has q zero eigenvalues at the origin. Then there exists a complex-linear change of coordinates in $\mathbf{C}^n$ such that*

$$\varphi(z) = \sum_{j=1}^{r}(1+\lambda_j)x_j^2 + \sum_{j=1}^{r}(1-\lambda_j)y_j^2 + O(|z|^3)$$

where $1 \geq \lambda_j \geq 0$, $z = x + iy$, $r = n - q$.

REMARK 1. Lemma 1 is a more complete form of Lemma 4 of **[10]**. For strictly plurisubharmonic functions the result was obtained in **[9]**.

Proof of Lemma 1. The proof is similar to the proof of Lemma 4 of **[10]** and most of it is presented there. The point 0 is a local minimum for φ so $\operatorname{grad}\varphi(0) = 0$ and the real Hessian of φ at 0 is semi-positive definite. By **[18]** it follows that the complex Hessian of φ is semi-positive definite at 0. We denote

$$x' = (x_1, \dots, x_r), \quad x'' = (x_{r+1}, \dots, x_n), \quad y' = (y_1, \dots, y_r),$$
$$y'' = (y_{r+1}, \dots, y_n), \quad z' = x' + iy', \quad z'' = x'' + iy''.$$

We have

$$\varphi(z) = \operatorname{Re}\sum_{i,j=1}^{n}\frac{\partial^2\varphi}{\partial z_i\,\partial z_j}(0)z_i z_j + \sum_{i,j=1}^{n}\frac{\partial^2\varphi}{\partial z_i\,\partial \bar z_j}(0)z_i\bar z_j + O(|z|^3).$$

By making a complex-linear change of coordinates in $\mathbf{C}^n$ we may suppose that

$$\left[\frac{\partial^2\varphi}{\partial z_i\,\partial\bar z_j}(0)\right]_{1\le i,j\le n} = \begin{array}{cl} \begin{array}{cc} \hphantom{\begin{matrix}1 & 0\end{matrix}} & \overbrace{\qquad}^{q} \end{array} & \\ \left[\begin{array}{cc} \begin{matrix}1 & 0\\ 0 & 1\end{matrix} & 0 \\ 0 & 0 \end{array}\right] & \left.\vphantom{\begin{matrix}1\\1\end{matrix}}\right\} r \end{array}$$

and $\varphi(z) = |z'|^2 + \mathrm{Re}({}^t zSz) + O(|z|^3)$ where

$$S = \left[\frac{\partial^2 \varphi}{\partial z_i \partial z_j}(0)\right]_{1 \le i,j \le n}.$$

Let $s = [\binom{x}{y}]$ be a real $2n$-vector in $\mathbf{R}^{2n}$, where $x, y \in \mathbf{R}^n$,

$$E' = \{s \in \mathbf{R}^{2n} | x'' = 0, y'' = 0\}, \quad E'' = \{s \in \mathbf{R}^{2n} | x' = 0, y' = 0\}.$$

We shall identify E' with $\mathbf{R}^{2r}$ and E'' with $\mathbf{R}^{2(n-r)}$. E' and E'' are complex subspaces of $\mathbf{C}^n = E' \oplus E''$ and for $s \in \mathbf{C}^n$ we obtain $s = s' + s''$ with $s' \in E'$, $s'' \in E''$. With these notations we obtain that

$$\varphi(s) = |s'|^2 + {}^t sTs + O(|s|^3) = |s'|^2 + \langle Ts, s\rangle + O(|s|^3)$$

where $\langle\, ,\, \rangle$ is the inner product in $\mathbf{R}^{2n}$ and $T = \left[\begin{smallmatrix} A & -B \\ -B & -A \end{smallmatrix}\right]$ with $S = A + iB$, A and B real symmetric matrices. In **[10]** we prove that

$$\langle Ts, s\rangle = \langle T_1' s', s'\rangle + \langle T_2' s', s''\rangle + \langle T_1'' s'', s'\rangle + \langle T_2'' s'', s''\rangle$$

where

$$T_1' = \begin{bmatrix} A_1' & 0 & -B_1' & 0 \\ 0 & 0 & 0 & 0 \\ -B_1' & 0 & -A_1' & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad T_2'' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & A_2'' & 0 & -B_2'' \\ 0 & 0 & 0 & 0 \\ 0 & -B_2'' & 0 & -A_2'' \end{bmatrix}$$

and A_2'', B_2'' are the $r \times r$ $(n - r \times n - r)$ matrices obtained by taking the first r (the last $n - r$) rows and columns of A and B respectively.

Let J be the real orthogonal matrix representing the multiplication by $i = \sqrt{-1}$, i.e., $J[\binom{x}{y}] = [\binom{-y}{x}]$. If $v' \in E$ ($v'' \in E''$) is an eigenvector for T_1' (T_2'') with eigenvalue λ, then Jv' (Jv'') is an eigenvector for T_1' (T_2'') with eigenvalue $-\lambda$. Because A and B are symmetric matrices, it follows that T_1', T_2'' are symmetric matrices. We may therefore consider an orthonormal basis of $\mathbf{R}^{2n}$ by the form $v_1', \dots, v_r', v_{r+1}'', \dots, v_n''$, $Jv_1', \dots, Jv_r', Jv_{r+1}'', \dots, Jv_n''$, where v_j', Jv_j', v_j'', Jv_j'', are eigenvectors for T_1', respectively T_2''. If λ_j is the eigenvalue of v_j' (v_j''), by interchanging v_j and Jv_j if necessary we may assume each $\lambda_j \ge 0$.

We have in fact a complex-linear change of coordinates in $\mathbf{C}^n$ and if the new coordinates are denoted also by $(z_1, \dots, z_n)$, we have

$$\begin{aligned}\varphi(z) = &\sum_{j=1}^{r}(1+\lambda_j)x_j^2 + \sum_{j=1}^{r}(1-\lambda_j)y_j^2 \\ &+ \sum_{i=1}^{r}\sum_{j=r+1}^{n}(a_{ij}x_i x_j + b_{ij}x_i y_j + c_{ij}x_j y_i + d_{ij}y_i y_j) \\ &+ \sum_{j=r+1}^{n}\lambda_j x_j^2 - \sum_{j=r+1}^{n}\lambda_j y_j^2 + O(|z|^3).\end{aligned}$$

Because the real Hessian of φ at 0 is semi-positive definite, it follows that $\lambda_j \le 1$ for $j = 1, \dots, r$ and $\lambda_j = 0$ for $j = r+1, \dots, n$. If for some $1 \le i \le r$ we have $\lambda_i = 1$, then $c_{ij} = d_{ij} = 0$ for $j = r+1, \dots, n$, because $c_{ij}x_j y_i$ and $d_{ij}y_i y_j$ change sign at the origin if $c_{ij} \neq 0$, $d_{ij} \neq 0$. Thus

$$\begin{aligned}\varphi(z) = &\sum_{i=1}^{r}\left[\sqrt{(1+\lambda_i)}x_i + \sum_{j=r+1}^{n}\frac{a_{ij}}{2\sqrt{1+\lambda_i}}x_j + \sum_{j=r+1}^{n}\frac{b_{ij}}{2\sqrt{1+\lambda_i}}y_j\right]^2 \\ &+ {\sum_{i=1}^{r}}' \left[\sqrt{(1-\lambda_i)}y_i + \sum_{j=r+1}^{n}\frac{c_{ij}}{2\sqrt{1-\lambda_i}}x_j + \sum_{j=r+1}^{n}\frac{d_{ij}}{2\sqrt{1-\lambda_i}}y_j\right]^2 \\ &- \frac{1}{4}\sum_{i=1}^{r}\frac{1}{1+\lambda_i}\left[\sum_{j=r+1}^{n}(a_{ij}^2x_j^2 + b_{ij}^2y_j^2)\right. \\ &\qquad\left. + \sum_{j,k=r+1}^{n}(a_{ij}a_{ik}x_j x_k + b_{ij}b_{ik}y_j y_k + 2a_{ik}b_{ik}x_j y_k)\right] \\ &- \frac{1}{4}{\sum_{i=1}^{r}}'\frac{1}{1-\lambda_i}\left[\sum_{j=r+1}^{n}(c_{ij}^2x_j^2 + d_{ij}^2y_j^2)\right. \\ &\qquad\left. + \sum_{j,k=r+1}^{n}(c_{ij}c_{ik}x_j x_k + d_{ij}d_{ik}y_j y_k + 2c_{ij}d_{ik}x_j y_k)\right] \\ &+ O(|z|^3),\end{aligned}$$

where $\sum'$ means that we take the sum over the indices i for which $\lambda_i < 1$. Because $\varphi \ge 0$ in the neighborhood of the origin, we obtain that $a_{ij} = b_{ij} = c_{ij} = d_{ij} = 0$ for each $i = 1, \dots, r$, $j = r+1, \dots, n$.

3. Local properties of strong support functions.

LEMMA 2. *Let $D \subset \mathbf{C}^n$ be a pseudoconvex domain with smooth boundary, $E \subset \partial D$ a peak set for $A^\infty(D)$, f a strong support function for E and $p \in E$. Let ρ be a local defining function for ∂D in the neighborhood of p. We denote $C_p(\rho, f) = -(\partial \operatorname{Re} f/\partial n)(p)\rho + \operatorname{Re} f$, where $\partial/\partial n$ is the derivative with respect to the normal direction at p. Then:*

(a) *$H^r C_p(\rho, f)(p)$ is semi-positive definite, where H^r represents the real Hessian restricted to the complex-tangent space $TC_p(\partial D)$;*

(b) *$H^c C_p(\rho, f)(p) = -(\partial \operatorname{Re} f/\partial n)(p)\, L_p$ where H^c is the complex Hessian restricted to $TC_p(\partial D)$ and L_p is the Levi form at p;*

(c) *Suppose that L_p has q zero-eigenvalues and $r = n - q - 1$ strictly positive eigenvalues at p. Let $e_1, \dots, e_r$ be the eigenvectors corresponding to the strictly positive eigenvalues and V_p' the real subspace generated by $e_1, \dots, e_r$. If V_p^+ is the subspace of $TC_p(\partial D)$ generated by the eigenvectors corresponding to the strictly positive eigenvalues of $H^r C_p(\rho, f)(p)$, then $V_p' \subset V_p^+$.*

REMARK 2. By the Hopf lemma we have $(\partial \operatorname{Re} f/\partial n)(p) > 0$.

Proof. The proof of Lemma 2 is similar to the proof of Proposition 9 of [3] and we shall repeat the arguments from the beginning of it.

By making a complex-linear change of coordinates in $\mathbf{C}^n$ we may suppose that p is the origin and in the neighborhood U_1 of the origin D is given by $D \cap U_1 = \{(z', w) \in U_1 | \rho(z', w) < 0\}$ where $z' = (z_1, \dots, z_{n-1})$, $z_j = x_j + iy_j$, $w = u + iv$ and $\rho(z', w) = u + R_1(z) + R_2(z', w)$, where $R_1(z')$ is a second order homogeneous polynomial in z', $\bar{z}'$, and $R_2(z', w) = O(|z'||w| + |w|^2 + |z'|^3)$.

Because $(0, 0)$ is a local minimum for $\operatorname{Re} f$, by the Hopf lemma we obtain that

$$\frac{\partial \operatorname{Re} f}{\partial u}(0,0) < 0, \quad \frac{\partial \operatorname{Re} f}{\partial v}(0,0) = 0,$$
$$\frac{\partial \operatorname{Re} f}{\partial x_j}(0,0) = \frac{\partial \operatorname{Re} f}{\partial y_j}(0,0) = 0, \qquad 1 \le j \le n-1.$$

It follows that in a neighborhood U_2 of the origin, $U_2 \subset U_1$, we have

$$\operatorname{Re} f(z', w) = \frac{\partial \operatorname{Re} f}{\partial u}(0,0)u + K_1(z', w) + K_2(z', w)$$

where $K_1(z', w)$ is a second order pluriharmonic polynomial in z', $\bar{z}$, w, $\bar{w}$ and $K_2(z', w) = O((|z'| + |w|)^3)$.

From the Cauchy-Riemann equations at the origin we obtain that

$$\frac{\partial \operatorname{Im} f}{\partial v}(0,0) < 0, \quad \frac{\partial \operatorname{Im} f}{\partial u}(0,0) = 0,$$
$$\frac{\partial \operatorname{Im} f}{\partial x_j}(0,0) = \frac{\partial \operatorname{Im} f}{\partial y_j}(0,0) = 0, \qquad j = 1, \dots, n-1.$$

Because

$$\frac{\partial(\rho, \operatorname{Im} f)}{\partial(w, \bar{w})}(0,0) = \frac{i}{2}\frac{\partial \operatorname{Re} f}{\partial u}(0,0) \neq 0$$

it follows that the set $\Sigma = \{(z', w) | \rho(z', w) = 0, \operatorname{Im} f(z', w) = 0\}$ is in a neighborhood U_3 of the origin, $U_3 \subset U_2$, a $2n-2$-dimensional C^∞-submanifold of the boundary which contains $E \cap U_3$.

So, there exists a C^∞-function $h = h(z')$ defined in a neighborhood V_1 of $0 \in C^{n-1}$ such that $\Sigma = \{(z', w) | w = h(z')\}$.

We have $\rho(z', h(z')) = 0 = \operatorname{Re} h(z') + R_1(z') + R_2(z', h(z'))$ and because the first order derivatives of h vanish at the origin we obtain that $\operatorname{Re} h(z') = -R_1(z') + O(|z'|^3)$.

We define

$$\begin{aligned}\Theta(z') &= \operatorname{Re} f(z', h(z')) \\ &= \frac{\partial \operatorname{Re} f}{\partial u}(0,0) \operatorname{Re} h(z') + K_1(z', h(z')) + K_2(z', h(z')) \\ &= -\frac{\partial \operatorname{Re} f}{\partial u}(0,0) R_1(z') + K_1(z', 0) + O(|z'|^3),\end{aligned}$$

and we obtain (b).

The complex tangent space of ∂D at (0,0) is $\{(z', w) | w = 0\}$, hence the complex Hessian of Θ has $n - q - 1$ strictly positive eigenvalues and q zero-eigenvalues at 0.

Because f is a strong support function for E we have $\Theta(z') \geq 0$ and $\Theta(z') = 0$ if and only if $(z', h(z')) \in E$. Because the origin is a minimum for Θ, we obtain (a).

We denote by $Z = \{z \in V_1 | \Theta(z') = 0\}$.

From Lemma 1 it follows that there exists a complex-linear change of coordinates in $\mathbf{C}^{n-1}$ such that in the new coordinates (which we shall denote also $z' = (z_1, \dots, z_{n-1})$) we have:

$$(1)\quad \Theta(z') = \sum_{j=1}^{n-q-1} (1 - \lambda_j) x_j^2 + \sum_{j=1}^{n-q-1} (1 - \lambda_j) y_j^2 + O(|z|^3), \qquad \lambda_j \geq 0,$$

and we obtain (c).

PROPOSITION 1. *Let $D \subset \mathbf{C}^n$ be a pseudoconvex domain with smooth boundary, $E \subset \partial D$ a peak set for $A^\infty(D)$, f a strong support function for E and $p \in E$ such that the Levi form has q zero-eigenvalues at p. We denote by Z_p the complex q-dimensional subspace of $TC_p(\partial D)$ generated by the eigenvectors corresponding to the zero-eigenvalues.*

Using the notations of Lemma 2, suppose that:

(i) *$H^r C_p(\rho, f)(p)$ has at least $n-1$ strictly positive eigenvalues;*

(ii) *There exists a neighborhood V of p and a $q+1$ codimensional generic submanifold S of ∂D such that $E \cap V \subset S$ and $TC_p(S) \oplus Z_p = TC_p(\partial D)$;*

(iii) *The tangent space $T_p(S)$ has a q dimensional complement V_p in $T_p(\partial D)$ which is contained in W_p, where $V_p' \oplus W_p = V_p^+$.*

Then there exists a neighborhood ω of p, an n-dimensional totally real submanifold of $\partial D \cap \omega$ and $c > 0$ such that $E \cap \omega \subset M$ and $\operatorname{Re} f(z) \geq c d(z, M)^2$ for each $z \in \overline{D} \cap \omega$.

REMARK 3. The conditions (ii) and (iii) mean that there exist $\rho_1, \ldots, \rho_q$ defined in the neighborhood of p such that

$$\frac{\partial(\rho_1, \ldots, \rho_q)}{\partial(z_1, \ldots, z_q)}(p) \quad \text{and} \quad \frac{\partial(\rho_1, \ldots, \rho_q)}{\partial(y_1', \ldots, y_q')}(p)$$

have maximal rank, where $z_1, \ldots, z_q$, respectively $y_1', \ldots, y_q'$ are the variables corresponding to Z_p, respectively to V_p.

Proof. We shall use the notations from the proof of Lemma 2 and continue the proof with the methods used in the proof of Proposition 9 of **[3]** and Proposition 3 of **[11]**.

The set

$$N = \left\{ z' \in V_1 \mid \frac{\partial \Theta}{\partial x_j}(z') = 0, 1 \leq j \leq n-q-1 \right\}$$

is in a neighborhood $V_2 \subset V_1$ of $0 \in \mathbf{C}^{n-1}$ an $n+q-1$-dimensional generic submanifold of C^{n-1} which contains $Z \cap V_2$.

We denote by $\tau(z) = J(\operatorname{grad} \rho(z))$ where J represents the complex structure on $\mathbf{C}^n = \mathbf{R}^{2n}$. Because $T_0(\Sigma) = \{(z, w) \mid w = 0\}$, it follows that τ is transversal to Σ at $(0, 0)$, hence there exists a neighborhood $U_4 \subset U_3$ such that τ is transversal to Σ on U_4.

Therefore there exists a $\mathbf{C}^\infty$-diffeomorphism φ defined on

$$0_\varepsilon = \{(z', t) \mid z' \in V_2, t \in (-\varepsilon, \varepsilon)\}$$

with values in ∂D such that

$$\varphi(z',0) = (z',h(z')) \quad \text{and} \quad \frac{\partial\varphi}{\partial t}(z',0) = \tau(z',h(z')). \tag{2}$$

Because $Z \cap V_2 \subset N$ we have

$$E \cap U_4 \subset \varphi(Z \times \{0\}) \subset \varphi(N \times \{0\}). \tag{3}$$

We denote by $\Phi(z',t) = \operatorname{Re} f(\varphi(z',t))$ and by

$$\tilde{N} = \{(z',t) \in 0_\varepsilon | r_j(z',t) = 0, 1 \le j \le n-q-1, \rho_j(\varphi(z',t)) = 0, \\ j = 1, \dots, q\}.$$

where $r_j(z',t) = (\partial\Phi/\partial x_j)(z',t)$ and ρ_j are obtained by Remark 3.

Let us suppose that $0 \le \lambda_j < 1$ for $1 \le j \le q$ and denote $h_j = \rho_j \circ \varphi$, $j = 1, \dots, q$. Let $\{e_1, \dots, e_n\}$ be the standard basis in $\mathbf{C}^n$ and let S_0 be the real space generated by $e_1, \dots, e_{n-q-1}, Je_1, \dots, Je_q$. Because

$$r_j(z',0) = \frac{\partial\Phi}{\partial x_j}(z',0) = \frac{\partial\Theta}{\partial x_j}(z') \tag{4}$$

from (1) we conclude that

$$(\operatorname{grad} r_j)(0,0) = 2(1+\lambda_j)e_j.$$

By Remark 3 we obtain that

$$\frac{\partial(r_1, \dots, r_{n-q-1}, h_1, \dots, h_q)}{\partial(x_1, y_1, \dots, y_{n-1}, t)}(0)$$

has maximal rank $n-1$ and $\tilde{N}$ is in the neighborhood of the origin an n-dimensional submanifold of 0_ε.

From (1) and (4) we obtain that the restriction to S_0 of the Hessian of Φ at the origin is strictly positive definite. From (iii) we obtain that $S_0 \oplus T_{(0,0)}(\tilde{N}) = \mathbf{R}^{2n-1} \times \mathbf{R}$ and the proof continues as in the proof of Proposition 3 of **[11]**, the genericity being obtained by (ii).

LEMMA 3. *Let D be a bounded pseudoconvex domain in $\mathbf{C}^n$, $\{E_n\}_{n\in m}$ a family of peak sets for $A^\infty(D)$ with strong support functions f_n which satisfy* (i) *of Proposition* 1. *Then $E = \bigcap_n E_n$ is a peak set for $A^\infty(D)$ with a strong support function which satisfies* (i).

Proof. A strong support function for E is $f = 1 - \sum_{n\in N}(1/2^n)e^{-f_n}$.

$$H_p^r(\operatorname{Re} f) = \sum_{n\in N} \frac{1}{2^n} H_p^r(\operatorname{Re} f_n)$$

and

$$\frac{\partial \operatorname{Re} f}{\partial n} = \sum_{n\in N} \frac{1}{2^n} \frac{\partial \operatorname{Re} f_n}{\partial n}$$

and by Lemma 2(a) the lemma follows.

PROPOSITION 2. *Let $D \subset \mathbf{C}^n$ be a bounded pseudoconvex domain with smooth boundary, E a compact subset of ∂D, ω a neighborhood of E in $\mathbf{C}^n$ and ρ a continuous function on ω which vanishes on E. We suppose that there exists $G \in C^\infty(\omega \cap \overline{D})$ such that:*

(a) $\{z \in \overline{D} \cap \omega | G(z) = 0\} = E$,

(b) *for each $\alpha \in \mathbf{N}^n$, $\kappa \in \mathbf{N}$, there exists $C_{\alpha\kappa} > 0$ such that*

$$|D^\alpha \bar{\partial}(G(z))| \leq C_{\alpha\kappa}\rho(z)^\kappa$$

for each $z \in \overline{D} \cap \omega$,

(c) *there exists $c > 0$ such that* $\operatorname{Re} G(z) \geq c\rho(z)$ *for each $z \in \overline{D} \cap \omega$.*

Suppose that $\operatorname{Re} G$ verifies (i) of Proposition 1. Then E is a peak set for $A^\infty(D)$ with a strong support function which verifies (i).

Proof. We know from **[3]** that E is a peak set for $A^\infty(D)$ with strong support function $f = G/(t - uG)$ where $t = 1$ in the neighborhood of E and u is a solution of a $\bar{\partial}$ problem. It is easy to see that f verifies condition (i).

4. Peak points in weakly pseudoconvex domains. For simplicity, we shall say that a peak set E for $A^\infty(D)$ which verifies (i), (ii), and (iii) of Proposition 1, verifies the (GC) condition (GC=good convexity).

REMARK 4. The (GC) condition is obviously verified at the points of strong pseudoconvexity.

THEOREM 1. *Let D be a bounded pseudoconvex domain in $\mathbf{C}^n$ with smooth boundary, E a peak set for $A^\infty(D)$ which verifies the (GC) condition, and K a compact subset of E. Then K is a peak set for $A^\infty(D)$.*

Proof. The proof is identical with the proof of Theorem 11 of **[3]**, which uses only the conclusions of Proposition 1.

THEOREM 2. *Let D be a bounded pseudoconvex domain with smooth boundary such that the set of weakly pseudoconvex boundary points $w(\partial D)$ is contained in a peak set E which verifies the (GC) condition. Then each subset of $w(\partial D)$ is a peak set for $A^\infty(D)$.*

Proof. By Corollary 1 of **[11]**, $w(\partial D)$ is a peak set for $A^\infty(D)$. By the proof of Lemma 1, Lemma 2, Corollary 1 of **[11]** and by Lemma 3 and Proposition 2 above, $w(\partial D)$ verifies the (GC) condition and we obtain the result from Theorem 1.

From Theorem 2 we obtain the following:

THEOREM 3. *Let D be a bounded pseudoconvex domain with smooth boundary in $\mathbf{C}^n$ such that $w(\partial D)$ is contained in a peak set E which verifies the (GC) condition. Then each point of ∂D is a peak point for $A^\infty(D)$.*

REMARK 5. Using the same proof as in Lemma 2 of **[11]** we may suppose in Theorem 3 that the (GC) condition is verified except at a finite number of points.

EXAMPLE. Let $\rho(z) = |z_1|^4+|z_2|^4+|z_3|^4+|z_3|^2\,((\operatorname{Im} z_1)^2+(\operatorname{Im} z_2)^2-\operatorname{Re} z_3^2)$ and $D = \{z \in \mathbf{C}^3 | \rho(z) < 1\}$. D is a bounded pseudoconvex domain in $\mathbf{C}^3$ with real analytic boundary which does not have the (NP) property (it is a slightly modified version of the domain considered in Example 3 of **[12]**). We have $w(\partial D) = C_1 \cup C_2 \cup C_3$, where

$$C_1 = \{z | \, |z_1| = 1, z_2 = z_3 = 0\}, \quad C_2 = \{z | \, |z_2| = 1, z_1 = z_3 = 0\},$$
$$C_3 = \{z | y_1 = y_2 = z_3 = 0, x_1^4 + x_2^4 = 1\}.$$

The points of C_3 are not of strict type in the sense of **[2]** or **[8]**.

Let $E = \{z \in \partial D \mid z_1^4 + z_2^4 = 1\}$, which is a peak set for $A^\infty(D)$ and $C_3 \subset E$. At each point of C_3 with $x_1 \neq 0$, $x_2 \neq 0$ we obtain that

$$\begin{aligned} H^r C_p(\rho, f) = 12\left(4\sqrt{x_1^6+x_2^6}-1\right)(x_1^2t_1^2+x_2^2t_3^2) \\ +4\left(\sqrt{x_1^6+x_2^6}+3\right)(x_1^2t_2^2+x_2^2t_4^2) \end{aligned}$$

has 4 strictly positive eigenvalues and in the neighborhood of p, C_3 is contained in $M = \{z | \rho(z) = 1,\ x_1^4 + y_1 + x_2^4 + x_3 = 1\}$. Because each point of C_1 and C_2 is obviously a peak point for $A^\infty(D)$, it follows that each point of ∂D is a peak point for $A^\infty(D)$.

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