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ANNULAR BUNDLES

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This paper is devoted to the study of a particular kind of complex manifold with non-trivial topology: holomorphic fiber bundles with fibers biholomorphic to plane annuli.

0. Introduction. In recent years, some work has been done on function theory in complex manifolds with non-trivial topology. Two different approaches have been developed, a variational one and a purely complex-theoretical one.

The origins of the former (due essentially to Bedford and Burns; cf. **[BBu]** and **[B1]**) lie in the work of Landau and Osserman **[LO1, LO2]** on multiply connected Riemann surfaces. They defined an invariant norm on the homology group of the surface, using a particular family of harmonic functions. The solution of an associated extremal problem is a *harmonic measure* of the surface. The invariance properties of this function can be used to get several results in function theory, for instance the classification of the plane annuli.

Bedford and Burns, in **[BBu]**, developed a similar theory in bounded domains of $\mathbf{C}^n$ of the form $D_1 \backslash D_2$, where D_1 and D_2 were smooth strongly pseudoconvex domains with $D_2 \subset\subset D_1$. They used an invariant norm on the homology groups defined by Chern et al. in **[CLN]**, and the solution of a particular complex Monge-Ampère equation as harmonic measure. Bedford, in **[B1]**, studied complex manifolds of (complex) dimension n with $H_n(X, \mathbf{R}) \neq (0)$ using several other invariant norms on $H_n(X, \mathbf{R})$.

The second approach (due essentially to Bedford, again, and Mok; cf. **[B2]** and **[Mo]**) is based on the classical theory of Stein manifolds, and is devoted to the study of Stein manifolds of (complex) dimension n with $H_n(X, \mathbf{R}) \neq (0)$. In particular, Mok proved that, under some mild assumptions, a holomorphic map of such a Stein manifold into itself inducing an isomorphism of $H_n(X, \mathbf{R})$ is an automorphism.

These methods do not work on manifolds with non-trivial homology only in low dimensions. For instance, we do not get any result on the simplest example of non-contractible strongly pseudoconvex domain, that is

$$D = \{(z, w) \in \mathbf{C}^2 \mid |z|^2 + |w|^2 + |w|^{-2} < 3\}.$$

Indeed, the map $\tau : D \to D_0 = D \cap (\{0\} \times \mathbf{C})$ given by $\tau(z, w) = (0, w)$ is a retraction of deformation of D onto D_0. Therefore $H_*(D, \mathbf{R}) = H_*(D_0, \mathbf{R})$; in particular, $H_2(D, \mathbf{R}) = (0)$.

As we shall see in Example 3.2, if we consider only generic holomorphic maps of D into itself, we cannot hope to find any result like Mok's. However, we may exploit the topology of D in another way.

Let $p: D \to \mathbf{C}$ be the projection on the first coordinate; then the fibers of p are biholomorphic to plane annuli. In other words, topologically D is a (trivial) fiber bundle in annuli, with non-constant complex structure on the fibers, a first example of what we call an *annular bundle*. On this kind of object it is possible to develop a theory generalizing the methods of Landau and Osserman, and we shall be able to study thoroughly the holomorphic fiber maps between annular bundles. Moreover, the study of such bundles leads to some interesting results regarding line bundles and hermitian metrics.

The content of this paper is the following. In the first section, we shall define the notion of annular bundle and several associated invariants. In particular, the curvature form of a hermitian metric on a particular line bundle correlated to the bundle, and a real function measuring the variation of the complex structure of the fibers will turn out to be essential to describe the structure of the bundle, deserving the specific name of modular data. We shall also define other related concepts and produce some examples.

In the second section we shall study the following problem: do the modular data determine the annular bundle? Along the way, we shall characterize the differential forms that may arise as curvature forms of a hermitian metric on a specific holomorphic line bundle.

In the third, and last, section we shall study the fiber maps between annular bundles. We shall define the notion of harmonic measure of an annular bundle, which we shall exploit to develop a theory parallel to the plane case. We shall be able to prove a rigidity theorem in the spirit of Mok's result for a large class of annular bundles (Theorem 3.8). Finally, we shall study in some details the group of fiber automorphisms of an annular bundle.

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1. Annular bundles: definitions and examples. A triple (A, p, B) will be called an *annular bundle* if

(i) A and B are connected complex manifolds and $p\colon A \to B$ is a holomorphic map;

(ii) there exist an open cover $\{U_\alpha\}$ of B, smooth functions $\rho_1^\alpha, \rho_2^\alpha\colon U_\alpha \to \mathbf{R}^+$ with $\rho_1^\alpha < \rho_2^\alpha$ and biholomorphisms

$$\chi_\alpha\colon p^{-1}(U_\alpha) \to \{(z, w) \in U_\alpha \times \mathbf{C} \mid \rho_1^\alpha(z) < |w| < \rho_2^\alpha(z)\}$$

such that the diagram

$$\begin{tikzcd} p^{-1}(U_\alpha) \arrow[r, "\chi_\alpha"] \arrow[d, "p"] & \{\rho_1^\alpha(z) < |w| < \rho_2^\alpha(z)\} \subset U_\alpha \times \mathbf{C} \arrow[dl, "\pi"] \\ U_\alpha & \end{tikzcd}$$

commutes, where π is the projection on the first coordinate;

(iii) there are holomorphic maps $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathbf{C}^*$ so that $\chi_\alpha \circ \chi_\beta^{-1}\colon \chi_\beta(U_\alpha \cap U_\beta) \to \chi_\alpha(U_\alpha \cap U_\beta)$ is given by

$$\chi_\alpha \circ \chi_\beta^{-1}(z, w) = (z, g_{\alpha\beta}(z)w). \tag{1.1}$$

A is called the *total space*, B the *base* and $\{U_\alpha\}$ a *trivializing cover.* Unless otherwise stated, we assume that every trivializing cover is a *good cover*, that is that any non-empty finite intersection of U_α is contractible. For every $z \in B$ we shall denote the fiber $p^{-1}(z)$ by A_z, and we shall often write A instead of (A, p, B).

Using (ii), we can construct the real line bundle $p^*\colon \mathscr{H}_1(p, \mathbf{R}) \to B$, with fiber $H_1(A_z, \mathbf{R})$ at $z \in B$. Then (iii) immediately implies that this bundle is trivial (cf. also the beginning of §3).

Actually, (iii) is equivalent to the triviality of this bundle. In fact, for any $z \in B$, $\chi_\alpha \circ \chi_\beta^{-1}(z, \cdot)$ is an isomorphism between two plane annuli; hence we should have either $\chi_\alpha \circ \chi_\beta^{-1}(z, w) = (z, g_{\alpha\beta}(z)w)$ or $\chi_\alpha \circ \chi_\beta^{-1}(z, w) = (z, g_{\alpha\beta}(z)w^{-1})$.

If $p^*\colon \mathscr{H}_1(p, \mathbf{R}) \to B$ is trivial, we can consistently choose generators for $H_1(A_z, \mathbf{R})$. Moreover, $\chi_\alpha \circ \chi_\beta^{-1}$ must send this generator in a positive multiple of itself, and this is possible iff $\chi_\alpha \circ \chi_\beta^{-1}$ is of the form (1.1), and we are done.

A *fiber map* between two annular bundles (A_1, p_1, B_1) and (A_2, p_2, B_2) is a holomorphic map $f\colon A_1 \to A_2$ such that there is a holomorphic map $\varphi\colon B_1 \to B_2$ so that the following diagram commutes:

$$\begin{array}{ccc} A_1 & \xrightarrow{f} & A_2 \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} \\ B_1 & \xrightarrow{\varphi} & B_2 \end{array}$$

An *isomorphism* of annular bundles is a fiber map with a fiber inverse, as usual. Later, we shall define an equivalence of annular bundles, a particular kind of isomorphism.

Now we shall define several objects canonically associated to an annular bundle.

First of all, the system of nowhere-vanishing holomorphic functions $\{g_{\alpha\beta}\}$ defined in (1.1) is a 1-cocycle with coefficients in $\mathscr{O}^*$, and so it determines an element of $H^1(B, \mathscr{O}^*)$. The corresponding line bundle E_A over B is the *filling bundle* of A.

The $\{g_{\alpha\beta}\}$ satisfy a very important relation. Since $\chi_\alpha \circ \chi_\beta^{-1}$ is a fiber preserving biholomorphism, we must have

$$\text{(1.2)} \qquad \rho_1^\alpha = |g_{\alpha\beta}|\rho_1^\beta \quad \text{and} \quad \rho_2^\alpha = |g_{\alpha\beta}|\rho_2^\beta \quad \text{on } U_\alpha \cap U_\beta.$$

In particular, $\rho_2^\alpha/\rho_1^\alpha = \rho_2^\beta/\rho_1^\beta$ on $U_\alpha \cap U_\beta$; hence we may define the *modular function* $r\colon B \to (1, +\infty)$ by

$$r(z) = \rho_2^\alpha(z)/\rho_1^\alpha(z) \quad \text{if } z \in U_\alpha.$$

In other words, $r(z)$ is just the modulus of the annulus A_z, and so r measures the variation of the complex structure on the fibers. Later, we shall often identify r and $r \circ p$, if no confusion will arise (cf. for instance (1.3)).

There is another consequence of (1.2): $\rho_1^\alpha/\rho_1^\beta$ is the modulus of a holomorphic function on $U_\alpha \cap U_\beta$. Therefore $\log(\rho_1^\alpha/\rho_1^\beta)$ is pluriharmonic, that is $dd^c \log \rho_1^\alpha = dd^c \log \rho_1^\beta$ on $U_\alpha \cap U_\beta$. Hence we may define the *modular form* ω of the annular bundle by

$$\omega = dd^c \log \rho_1^\alpha \quad \text{on } U_\alpha;$$

ω is a real closed (1,1)-form on B. The modular function and the modular form are the *modular data* of the annular bundle.

To understand the meaning of ω we need another element. (1.2) yields

$$(\rho_1^\alpha)^{-2} = |g_{\alpha\beta}|^2(\rho_1^\beta)^{-2} \quad \text{on } U_\alpha \cap U_\beta.$$

So we may define a hermitian metric h on the filling bundle E_A, called the *filling metric*, setting $h_\alpha = (\rho_1^\alpha)^{-2}$ on U_α. The curvature form of this metric is

$$\partial\overline{\partial}\log(\rho_1^\alpha)^{-2} = -idd^c \log \rho_1^\alpha = -i\omega.$$

Therefore ω essentially is the curvature form of the filling metric.

The annular bundle may be recovered starting from the filling metric and the modular function:

PROPOSITION 1.1. *Let (A, p, B) be an annular bundle with filling bundle E, filling metric h and modular function r. Then*

$$A = \{\zeta \in E \mid 1 < h(\zeta) < r(\zeta)^2\}. \tag{1.3}$$

Conversely, given a line bundle (E, p, B) with a hermitian metric h and a smooth function $r\colon B \to (1, +\infty)$, the subset A of E defined by (1.3) *is an annular bundle with filling bundle E, filling metric h and modular function r.*

Proof. Let $\{U_\alpha\}$ be a trivializing cover of B. Using the corresponding local coordinates on E, if we set $\zeta = (z, w) \in p^{-1}(U_\alpha)$, we have

$$h(\zeta) = |w|^2/\rho_1^\alpha(z)^2 \quad \text{and} \quad r(\zeta)^2 = (\rho_2^\alpha(z)/\rho_1^\alpha(z))^2,$$

and A is given by (1.3).

Conversely, if we have E, h and r, set $\rho_1^\alpha = (h_\alpha)^{-2}$ and $\rho_2^\alpha = r\rho_1^\alpha$ on U_α; it is immediate to check that A is the annular bundle locally defined by ρ_1^α and ρ_2^α. □

Let E_A^* be the dual bundle of E_A(defined by the 1-cocycle $\{(g_{\alpha\beta})^{-1}\}$). On E_A^* we may put the *dual metric* h^* given by $(h^*)_\alpha = (\rho_2^\alpha)^2$ on U_α. Since (by (1.2))

$$(\rho_2^\alpha)^2 = |g_{\beta\alpha}|^{-2}(\rho_2^\beta)^2 \quad \text{on } U_\alpha \cap U_\beta,$$

h^* is well defined. The *dual annular bundle* A^* is given by

$$A^* = \{\zeta \in E_A^* \mid 1 < h^*(\zeta) < r(\zeta)^2\}.$$

A^* is an annular bundle with filling bundle E_A^*, modular function r and modular form ω^* given by

$$\omega^* = -dd^c \log \rho_2^\alpha \quad \text{on } U_\alpha.$$

ω and ω^* are related by

$$\omega + \omega^* = -dd^c \log r. \tag{1.4}$$

There is a natural isomorphism $*\colon A \to A^*$ given locally by

$$*(z, w) = (z, w^{-1}).$$

It is immediate to check that $*$ is well defined.

A fiber map f between two annular bundles is an *isometry* if it is an isometry for the filling metrics; it is a *dual isometry* if $*f$ is an isometry. An isometric isomorphism between two annular bundles over the same manifold B inducing the identity on B is an *equivalence* of annular bundles.

An equivalence induces biholomorphisms between the fibers and preserves the filling metrics; therefore it must be linear on the fibers, and it extends to an equivalence of the filling bundles. In particular, an equivalence preserves both the modular data and the filling bundle with its filling metric. Therefore, we shall identify two equivalent annular bundles. Finally, remark that $*$ is not an isometry.

At this point, some examples are mandatory.

EXAMPLE 1.1. Let B be a connected complex manifold, $\rho_1, \rho_2\colon B \to \mathbf{R}^+$ smooth functions with $\rho_1 < \rho_2$, and set $A = \{(z, w) \in B \times \mathbf{C} \mid \rho_1(z) < |w| < \rho_2(z)\}$. (A, p, B), where p is the projection on the first coordinate, is an annular bundle over B. Every annular bundle equivalent to A is called a *trivial bundle*. In this case, the filling bundle is the trivial bundle $B \times \mathbf{C}$, the modular function is ρ_2/ρ_1 and the modular form is $dd^c \log \rho_1$.

EXAMPLE 1.2. An annular bundle (A, p, B) is *self-dual* if there is a trivializing cover $\{U_\alpha\}$ of B and a smooth function $r\colon B \to (1, +\infty)$ such that $p^{-1}(U_\alpha)$ is fiber biholomorphic to

$$\{(z, w) \in U_\alpha \times \mathbf{C} \mid r(z)^{-1/2} < |w| < r(z)^{1/2}\}.$$

In this case, the modular function is r and the modular form is $-dd^c \log r^{-1/2}$. In particular, (1.4) yields

$$\omega = \omega^*. \tag{1.5}$$

As we shall see later, (1.5) characterizes self-dual bundles up to equivalence.

If $\{g_{\alpha\beta}\}$ is the 1-cocycle representing E_A, (1.2) yields $|g_{\alpha\beta}| r = r$ on $U_\alpha \cap U_\beta$, and $|g_{\alpha\beta}| \equiv 1$. Therefore $\{g_{\alpha\beta}\}$ is a 1-cocycle with coefficients in $\mathbf{S}^1$, and E_A belongs to the image of $H^1(B, \mathbf{S}^1)$ in $H^1(B, \mathcal{O}^*)$ through the map induced by the natural inclusion of sheaves $\mathbf{S}^1 \to \mathcal{O}^*$. In particular, the map $f\colon A \to A^*$ defined locally by $f(z, w) = (z, \overline{w})$ is a *real* equivalence between A and A^*, explaining the name of this kind of bundle.

One may wonder if, for self-dual bundles, A is equivalent to A^*. As we shall see later, this is equivalent to a strong property of the automorphism group of the annular bundle.

EXAMPLE 1.3. Let $\mathbf{B}^n$ be the unit ball of $\mathbf{C}^n$, and define an annular bundle over $\mathbf{B}^n$ by

$$A = \{(z, w) \in \mathbf{B}^n \times \mathbf{C} \mid \|z\|^2 + |w|^2 + |w|^{-2} < 3\}.$$

A is a self-dual bundle over $\mathbf{B}^n$ with modular function

$$r(z) = \left[k(z) + \sqrt{k(z)^2 - 4}\right]/2,$$

where $k(z) = 3 - \|z\|^2$. In particular, r is a strictly decreasing function of $\|z\|$ with the origin as unique maximum point. As we shall see, using this information we shall be able to compute the automorphism group of A.

The modular form is

$$\begin{aligned}\omega &= -\frac{1}{2} dd^c \log r \\ &= -i\frac{k(z)}{(k(z)^2 - 4)^{3/2}} \sum_{\mu,\nu=1}^{n} \left[z^\mu \bar{z}^\nu + \left(k(z) - \frac{4}{k(z)}\right)\delta_{\mu\nu}\right] d\bar{z}^\mu \wedge dz^\nu,\end{aligned}$$

where $\delta_{\mu\nu}$ is the Kronecker delta.

These are the examples we need later on; anyway, using Proposition 1.1, it is very easy to construct several other examples.

2. The modular theory. Among the several objects attached to an annular bundle, the easiest to work with are the modular data. Therefore we want to address the following problem: do the modular data determine the annular bundle? As we shall see, the answer is linked to the topological structure of the base.

We need some preliminary definitions and results. Let B be a complex manifold: a smooth function $\rho\colon B \to \mathbf{R}^+$ is *logarithmically pluriharmonic* if $\log \rho$ is pluriharmonic, that is if $dd^c \log \rho \equiv 0$ on B. If $f\colon B \to \mathbf{C}^*$ is a non-vanishing holomorphic function, then $\rho = |f|$ is logarithmically pluriharmonic; indeed, locally we may define $\log f$, and we have $\log \rho = \operatorname{Re}(\log f)$.

Essentially, all the logarithmically pluriharmonic functions are of this kind. Let $\mathscr{L}^+$ be the sheaf of germs of logarithmically pluriharmonic functions; then we may define a sheaf morphism $m\colon \mathscr{O}^* \to \mathscr{L}^+$ sending, at the presheaf level, a nowhere-vanishing holomorphic function to its modulus. Then

PROPOSITION 2.1. *The sequence of sheaves*

$$0 \to \mathbf{S}^1 \xrightarrow{i} \mathscr{O}^* \xrightarrow{m} \mathscr{L}^+ \to 0$$

is exact.

Proof. The exactness at $\mathbf{S}^1$ is obvious. Let $\mathbf{f} \in \mathscr{O}^*$ be in the kernel of m. Then there is a representative f of $\mathbf{f}$ such that $|f| \equiv 1$. Then f is a constant of modulus 1, and $\mathbf{f} \in i(\mathbf{S}^1)$. Finally, we must prove the exactness at $\mathscr{L}^+$, that is that every logarithmically pluriharmonic function is locally the modulus of a nowhere-vanishing holomorphic function. Taking (locally) logarithms, this is equivalent to proving that every pluriharmonic function is locally the real part of a holomorphic function, a classical result. For sake of completeness, we report the proof.

Let $\varphi \colon U \to \mathbf{R}$ be a pluriharmonic function, where $U \subset B$ is a contractible open set. Since

$$0 = dd^c\varphi = -2i\overline{\partial}\partial\varphi = -2id(\partial\varphi)$$

and $H^1(U, \mathbf{R}) = (0)$, there is a $g \in C^\infty(U)$ such that $\partial\varphi = dg = \partial g + \overline{\partial} g$. Comparing bidegrees, we get $\overline{\partial} g = 0$, that is $g \in \mathscr{O}(U)$ and $\partial g = \partial\varphi$. Now

$$d\varphi = \partial\varphi + \overline{\partial}\varphi = \partial\varphi + \overline{\partial\varphi} = \partial g + \overline{\partial g} = dg + \overline{dg} = 2d(\operatorname{Re} g).$$

Therefore there is $a \in \mathbf{R}$ such that $\varphi = \operatorname{Re}(2g + a)$, and the assertion follows. □

COROLLARY 2.2. *The following long sequence is exact:*

$$\begin{aligned} 0 \to \mathbf{S}^1 \to \mathscr{O}^*(B) \to \mathscr{L}^+(B) \xrightarrow{\partial^*} H^1(B, \mathbf{S}^1) \xrightarrow{i^*} H^1(B, \mathscr{O}^*) \\ \xrightarrow{m^*} H^1(B, \mathscr{L}^+) \to H^2(B, \mathbf{S}^1) \to \cdots \\ \to H^k(B, \mathbf{S}^1) \to H^k(B, \mathscr{O}^*) \to H^k(B, \mathscr{L}^+) \to \cdots . \end{aligned}$$

Now we would like to examine more closely the group $H^1(B, \mathscr{L}^+)$. We need

LEMMA 2.3. *Let ω be a real closed $(1,1)$-form on a complex manifold B. Then in every open contractible subset U of B there exists a smooth function $u \colon U \to \mathbf{R}^+$ such that $\omega = dd^c \log u$ in U.*

Proof. Since ω is closed and U is contractible, there is a 1-form $\psi = \psi^{(1,0)} + \psi^{(0,1)}$ such that $\omega = d\psi$. Comparing bidegrees, we get

$$\omega = \overline{\partial}\psi^{(1,0)} + \partial\psi^{(0,1)},$$
$$\partial\psi^{(1,0)} = \overline{\partial}\psi^{(0,1)} = 0.$$

Then there exist $v_1, v_2 \in C^\infty(U)$ such that $\psi^{(1,0)} = \partial v_1$ and $\psi^{(0,1)} = \overline{\partial} v_2$. Therefore

$$\omega = \overline{\partial}\partial v_1 + \partial\overline{\partial} v_2 = dd^c[(v_1 - v_2)/2i].$$

Put $v = \mathrm{Re}[(v_1 - v_2)/2i]$; since ω is real,

$$dd^c v = \mathrm{Re}[dd^c((v_1 - v_2)/2i)] = \mathrm{Re}\,\omega = \omega.$$

Finally, $u = \exp(v)$ is as required. □

Denote by $\Lambda^{1,1}(B, \mathbf{R})$ the group of the real closed (1,1)-forms modulo the real forms of the kind $dd^c \log \rho$ on B. Then we have

THEOREM 2.4. *Let B be a complex manifold. Then $\Lambda^{1,1}(B, \mathbf{R})$ and $H^1(B, \mathscr{L}^+)$ are isomorphic.*

Proof. We want to define a homomorphism $\tilde{\mu}$ from the group $Z^{1,1}(B, \mathbf{R})$ of the real closed (1,1)-forms onto $H^1(B, \mathscr{L}^+)$.

Fix a locally finite good cover $\{U_\alpha\}$ of B, and let ω be a real closed (1,1)-form on B. By Lemma 2.3, on every U_α we find a smooth function $\rho_\alpha \colon U_\alpha \to \mathbf{R}^+$ such that $\omega = dd^c \log \rho_\alpha$ on U_α. Moreover, if $\rho'_\alpha \colon U_\alpha \to \mathbf{R}^+$ is another function with the same property, then ρ'_α/ρ_α is logarithmically pluriharmonic on U_α, and therefore (cf. the proof of Proposition 2.1) there exists a holomorphic function $h_\alpha \colon U_\alpha \to \mathbf{C}^*$ such that $\rho'_\alpha = \rho_\alpha |h_\alpha|$.

On $U_\alpha \cap U_\beta$, we have $dd^c \log \rho_\alpha = dd^c \log \rho_\beta$; therefore $\{\rho_\alpha/\rho_\beta\}$ is a 1-cocycle with coefficients in $\mathscr{L}^+$. Moreover, by the previous remark, if $\omega = dd^c \log \rho'_\alpha$ is another local representation of ω in U_α, there exist holomorphic functions $h_\alpha \colon U_\alpha \to \mathbf{C}^*$ such that

$$(\rho'_\alpha/\rho'_\beta)|h_\beta| = |h_\alpha|(\rho_\alpha/\rho_\beta) \quad \text{on } U_\alpha \cap U_\beta.$$

So ω determines a well defined element $\tilde{\mu}(\omega) \in H^1(B, \mathscr{L}^+)$, and $\tilde{\mu}$ is obviously a homomorphism from $Z^{1,1}(B, \mathbf{R})$ to $H^1(B, \mathscr{L}^+)$.

Now we want to prove that our map is onto. Let $\{\rho_{\alpha\beta}\}$ be a 1-cocycle with coefficients in $\mathscr{L}^+$; choose a partition of unity $\{\tau_\alpha\}$ subordinate to $\{U_\alpha\}$, and define $\rho_\alpha \colon U_\alpha \to \mathbf{R}^+$ by

$$\rho_\alpha = \prod_\sigma (\rho_{\alpha\sigma})^{\tau_\sigma}.$$

On $U_\alpha \cap U_\beta$ we have

$$\rho_\alpha/\rho_\beta = \prod_\sigma (\rho_{\alpha\sigma}/\rho_{\beta\sigma})^{\tau_\sigma} = \prod_\sigma (\rho_{\alpha\beta})^{\tau_\sigma} = \rho_{\alpha\beta}.$$

Set $\omega = dd^c \log \rho_\alpha$ on U_α. Since

$$\begin{aligned} dd^c \log \rho_\alpha - dd^c \log \rho_\beta &= dd^c \log(\rho_\alpha/\rho_\beta) \\ &= dd^c \log \rho_{\alpha\beta} = 0 \quad \text{on } U_\alpha \cap U_\beta, \end{aligned}$$

ω is a well defined real closed (1,1)-form on B, and obviously $\tilde{\mu}(\omega) = \{\rho_{\alpha\beta}\}$.

We are left with the computation of $\ker \tilde{\mu}$. If $\omega = dd^c \log \rho$ for some $\rho\colon B \to \mathbf{R}^+$, then obviously $\omega \in \ker \tilde{\mu}$. Conversely, choose $\omega \in \ker \tilde{\mu}$, and write $\omega = dd^c \log \rho_\alpha$ on U_α. Since $\tilde{\mu}(\omega) = 1$, there exist $\sigma_\alpha\colon U_\alpha \to \mathbf{R}^+$ with $dd^c \log \sigma_\alpha = 0$ and

$$\sigma_\alpha/\sigma_\beta = \rho_\alpha/\rho_\beta \quad \text{on } U_\alpha \cap U_\beta. \tag{2.1}$$

Let $\rho\colon B \to \mathbf{R}^+$ be given by $\rho = \rho_\alpha/\sigma_\alpha$ on U_α; by (2.1) ρ is well defined, and

$$dd^c \log \rho = dd^c \log \rho_\alpha - dd^c \log \sigma_\alpha = dd^c \log \rho_\alpha \quad \text{on } U_\alpha,$$

that is $\omega = dd^c \log \rho$.

Hence $\tilde{\mu}$ defines an isomorphism between $\Lambda^{1,1}(B, \mathbf{R})$ and $H^1(B, \mathscr{L}^+)$, as required. □

$\Lambda^{1,1}(B, \mathbf{R})$ is a real "Aeppeli group" of the manifold B; compare Bigolin **[Bi]** for definitions and properties of the Aeppeli groups.

We shall denote by $\mu\colon \Lambda^{1,1}(B, \mathbf{R}) \to H^1(B, \mathscr{L}^+)$ the isomorphism defined in Theorem 2.4. Furthermore, if ω is a real closed (1,1)-form, we shall denote by $[\omega]$ its class in $\Lambda^{1,1}(B, \mathbf{R})$.

Corollary 2.2 and Theorem 2.4 are what we need to solve our problem:

THEOREM 2.5. *Let B be a complex manifold. Then a real closed $(1,1)$-form ω is the modular form of an annular bundle over B with filling bundle E iff*

$$\mu[\omega] = m^*(E).$$

Proof. Fix a good open cover $\{U_\alpha\}$ of B trivializing E. Suppose that ω and r are the modular data for an annular bundle (A, p, B) with filling bundle E, and write $\omega = dd^c \log \rho_\alpha$ on U_α. Let $\{g_{\alpha\beta}\}$ be the 1-cocycle representing E. Then (1.2) yields

$$|g_{\alpha\beta}| = \rho_\alpha/\rho_\beta \quad \text{on } U_\alpha \cap U_\beta, \tag{2.2}$$

and therefore $m^*(E) = \mu[\omega]$, as required.

Conversely, assume that $\mu[\omega] = m^*(E)$, and let $\{g_{\alpha\beta}\}$ be the 1-cocycle representing E. This means that (cf. Lemma 2.3) we may write $\omega = dd^c \log \rho_\alpha$ on U_α, and (2.2) holds. So, define a hermitian metric h on E setting $h_\alpha = (\rho_\alpha)^{-2}$ on U_α (which is possible by (2.2)). Then

$$A = \{\zeta \in E \mid 1 < h(\zeta) < r(\zeta)^2\},$$

where $r\colon B \to (1, +\infty)$ is an arbitrary smooth function, is an annular bundle with modular form ω and filling bundle E. □

Actually, Theorem 2.5 proved the following statement:

THEOREM 2.5′. *Let B be a complex manifold, and ω a real closed $(1,1)$-form on B. Then $-i\omega$ is the curvature form of a hermitian metric on a given holomorphic line bundle E over B iff*

$$\mu[\omega] = m^*(E).$$

So the map $m^*\colon H^1(B, \mathscr{O}^*) \to H^1(B, \mathscr{L}^+)$ is at the core of our problem: the existence or the unicity of an annular bundle with given modular data is linked to the surjectivity or injectivity of this map. Therefore it is natural to ask when the groups $H^k(B, \mathbf{S}^1)$ vanish. It turns out that the answer is very natural:

PROPOSITION 2.6. *Let B be a manifold. Then for every $k \geq 0$ we have $H^k(B, \mathbf{S}^1) = (0)$ iff $H_k(B, \mathbf{Z}) = (0)$.*

Proof. By the universal coefficient theorem, for every $k \geq 0$ there is an exact sequence

$$0 \longrightarrow \operatorname{Ext}(H_{k-1}(B, \mathbf{Z}), \mathbf{S}^1) \to H^k(B, \mathbf{S}^1) \to \operatorname{Hom}(H_k(B, \mathbf{Z}), \mathbf{S}^1) \to 0.$$

Now, $\mathbf{S}^1$ is a divisible group; therefore (cf. [M]) $\operatorname{Ext}(H_{k-1}(B, \mathbf{Z}), \mathbf{S}^1) = (0)$, and $H^k(B, \mathbf{S}^1)$ is isomorphic to $\operatorname{Hom}(H_k(B, \mathbf{Z}), \mathbf{S}^1)$, the character group of $H_k(B, \mathbf{Z})$. By Pontrjagin's theorem (see [P]), where we are considering $H_k(B, \mathbf{Z})$ endowed with the discrete topology and $H^k(B, \mathbf{S}^1)$ endowed with the compact-open topology, the topological character group of $H^k(B, \mathbf{S}^1)$ is naturally isomorphic to $H_k(B, \mathbf{Z})$. In particular, $H^k(B, \mathbf{S}^1) = (0)$ iff $H_k(B, \mathbf{Z}) = (0)$. □

So, here is our result.

THEOREM 2.7. *Let B be a complex manifold, $r\colon B \to (1, +\infty)$ a smooth function, and ω a real closed $(1,1)$-form on B. Then:*

(i) *r and ω are the modular data of an annular bundle over B iff $\mu[\omega] \in m^*(H^1(B, \mathcal{O}^*))$. In particular, this happens if $H_2(B, \mathbf{Z}) = (0)$;*

(ii) *if $H_1(B, \mathbf{Z}) = (0)$, then r and ω are the modular data of at most one annular bundle over B.*

Proof. (i) By Theorem 2.5, r and ω are the modular data of an annular bundle over B iff $\mu[\omega]$ is in the image of m^*. When $H_2(B, \mathbf{Z}) = (0)$, by Proposition 2.6 also $H^2(B, \mathbf{S}^1)$ vanishes, and therefore m^* is surjective (by Corollary 2.2).

(ii) When $H_1(B, \mathbf{Z}) = (0)$, by Proposition 2.6 we have $H^1(B, \mathbf{S}^1) = (0)$, and therefore (Corollary 2.2) m^* is injective. Hence the assertion follows from Theorem 2.5. □

So the modular data determine the annular bundle when $H_1(B, \mathbf{Z}) = H_2(B, \mathbf{Z}) = (0)$. In this case we have

COROLLARY 2.8. *Let B be a complex manifold with $H_1(B, \mathbf{Z}) = H_2(B, \mathbf{Z}) = (0)$. Then the line bundles over B are parametrized by $\Lambda^{1,1}(B, \mathbf{R})$.*

Proof. In fact, in this case $H^1(B, \mathcal{O}^*)$ is isomorphic to $H^1(B, \mathcal{L}^+)$ (by Corollary 2.2 and Proposition 2.6) which, in turn, is isomorphic to $\Lambda^{1,1}(B, \mathbf{R})$ by Theorem 2.5. □

We end this section with the characterization of the self-dual bundles:

PROPOSITION 2.9. *Let B be a complex manifold, and (A, p, B) an annular bundle with modular data ω and r. Then:*

(i) *if A is trivial, $[\omega] = 0$. The converse is true if $H_1(B, \mathbf{Z}) = (0)$;*

(ii) *A is self-dual iff $\omega = \omega^*$. In particular, if $H_1(B, \mathbf{Z}) = (0)$ then every self-dual bundle is trivial.*

Proof. The only non-trivial part is to prove that $\omega = \omega^*$ implies A self-dual. Fix a trivializing cover $\{U_\alpha\}$ of B, and let h be the filling metric on E_A, so that A is given by (1.3). On U_α, h is given by some $h_\alpha\colon U_\alpha \to \mathbf{R}^+$. Then, setting $\rho_\alpha = (h_\alpha)^{-2}$, our hypothesis (coupled with (1.4)) yields

$$dd^c \log \rho_\alpha = dd^c \log r^{-1/2} \quad \text{on } U_\alpha.$$

Therefore $dd^c \log(\rho_\alpha r^{1/2}) = 0$ on U_α, that is there exist holomorphic functions $g_\alpha\colon U_\alpha \to \mathbf{C}^*$ such that $\rho_\alpha r^{1/2} = |g_\alpha|$ on U_α.

Let E' be the line bundle over B represented by the 1-cocycle $\{g_\alpha g_\beta^{-1} g_{\alpha\beta}\}$, where $\{g_{\alpha\beta}\}$ is the 1-cocycle representing E_A. The fiber map $g\colon E' \to E_A$ defined locally by

$$g(z, w) = (z, g_\alpha(z) w)$$

is an equivalence between E' and E_A. Let $h' = g^* h$, and set

$$A' = \{\zeta \in E' \mid 1 < h'(\zeta) < r(\zeta)^2\},$$

where r is extended to E' in the standard way. Then A' is an annular bundle over B equivalent (through g) to A; moreover, locally h' is given by

$$h'_\alpha(z, w) = h(z, g_\alpha(z) w) = \rho_\alpha(z)^{-2} |g_\alpha(z)|^2 |w|^2 = r(z)|w|^2.$$

Hence A' is locally fiber-biholomorphic to

$$\{(z, w) \in U_\alpha \times \mathbf{C} \mid r(z)^{-1/2} < |w| < r(z)^{1/2}\}.$$

and we are done. □

3. Fiber maps and automorphisms. In this section we want to study the fiber maps between two annular bundles, devoting particular attention to the automorphisms.

Two tools are used studying maps between plane annuli: the topological degree and some kind of variational instrument (like Schwarz lemma [**H**], invariant length [**K**], harmonic measure [**LO1**, **LO2**], and so on). So, we would like to construct these tools in our setting.

Let $(f, \varphi)\colon (A, B) \to (A', B')$ be a fiber map. Then f maps the annulus A_z into the annulus $A'_{\varphi(z)}$ for every $z \in B$; therefore f_* sends $H_1(A_z, \mathbf{Z})$ into $H_1(A'_{\varphi(z)}, \mathbf{Z})$. Let $\gamma_z\colon [0, 1] \to A_z$ be given by

$$\gamma_z(t) = (z, \sqrt{\rho_1(z)\rho_2(z)}\, e^{2\pi i t}),$$

where $z \in p^{-1}(U) \cong \{(z, w) \in U \times \mathbf{C} \mid \rho_1(z) < |w| < \rho_2(z)\}$ and U is an open subset of B trivializing A. If we change the trivialization, the only change in γ_z is the origin; therefore γ_z defines a generator of $H_1(A_z, \mathbf{Z})$.

Let $\deg_z f$ be the integer defined by the formula

$$f_*[\gamma_z] = \deg_z f \cdot [\gamma_{\varphi(z)}],$$

where $[\gamma]$ denotes the homology class represented by γ. The interesting fact is

LEMMA 3.1. *Let $(f, \varphi)\colon (A, B) \to (A', B')$ be a fiber map. Then* $\deg_z f$ *is constant.*

Proof. Let $z \in B$, $U \subset B$ a neighbourhood of z trivializing A and $U' \subset B'$ a neighborhood of $\varphi(z)$ trivializing A' with $\varphi(U) \subset U'$. On $p^{-1}(U)$, f may be expressed as $f(z, w) = (\varphi(z), \psi(z, w))$, where $\psi\colon p^{-1}(U) \to \mathbf{C}$ is holomorphic. Let γ_z be the generator of $H_1(A_z, \mathbf{Z})$ previously defined; then

$$\deg_z f = \frac{1}{2\pi i}\int_{\gamma_z} \frac{\frac{\partial\psi}{\partial w}(z, w)}{\psi(z, w)}\, dw.$$

Therefore $\deg_z f$ is continuous in z; since it is an integer, and B is connected, $\deg_z f$ is constant. □

So the *degree* $\deg f$ of a fiber map f is just $\deg_z f$ for any $z \in B$.

Later on we shall need the following general construction of fiber maps of degree 1. Let $\varphi\colon B_1 \to B_2$ be a holomorphic map, and A an annular bundle over B_2 with modular data r and ω, filling bundle E and filling metric h. Then we can define a new annular bundle $\varphi^* A$ over B_1 and a fiber map $i_\varphi\colon \varphi^* A \to A$ so that the following diagram

$$\begin{array}{ccc} \varphi^* A & \xrightarrow{i_\varphi} & A \\ \downarrow & & \downarrow \\ B_1 & \xrightarrow{\varphi} & B_2 \end{array}$$

commutes. Indeed, it is enough to define

$$\varphi^* A = \{\zeta \in \varphi^* E \mid 1 < \varphi^* h(\zeta) < \varphi^* r(\zeta)^2\},$$

and let i_φ be the restriction to $\varphi^* A$ of the usual map between $\varphi^* E$ and E. Obviously, $\deg i_\varphi = 1$.

The second tool will be a variational one; to introduce it, we need some preliminary definitions.

Let (A, p, B) be an annular bundle. A *vertical* k-cycle is a k-cycle Γ with support contained in a fiber A_z. We shall say that a k-form η on A is *horizontal* if for every vertical k-cycle Γ we have $\int_\Gamma \eta = 0$. In the usual local coordinates on A, this is equivalent to requiring that η has no components involving only dw and $d\overline{w}$. In particular, the notion of horizontal k-form is interesting only if $k < 3$.

Let us define $\mathscr{H} = \{\eta \in C^\infty(A) \mid 0 < \eta < 1$ and $dd^c\eta$ is horizontal$\}$. If $\eta \in \mathscr{H}$, the quantity $\int_{\gamma_z} d^c\eta$ is independent of the trivialization. In fact, if γ is another cycle in A_z homologous to γ_z *in* A_z, we have

$$\int_\gamma d^c\eta = \int_{\gamma_z} d^c\eta,$$

by Stokes' theorem, for $dd^c\eta$ is horizontal.

Now we may define a function N on B by

$$\forall z \in B \quad N(z) = \sup\left\{\int_{\gamma_z} d^c\eta \,\middle|\, \eta \in \mathscr{H}\right\};$$

N essentially is a norm on the real line bundle $\mathscr{H}_1(p, \mathbf{R})$ mentioned in the first section.

Furthermore, define $\chi\colon A \to (0, 1)$ by

$$\chi(\zeta) = \frac{\log h(\zeta)}{\log r(\zeta)^2}, \tag{3.1}$$

where h is the filling metric and r the modular function of A. χ is the *harmonic measure* of A. Now we may prove

THEOREM 3.2. *Let* (A, p, B) *be an annular bundle. Then* $\chi \in \mathscr{H}$ *and*

$$\forall z \in B \quad N(z) = \int_{\gamma_z} d^c\chi. \tag{3.2}$$

Moreover, χ *is the unique function of* $\mathscr{H}$ *such that* (3.2) *holds.*

Proof. First of all we have to check that $\chi \in \mathscr{H}$. By (1.3), it follows at once that $0 < \chi < 1$. Next, using the formulas for $h(\zeta)$ and $r(\zeta)$ given in the proof of Proposition 1.1, we see that in local coordinates

$$\begin{aligned} d^c\chi &= \tau_1 - \frac{i}{\log r(z)^2}\left[\frac{1}{w}dw - \frac{1}{\overline{w}}d\overline{w}\right], \\ dd^c\chi &= d\tau_1 + \tau_2, \end{aligned}$$

where τ_1 and τ_z are horizontal; therefore $\chi \in \mathscr{H}$.

Now, set

$$\forall z \in B \quad \tilde{N}(z) = \sup\left\{\int_{\gamma_z} d^c\tilde{\eta} \,\middle|\, \tilde{\eta} \in C^\infty(A_z), 0 < \tilde{\eta} < 1, \tilde{\eta} \text{ harmonic}\right\},$$

where d^c is the differential operator of A_z. $\tilde{N}(z)$ is the norm on $H_1(A_z, \mathbf{R})$ defined by Landau and Osserman we mentioned in the introduction. Their theorem (cf. **[LO2]**) states that the supremum is achieved by exactly one function, the harmonic measure of the annulus. Now, if $\eta \in \mathscr{H}$, its restriction to A_z is harmonic, because $dd^c\eta$ is horizontal; therefore $\tilde{N} \geq N$, for restriction commutes with d^c and integration. On the other hand, χ restricted to A_z is the harmonic measure of the annulus A_z; so $N(z) \geq \int_{\gamma_z} d^c\chi = \tilde{N}(z)$, and we get (3.2). The unicity statement follows from the corresponding result of Landau and Osserman. □

So we may explicitly compute N:

COROLLARY 3.3. *Let* (A, p, B) *be an annular bundle with modular function* r. *Then*

$$N = \frac{2\pi}{\log r}.$$

Proof. In fact,

$$N(z) = \int_{\gamma_z} d^c\chi = \frac{-i}{2\log r(z)} \int_{\gamma_z} \left(\frac{1}{w}dw - \frac{1}{\overline{w}}d\overline{w}\right) = \frac{2\pi}{\log r(z)}.$$

□

Now we have enough material to begin our study of the fiber maps between annular bundles. The important result is:

THEOREM 3.4. *Let* $(f, \varphi)\colon (A, B) \to (A', B')$ *be a fiber map of annular bundles. Then*

$$N \geq |\deg f| \varphi^* N'.$$

Proof. First of all, we want to prove that $f^*\mathscr{H}' \subset \mathscr{H}$. If $\eta \in \mathscr{H}'$ and Γ is a vertical 2-cycle in A, we have

$$\int_\Gamma dd^c(f^*\eta) = \int_\Gamma f^*(dd^c\eta) = \int_{f_*\Gamma} dd^c\eta = 0,$$

for f is a holomorphic fiber map and therefore $f_*\Gamma$ is a vertical 2-cycle in A'. Therefore Theorem 3.2 yields

$$\begin{aligned} N(z) &= \int_{\gamma_z} d^c\chi \geq \int_{\gamma_z} d^c(f^*\chi') = \int_{\gamma_z} f^*(d^c\chi') = \int_{f_*\gamma_z} d^c\chi' \\ &= \deg f \int_{\gamma'_{\varphi(z)}} d^c\chi' = \deg f \cdot N'(\varphi(z)). \end{aligned}$$

Now, $\eta \in \mathscr{H}$ implies $1 - \eta \in \mathscr{H}$, and $\int_{\gamma_z} d^c(1 - \eta) = -\int_{\gamma_z} d^c\eta$. Then

$$-N(z) = \int_{\gamma_z} d^c(1 - \chi) \leq \int_{\gamma_z} d^c(f^*\chi') = \deg f \cdot N'(\varphi(z)),$$

and the assertion follows. □

Corollary 3.5. *Let $(f, \varphi)\colon (A, B) \to (A', B')$ be a fiber map of annular bundles. Then*

$$\log(\varphi^* r') \geq |\deg f| \log r.$$

A map of a plane annulus into itself cannot have degree greater than one in absolute value. A similar result is true in our setting:

Corollary 3.6. *Let (A, p, B) be an annular bundle with bounded modular function r, and take a fiber map $(f, \varphi)\colon (A, B) \to (A, B)$. Then $|\deg f| \leq 1$.*

Proof. Let $M = \sup \log r < \infty$. If $|\deg f| > 1$, there is $z \in B$ such that $\log r(z) > M / |\deg f|$. Then

$$\log r(\varphi(z)) \geq |\deg f| \log r(z) > M,$$

impossible. □

Example 3.1. Let $\mathbf{B}^n$ be the unit ball of $\mathbf{C}^n$, and A the annular bundle over $\mathbf{B}^n$ defined in Example 1.3. In this case, the modular function r is strictly decreasing in the radius; therefore, if $(f, \varphi)\colon (A, \mathbf{B}^n) \to (A, \mathbf{B}^n)$ is a fiber map, Corollaries 3.5 and 3.6 imply that $|\deg f| \leq 1$ and $\varphi^* r \geq r$, that is $\|\varphi(z)\| \leq \|z\|$ for all $z \in \mathbf{B}^n$. By Schwarz's lemma, the last condition is equivalent to requiring that $\varphi(0) = 0$. On the other hand, if $\varphi\colon \mathbf{B}^n \to \mathbf{B}^n$ is a holomorphic map with $\varphi(0) = 0$, then $f\colon A \to A$ given by $f(z, w) = (\varphi(z), w)$ is a fiber map.

Now, let us examine the isomorphisms between two annular bundles. If $f\colon A \to A'$ is an isomorphism, obviously $|\deg f| = 1$. But we have something more:

Proposition 3.7. *Let $(f, \varphi)\colon (A, B) \to (A', B')$ be an isomorphism of annular bundles. Then*

(i) *$\varphi^* N' = N$ and $\varphi^* r' = r$;*
(ii) *if $\deg f = 1$, then $f^* \chi' = \chi$, f is an isometry and $\varphi^* \omega' = \omega$;*
(iii) *if $\deg f = -1$, then $f^* \chi' = 1 - \chi$, f is a dual isometry and $\varphi^* \omega' = \omega^*$.*

Proof. (i) By Theorem 3.4, $N \geq \varphi^* N'$ and $N' \geq (\varphi^{-1})^* N$; therefore $N = \varphi^* N'$. The same argument, using Corollary 3.5, works for the modular functions.

(ii) We have

$$\int_{\gamma_z} d^c\chi \geq \int_{\gamma_z} d^c(f^*\chi') = \int_{\gamma'_{\varphi(z)}} d^c\chi' \geq \int_{\gamma'_{\varphi(z)}} d^c((f^{-1})^*\chi) = \int_{\gamma_z} d^c\chi.$$

Therefore for all $z \in B$ we have $\int_{\gamma_z} d^c\chi = \int_{\gamma_z} d^c(f^*\chi')$, and Theorem 3.2 yields $f^*\chi' = \chi$. By (i), this implies that f is an isometry and, since the modular forms are essentially the curvature forms of the filling metrics, that $\varphi^*\omega' = \omega$.

(iii) We have

$$\begin{aligned}\int_{\gamma_z} d^c(1-\chi) \leq \int_{\gamma_z} d^c(f^*\chi') = -\int_{\gamma'_{\varphi(z)}} d^c\chi' = \int_{\gamma'_{\varphi(z)}} d^c(1-\chi') \\ \leq \int_{\gamma'_{\varphi(z)}} d^c((f^{-1})^*\chi) = \int_{\gamma_z} d^c(1-\chi).\end{aligned}$$

Therefore for all $z \in B$ we have $\int_{\gamma_z} d^c(1-\chi) = \int_{\gamma_z} d^c(f^*\chi')$ and, by the unicity statement of Theorem 3.2, $f^*\chi' = 1-\chi$. By definition, this means that for all $\zeta \in A$

$$\frac{\log(h')^*(*f(\zeta))}{\log \varphi^* r'(\zeta)^2} = 1 - \frac{\log f^*h'(\zeta)}{\log \varphi^* r'(\zeta)^2} = \frac{\log h(\zeta)}{\log r(\zeta)^2},$$

that is, by (i), $(*f)^*(h')^* = h$, and f is a dual isometry. Looking at the curvature forms we get $\varphi^*\omega' = \omega^*$. □

In one variable, a holomorphic map of a plane annulus into itself of degree 1 is an automorphism. In our setting, this is in general not true.

EXAMPLE 3.2. Let $(A, p, \mathbf{B}^n)$ be the annular bundle of Example 1.3. The map $\tau: A \to A_0$ given by $\tau(z, w) = (0, w)$ is a retraction of deformation of A onto A_0 (the homotopy is $H(t, (z, w)) = (tz, w)$). Hence the immersion $\iota_0: A_0 \to A$ induces an isomorphism $\iota_{0*}: H_1(A_0, \mathbf{Z}) \to H_1(A, \mathbf{Z})$, and therefore every immersion $\iota_z: A_z \to A$ induces an isomorphism $\iota_{z*}: H_1(A_z, \mathbf{Z}) \to H_1(A, \mathbf{Z})$. So a fiber map $f: A \to A$ is of degree 1 iff it induces the identity on $H_1(A, \mathbf{Z})$. Now we shall construct a large family of fiber (and non-fiber) maps of A into itself which are the identity on $H_1(A, \mathbf{Z})$ without being surjective.

Let $f: A \to A$ be a holomorphic map, and write $f(z, w) = (\varphi(z, w), \psi(z, w))$. We claim that the induced homomorphism $f_*: H_1(A, \mathbf{Z}) \to H_1(A, \mathbf{Z})$ is the identity iff $\psi(0, w) = e^{i\theta}w$ for some $\theta \in \mathbf{R}$. In one direction it is obvious. Conversely, assume that f_* is the identity, and let $\tau: A \to A_0$ be the previously defined retraction. Then $\tau \circ f \circ \iota$ is holomorphic and $(\tau \circ f \circ \imath)_*$ is the identity. Since A_0 is a plane annulus, this implies that $\tau \circ f \circ \imath(0, w) = (0, e^{i\theta} w)$ for some $\theta \in \mathbf{R}$, which is our claim.

Now, choose $0 < \alpha < 1$ and holomorphic functions $\varphi \colon A \to \mathbf{C}^n$, $g \colon A \to \mathbf{C}$ such that for all $(z, w) \in A$

$$\|\varphi(z, w)\| \leq \alpha \|z\| \quad \text{and} \quad |g(z, w)| \leq \beta \|z\|^2,$$

where $0 < \beta < 1$ satisfies

$$\frac{\beta(2 - \beta)}{(1 - \beta)^2} \leq \frac{1 - \alpha^2}{3}$$

(such a β always exists). We claim that $f \colon A \to \mathbf{C}^{n+1}$ given by

$$f(z, w) = (\varphi(z, w), e^{i\theta}(1 + g(z, w))w)$$

(where $\theta \in \mathbf{R}$) is a non-surjective map from A into A inducing the identity in homology. Assume for a moment that $f(A) \subset A$. Then, since $p(f(A))$ is strictly contained in $\mathbf{B}^n$, f is not surjective; moreover, since $f(0, w) = (0, e^{i\theta} w)$, f_* is the identity.

So we have to prove that $f(A) \subset A$. Let $\tilde{z} = \varphi(z, w)$ and $\tilde{w} = e^{i\theta}(1 + g(z, w))w$. Then

$$\begin{aligned}(3.3)\quad & \|\tilde{z}\|^2 + |\tilde{w}|^2 + |\tilde{w}|^{-2} \\ & \leq \alpha^2 \|z\|^2 + |1 + g(z, w)|^2 |w|^2 + |1 + g(z, w)|^{-2} |w|^{-2} \\ & \leq \alpha^2 \|z\|^2 + (1 + \beta \|z\|^2)^2 |w|^2 + (1 - \beta \|z\|^2)^{-2} |w|^{-2}.\end{aligned}$$

We claim that if $x \in [0, \beta]$ then

$$(3.4) \qquad (1 + x)^2 \leq 1 + kx \quad \text{and} \quad (1 - x)^{-2} \leq 1 + kx,$$

where $k = (2 - \beta)/(1 - \beta)^2$. The first one is easy, for

$$(1 + x)^2 - (1 + kx) = x(x + 2 - k) \leq 0$$

in $[0, \beta]$ iff $k - 2 \geq \beta$, which is true. The second one is just a bit harder, for

$$(1 + kx) - (1 - x)^{-2} = x(1 - x)^{-2}[kx^2 + (1 - 2k)x + k - 2] \geq 0$$

in $[0, \beta]$ iff $k\beta^2 + (1 - 2k)\beta + k - 2 \geq 0$, since $k \geq 2$. But k was chosen so that the last expression vanishes, and the claim is proven.

Putting together (3.3) and (3.4) we get

$$\begin{aligned}\|\tilde{z}\|^2 + |\tilde{w}|^2 + |\tilde{w}|^{-2} &\leq \alpha^2\|z\|^2 + (1 + k\beta\|z\|^2)(|w|^2 + |w|^{-2}) \\ &< (\alpha^2 - 1 + 3k\beta)\|z\|^2 - k\beta\|z\|^4 + 3 \leq 3 - k\beta\|z\|^4 \leq 3,\end{aligned}$$

by definition of β, and so $(\tilde{z}, \tilde{w}) \in A$.

Example 3.2 depends on a particular feature of the modular function: it is decreasing toward the boundary, allowing us to shrink the base to construct a non-surjective map of degree 1. On the other hand, if we choose a modular function with completely different behavior, we get rid of this unpleasant possibility. We shall say that a function $r: B \to \mathbf{R}$ is an *exhaustion* if $\{z \in B \mid r(z) \leq c\}$ is compact for all $c \in r(B)$. Then

THEOREM 3.8. *Let (A, p, B) be an annular bundle such that the modular function r is an exhaustion with just one minimum point. Assume either*

(i) *B is a bounded pseudoconvex domain of $\mathbf{C}^n$ with analytic boundary (and $n > 1$), or*

(ii) *B is the unit disk in $\mathbf{C}$.*

Then every fiber map $f: A \to A$ of degree ± 1 is an automorphism.

Proof. (i) Let $\varphi: B \to B$ be the induced map. By Corollary 3.5, $\varphi^* r \geq r$; therefore, since r is an exhaustion, φ is proper. By a theorem of Bedford and Bell (see **[BB]**), φ is an automorphism. We claim that $\varphi^* r = r$.

Let $z_0 \in B$ be the minimum point of r. If $z = \varphi^{-1}(z_0)$, we have $r(z) \leq r(\varphi(z)) = r(z_0)$; therefore $z = z_0$, and z_0 is a fixed point of φ.

Now we need three classical theorems of Cartan (see **[N]** and **[Kr]**). First of all, from the sequence $\{\varphi^k\}$ of the iterates of φ we may extract a subsequence $\{\varphi^{k_\nu}\}$ converging to $\psi \in \mathrm{Aut}(B)$. Secondly, the eigenvalues of $d\varphi_{z_0}$ have absolute value 1; therefore we may choose $\{\varphi^{k_\nu}\}$ so that $d(\varphi^{k_\nu})_{z_0} = (d\varphi_{z_0})^{k_\nu}$ converges to the identity. In particular, $d\psi_{z_0} = \mathrm{id}$ and, by the third theorem, $\psi = \mathrm{id}_B$.

So, assume by contradiction that $r(\varphi(z)) > r(z)$ for some $z \in B$. Then for all $\nu \in \mathbf{N}$ we have $r(\varphi^{k_\nu}(z)) \geq r(\varphi^{k_\nu - 1}(z)) \geq \cdots \geq r(\varphi(z))$, and

$$r(z) = \lim_{\nu\to\infty} r(\varphi^{k_\nu}(z)) \geq r(\varphi(z)) > r(z),$$

contradiction.

In short, we have proven that $\varphi \in \mathrm{Aut}(B)$ and $\varphi^* r = r$. Hence f sends A_z biholomorphically onto $A_{\varphi(z)}$ for all $z \in B$, because $r(\varphi(z)) = r(z)$ and a holomorphic map of degree ± 1 of a plane annulus into itself is a biholomorphism. Therefore f is bijective and, by Osgood's theorem (see [N]), an automorphism.

(ii) In this case we have to prove directly that φ is an automorphism; the rest of the proof works in the same way.

Let $z_0 \in B$ be the minimum point of r, and $\tau \in \mathrm{Aut}(B)$ such that $\tau(z_0) = 0$. Then, working in $\tau^* A$ instead of A, we may assume that $z_0 = 0$.

Exactly as in (i), we have $\varphi^{-1}(0) = \{0\}$. In particular, since a proper map of the unit disk into itself is a finite Blaschke product, we must have $\varphi(z) = e^{i\theta} z^k$ for some $\theta \in \mathbf{R}$ and $k \in \mathbf{N}^*$. Assume, by contradiction, that $k > 1$, and set $\eta = \theta/(1-k)$; remark that $\varphi(\rho e^{i\eta}) = \rho^k e^{i\eta}$. Set $r_\eta(\rho) = r(\rho e^{i\eta})$; then for all $\nu \in \mathbf{N}^*$ and for all $0 < \rho < 1$ we have

$$\begin{aligned} \frac{r_\eta(\rho^{k^\nu}) - r_\eta(0)}{\rho^{k^\nu}} &= \frac{r(\varphi(\rho^{k^{\nu-1}} e^{i\eta})) - r(0)}{\rho^{k^\nu}} \\ &\geq \frac{r(\rho^{k^{\nu-1}} e^{i\eta}) - r(0)}{\rho^{k^{\nu-1}}} \geq \cdots \geq \frac{r(\rho e^{i\eta}) - r(0)}{\rho} > 0. \end{aligned}$$

Therefore

$$r'_\eta(0) = \lim_{\nu \to \infty} \frac{r_\eta(\rho^{k^\nu}) - r_\eta(0)}{\rho^{k^\nu}} \geq \frac{r(\rho e^{i\eta}) - r(0)}{\rho} > 0.$$

But 0 is a minimum point for r_η, contradiction. In conclusion, $k = 1$, and φ is an automorphism. □

Coming back to the general situation, let us denote by $\mathrm{Aut}_+(A)$ ($\mathrm{Aut}_-(A)$) the set of all automorphisms of degree $+1$ (-1) of the annular bundle A. There is a natural inclusion $\varepsilon \colon \mathbf{S}^1 \to \mathrm{Aut}_+(A)$, where $\varepsilon(e^{i\theta})$ is locally defined by $\varepsilon(e^{i\theta})(z, w) = (z, e^{i\theta} w)$. These are the only autoequivalences of A:

Proposition 3.9. *Let $f \colon A \to A$ be an equivalence. Then $f = \varepsilon(e^{i\theta})$ for some $\theta \in \mathbf{R}$.*

Proof. Since f restricted to a generic fiber A_z is an automorphism of A_z of degree 1, locally f must have the form

$$f(z, w) = (z, e^{i\theta(z)} w),$$

for some $\theta(z) \in \mathbf{R}$. Using the compatibility conditions and the holomorphy of f, we see at once that $e^{i\theta(z)}$ does not depend on z. □

In particular, an automorphism is essentially determined by the action on the base:

COROLLARY 3.10. *Let $f, \tilde{f}\colon A \to A$ be automorphisms of an annular bundle A of the same degree, inducing the same automorphisms on the base. Then $\tilde{f} = \varepsilon(e^{i\theta})f$ for some $\theta \in \mathbf{R}$.*

Proof. Since f and $\tilde{f}$ have the same degree, $\tilde{f}^{-1} \circ f$ is an equivalence (Proposition 3.7), and the assertion follows by Proposition 3.9. □

Conversely, given an automorphism φ of the base, we would like to know if there exists an automorphism of the bundle inducing it:

PROPOSITION 3.11. *Let (A, p, B) be an annular bundle, and $\varphi \in$ $\mathrm{Aut}(B)$. Then*

(i) *φ is induced by an element of* $\mathrm{Aut}_+(A)$ *iff φ^*A is equivalent to A;*

(ii) *φ is induced by an element of* $\mathrm{Aut}_-(A)$ *iff φ^*A is equivalent to A^*.*

Proof. (i) Since φ is an automorphism, $i_\varphi\colon \varphi^*A \to A$ is an isometric isomorphism. Therefore, if $f'\colon A \to \varphi^*A$ is an equivalence, $f = i_\varphi \circ f'$ is an element of $\mathrm{Aut}_+(A)$ inducing φ on B.

Conversely, if $f \in \mathrm{Aut}_+(A)$ induces φ on B, then $f^{-1} \circ i_\varphi$ is an isometric isomorphism between φ^*A and A inducing the identity on B, that is an equivalence.

(ii) If $f \in \mathrm{Aut}_-(A)$ induces φ on B, then $*(f^{-1} \circ i_\varphi)$ is an equivalence between φ^*A and A^*. Conversely, if $f'\colon A^* \to \varphi^*A$ is an equivalence, then $i_\varphi \circ f' \circ *$ is an element of $\mathrm{Aut}_-(A)$ inducing φ on B. □

Using the theory of the previous section, it is possible to state Proposition 3.11 in a more expressive fashion:

COROLLARY 3.12. *Let (A, p, B) be an annular bundle with annular data r and ω, and assume $H_1(B, \mathbf{Z}) = (0)$. Let φ be an automorphism of B. Then*

(i) *φ is induced by an element of* $\mathrm{Aut}_+(A)$ *iff $\varphi^*r = r$ and $\varphi^*\omega = \omega$;*

(ii) *φ is induced by an element of* $\mathrm{Aut}_-(A)$ *iff $\varphi^*r = r$ and $\varphi^*\omega = \omega^*$.*

Proof. One direction is contained in Proposition 3.7. Conversely, if $\varphi^* r = r$ and $\varphi^*\omega = \omega$ ($\varphi^*\omega = \omega^*$), then $\varphi^* A$ has the same modular data as A (A^*); therefore, by Theorem 2.7(ii), $\varphi^* A$ is equivalent to A (A^*). □

Now we are able to answer the question raised in Example 1.2, about the equivalence of A and A^*. We need a last definition: we shall say that an annular bundle A is *involutive* if $E_A \otimes E_A$ is trivial. Then we have the following

THEOREM 3.13. *Let (A, p, B) be an annular bundle. Then the following conditions are equivalent*:

(i) *A is self-dual involutive*;
(ii) *A is equivalent to A^**;
(iii) id_B *is induced by an element of* $\mathrm{Aut}_-(A)$.

Proof. (i) ⇒ (ii). Let A be an involutive self-dual bundle. Then we know that E_A is represented in $H^1(B, \mathscr{O}^*)$ by an element of the form $i^*(\mathbf{e})$, where $\mathbf{e} \in H^1(B, \mathbf{S}^1)$. Since A is involutive, $i^*(\mathbf{e}^2) = 1$ in $H^1(B, \mathscr{O}^*)$. We claim that we may represent E_A by an element of the form $i^*(\tilde{\mathbf{e}})$, where $\tilde{\mathbf{e}} \in H^1(B, \mathbf{S}^1)$ is such that $\tilde{\mathbf{e}}^2 = 1$ in $H^1(B, \mathbf{S}^1)$. In fact, $i^*(\mathbf{e}^2) = 1$ means that $\mathbf{e}^2 = \partial^*(\rho)$ for some $\rho \in \mathscr{L}^+(B)$, where $\partial^*\colon \mathscr{L}^+(B) \to H^1(B, \mathbf{S}^1)$ is the connecting homomorphism of Corollary 2.2. Set $\tilde{\mathbf{e}} = \mathbf{e}\partial^*(\rho^{-1/2})$; remark that $\rho^{-1/2} \in \mathscr{L}^+(B)$. Then $i^*(\tilde{\mathbf{e}}) = i^*(\mathbf{e})$ and $\tilde{\mathbf{e}}^2 = \mathbf{e}^2\partial^*(\rho^{-1}) = 1$, as claimed.

So, up to equivalence, we may assume that there exists a trivializing cover $\{U_\alpha\}$ of B such that E_A is represented by a 1-cocycle $\{e_{\alpha\beta}\} \in Z^1(\{U_\alpha\}, \mathbf{S}^1)$ with $(e_{\alpha\beta})^2 = e_\alpha / e_\beta$ for suitable $e_\alpha \in \mathbf{S}^1$. Then

$$e_{\alpha\beta} e_\beta = e_\alpha \overline{e_{\alpha\beta}} = e_\alpha (e_{\alpha\beta})^{-1},$$

that is E_A and E_A^* are equivalent, and it is immediate that the equivalence $f\colon E_A \to E_A^*$ defined locally by $f(z, w) = (z, e_\alpha w)$ restricts to an equivalence of A and A^*.

(ii) ⇒ (iii). Proposition 3.11(ii).

(iii) ⇒ (i). Let $f\colon A \to A$ be an automorphism of degree -1 inducing the identity on B. Then, by Proposition 3.7, $\omega = \omega^*$ and (Proposition 2.9) A is self-dual. Let $\{U_\alpha\}$ be a trivializing cover of B, r the modular function of A and $\{g_{\alpha\beta}\}$ the 1-cocycle associated to E_A. Locally, f may be expressed as

$$f(z, w) = (z, f_\alpha(z) w^{-1}) \quad \text{on } p^{-1}(U_\alpha),$$

where $f_\alpha \colon U_\alpha \to \mathbf{C}^*$ is holomorphic. Since A is self-dual, the f_α must satisfy the $|f_\alpha| r^{1/2} = r^{1/2}$, that is $f_\alpha \equiv e^{i\theta_\alpha} \in \mathbf{S}^1$. Now, on $U_\alpha \cap U_\beta$ we must have

$$(g_{\alpha\beta})^{-1} e^{i\theta_\alpha} = e^{i\theta_\beta} g_{\alpha\beta},$$

that is $(g_{\alpha\beta})^2 = e^{i\theta_\alpha}/e^{i\theta_\beta}$. This means that $\{(g_{\alpha\beta})^2\}$ is a 1-coboundary, and therefore A is involutive. □

For instance, the bundle of Example 1.3 is involutive. We may compute its automorphism group:

EXAMPLE 3.3. Let $(A, p, \mathbf{B}^n)$ be the annular bundle of Example 1.3, as usual. A is involutive, since it is trivial; an automorphism of degree -1 inducing the identity on $\mathbf{B}^n$ is

$$j(z, w) = (z, w^{-1}).$$

So we have to compute only $\mathrm{Aut}_+(A)$, since $\mathrm{Aut}_-(A) = j \cdot \mathrm{Aut}_+(A)$. Now, if $f \in \mathrm{Aut}_+(A)$ induces $\varphi \in \mathrm{Aut}(\mathbf{B}^n)$, by Example 3.1 the origin is a fixed point of φ, that is $\varphi(z) = Uz$, for some $U \in \mathbf{U}(n)$. Conversely, if $U \in \mathbf{U}(n)$, then $f_U \colon A \to A$ given by $f_U(z, w) = (Uz, w)$ is an element of $\mathrm{Aut}_+(A)$. Hence, by Corollary 3.10, every automorphism of A is of the form

$$f(z, w) = (Uz, e^{i\theta} w^{\pm 1}),$$

for some $U \in \mathbf{U}(n)$ and $\theta \in \mathbf{R}$.

We shall conclude this paper showing that, in some particular case, every automorphism of the total space of an annular bundle is a fiber map:

PROPOSITION 3.14. *Let B be a compact Riemann surface of genus greater than* 1, *and let*

$$A = \{(z, w) \in B \times \mathbf{C} \mid \rho_1(z) < |w| < \rho_2(z)\}$$

be a trivial bundle over B, where $\rho_1, \rho_2 \colon B \to \mathbf{R}^+$ are smooth functions. Assume that there are $r_1, r_2 \in \mathbf{R}^+$ such that $\rho_1 \leq r_1 < r_2 \leq \rho_2$ on B. Then every holomorphic injective map $f \colon A \to A$ is a fiber map.

Proof. Let us write $f(z, w) = (\varphi_w(z), \psi_w(z))$. Put $\Omega = \{w \in \mathbf{C} \mid r_1 < |w| < r_2\}$; then $B \times \Omega \subset A$. For fixed $w \in \Omega$, ψ_w is a holomorphic function on B, and hence it is constant. So $\partial\psi/\partial z \equiv 0$ on $B \times \Omega$, and therefore on A. Hence $f(z, w) = (\varphi_w(z), \psi(w))$.

For fixed $w \in \Omega$, $\varphi_w \colon B \to B$ is holomorphic. Since f is one-to-one and its second component does not depend on z, φ_w must be one-to-one. Therefore it is open and, since B is compact, onto. In other words, φ_w is an automorphism of B depending continuously on w. But, since the genus of B is greater than 1, $\mathrm{Aut}(B)$ is discrete, and so φ_w does not depend on w.

This means that $\partial\varphi/\partial w \equiv 0$ on $B \times \Omega$, and therefore on A. In conclusion, $f(z, w) = (\varphi(z), \psi(w))$ is a fiber map. □

If B is the torus $\mathbf{T}^2$, that is if the genus of B is 1, Proposition 3.14 does not hold. Indeed, we have:

PROPOSITION 3.15. *Let* $A = \{(z, w) \in \mathbf{T}^2 \times \mathbf{C} \mid r_1 < |w| < r_2\}$ *be a trivial bundle over* $\mathbf{T}^2$, *with* $r_1, r_2 \in \mathbf{R}^+, r_1 < r_2$. *Then every automorphism of the manifold* A *is of one of the following two forms*:

$$f(z, w) = (\tau_{\eta(w)}(z), e^{i\theta} w),$$
$$f(z, w) = (\tau_{\eta(w)}(z), \sqrt{r_1 r_2}\, e^{i\theta} w^{-1}),$$

where $\theta \in \mathbf{R}$, $\eta \colon \Omega \to \mathbf{T}^2$ *is holomorphic (and* $\Omega = \{w \in \mathbf{C} \mid r_1 < |w| < r_2\}$*), and for all* $z \in \mathbf{T}^2$ $\tau_z \in \mathrm{Aut}(\mathbf{T}^2)$ *is the right translation.*

Proof. Arguing as in the proof of Proposition 3.14, we find that f is of the form

$$f(z, w) = (\varphi_w(z), \psi(w)),$$

where φ_w is an automorphism of $\mathbf{T}^2$. Now, the connected component at the identity of $\mathrm{Aut}(\mathbf{T}^2)$ is isomorphic to $\mathbf{T}^2$, and the isomorphism associates each $z \in \mathbf{T}^2$ to the right translation τ_z. Therefore there exists $\eta \colon \Omega \to \mathbf{T}^2$ holomorphic such that $\varphi_w = \tau_{\eta(w)}$.

Finally, ψ is a map from Ω into Ω and, since it does not depend on z, it must be onto. Therefore, since a surjective map of a plane annulus into itself is an automorphism, $\psi \in \mathrm{Aut}(\Omega)$, and we are done. □

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