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PROPAGATION OF HYPO-ANALYTICITY ALONG BICHARACTERISTICS

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It is shown here the hypo-analytic singularities for solutions propagate along the bicharacteristics of hypo-analytic differential operators of principal type. This generalizes the well-known similar result for analytic differential operators.

0. Introduction. In [4] Hormander proved a result concerning the propagation of C^∞ singularities of solutions of $Pu = f$ for a smooth linear partial differential operator P whose leading symbol is real. The analytic version of this question was treated by Hanges in [3]. In this paper we prove a similar theorem for what we call hypo-analytic differential operators. The paper is organized as follows. In §1 we discuss the structures we work in and introduce our operators. In §2 we recall the definition of microlocal hypo-analyticity and give a statement of the main result. §3 discusses the Fourier transform criterion of microlocal hypo-analyticity due to Baouendi, Chang and Treves [1]. A theorem concerning this criterion is proved in the same section and then used in the proof of our main result.

1. Hypo-analytic differential operators. Our results deal with structures which are a special case of the hypo-analytic structures introduced in [1]. Let Ω be a C^∞ manifold of dimension m. A hypo-analytic structure of maximal dimension on Ω is the data of an open covering (U_α) of Ω and for each index α, of $m\, C^\infty$ functions $Z_\alpha^1, \dots, Z_\alpha^m$ satisfying the following two conditions:

(i) $dZ_\alpha^1, \dots, dZ_\alpha^m$ are linearly independent at each point of U_α;

(ii) if $U_\alpha \cap U_\beta \neq \emptyset$, there are open neighbors O_α of $Z_\alpha(U_\alpha \cap U_\beta)$ and O_β of $Z_\beta(U_\alpha \cap U_\beta)$ and a holomorphic map F_β^α of O_α onto O_β, such that

$$Z_\beta = F_\beta^\alpha \circ Z_\alpha \quad \text{on } U_\alpha \cap U_\beta.$$

We will use the notation $Z_\alpha = (Z_\alpha^1, \dots, Z_\alpha^m)\colon U_\alpha \to C^m$. A distribution h defined on an open neighborhood of a point p_0 of Ω is called hypo-analytic at p_0 if there is a local chart (U_α, Z_α) of the above type whose domain contains p_0 and a holomorphic function $\tilde{h}$ defined on an open neighborhood of $Z_\alpha(p_0)$ in C^m such that $h = \tilde{h} \circ Z_\alpha$ in a neighborhood of p_0. By a hypo-analytic local chart we mean an $m+1$-tuple $(U, Z^1, \dots, Z^m)$ [abbreviated (U, Z)] consisting of an open subset U of Ω and of m hypo-analytic functions $Z^1, \dots, Z^m$ whose differentials are linearly independent at every point of U. We note that when the Z^j are real valued these structures specialize to the real analytic ones. We introduce the following differential operators on Ω which are naturally associated with its hypo-analytic structure.

DEFINITION 1.1. A linear differential operator P on Ω is called a hypo-analytic differential operator if for every open set U and every hypo-analytic function f on U, Pf is hypo-analytic on U.

The following example shows that Definition 1.1 extends the standard definition in the analytic category.

EXAMPLE. Let Ω' be a real-analytic manifold. The real-analytic structure of Ω' can be viewed as a hypo-analytic structure. A differential operator $P\colon C^\infty(\Omega') \to C^\infty(\Omega')$ is said to be analytic if, in terms of local analytic coordinates on Ω', P is given by

$$P = \sum_{|\alpha| \le m_0} a_\alpha(x) D^\alpha$$

where the a_α are real-analytic and m_0 is some integer. This is the case if and only if P is a hypo-analytic differential operator in the sense of Definition 1.1.

Suppose now Ω is as before and P is a hypo-analytic differential operator on Ω. Let (U, Z) be a hypo-analytic local chart and assume U is a coordinate neighborhood. Choose vector fields $M_1, \dots, M_m$ satisfying $M_l Z^k = \delta_l^k$. Since $\{M_1, \dots, M_m\}$ is a linearly independent set, there are smooth functions a_α defined on U such that on $U, P = \sum_{|\alpha| \le m_0} a_\alpha M^\alpha$ for some integer m_0. For $|\alpha| \le m_0$, $PZ^\alpha = a_\alpha$ is hypo-analytic. Conversely, it is easy to see that if in every hypo-analytic local chart (U, Z) P can be written in the above form then P is a hypo-analytic differential operator.

2. Statement of the main result. We continue to work in a hypo-analytic local chart (U, Z) of the maximal structure Ω. Before proceeding to the statement of our theorem, we need to recall Sato's microlocalization as adopted in **[1]**. In the sequel Γ is a nonempty, acute and open cone in $R_m \backslash \{0\}$. For A an open subset of U, we shall use the notation:

$$N_\delta(A, \Gamma) = \{Z(x) + \sqrt{-1} Z_x(x) v \colon x \in A,\ v \in \Gamma,\ |v| < \delta\}.$$

DEFINITION 2.1. We denote by $B_\delta(A,\Gamma)$ the space of holomorphic functions f in $N_\delta(A,\Gamma)$ satisfying:

To every compact subset K of $N_\delta(A,\Gamma)$ there are an integer $k \geq 0$ and a constant $c > 0$ such that $|f(z)| \leq c(\mathrm{dist}[z, Z(A)])^{-k}$ for all z in K.

In [**1**] it is shown that if A is small enough and $f \in B_\delta(A,\Gamma)$ then for every $\psi \in C_c^\infty(A)$,

$$\lim_{t\to+0} \int_A f(Z(x) + \sqrt{-1} Z_x(x) t v)\psi(x)\, dZ(x)$$

exists and is independent of $v \in \Gamma$. We will denote the distribution we get in the limit by bf.

DEFINITION 2.2. Let $u \in D'(U)$ and (x,ξ) be a point in $U \times (R_m \backslash \{0\})$. We say that u is hypo-analytic at (x,ξ) if there are an open neighborhood $A \subseteq U$ of x, $\delta > 0$ and a finite collection of nonempty acute open cones Γ_k in $R_m\backslash\{0\}$ $(k = 1, \ldots, r)$ such that the following hold: for every k and every $v \in \Gamma_k$, $\xi \cdot v < 0$; for each k there is $f_k \in B_\delta(A, \Gamma_k)$ such that in A

$$u = bf_1 + \cdots + bf_k.$$

We remark that the above definition of microlocal hypo-analyticity does not depend on the choice of the chart (U, Z) (see [**1**]).

DEFINITION 2.3. Let $u \in D'(\Omega)$. The hypo-analytic wavefront set of the distribution u is denoted by $WF_{\mathrm{ha}}u$ and is defined as

$$WF_{\mathrm{ha}}u = \{(x,\xi) \in T^*\Omega \colon u \text{ is not hypo-analytic at } (x,\xi)\}$$

We now describe the curves in $T^*\Omega$ along which hypo-analytic singularities propagate. Let P_0 be a hypo-analytic differential operator defined near a point p_0 of Ω.

The previous section tells us that in a hypo-analytic local chart (U, Z) P takes the form $P = \sum_{|\alpha|\leq k} a_\alpha(x) M^\alpha$ where each $a_\alpha(x)$ is a hypo-analytic function on U.

After shrinking U if necessary, we may assume that U is the domain of local coordinates $x_1, \ldots, x_m$. Let $t \to (x(t), \xi(t)) = \gamma(t)$ be a curve in $T^*U\backslash 0$ and set

$$\tilde\gamma(t) = (\tilde x(t), \tilde\xi(t)) = (Z(x(t)), Z_x^*(x(t))\xi(t))$$

where Z_x^* denotes the transpose of the inverse of the Jacobian matrix Z_x. We will use the notation $p(z,\xi) = \sum_{|\alpha|=k} \tilde a_\alpha(z)\xi^\alpha$ where each $a_\alpha(x) = \tilde a_\alpha(Z(x))$ for holomorphic $\tilde a_\alpha$.

DEFINITION 2.4. The curve $\gamma(t)$ is said to be a bicharacteristic for P if the equations

$$\frac{d\tilde{x}}{dt} = \frac{\partial p}{\partial \zeta}(\tilde{x}(t), \tilde{\xi}(t)),$$
$$\frac{d\tilde{\xi}}{dt} = -\frac{\partial p}{\partial z}(\tilde{x}(t), \tilde{\xi}(t))$$

hold.

We can now state the main theorem of this paper.

THEOREM 2.1. *Assume $p(0,\xi^0) = 0$ and P is of principal type at $(0,\xi^0)$. Suppose $\gamma = \{(x(t),\xi(t))\}$ is a bicharacteristic for P through $(x(0),\xi(0)) = (0,\xi^0)$ and that Pu is hypo-analytic on γ. Then either u is hypo-analytic at every point of γ or u is not hypo-analytic at any point of γ.*

3. On the Fourier transform criterion of hypo-analyticity. We will work in a hypo-analytic local chart (U, Z) in Ω. We shall assume that the open set U has been contracted so that the map $Z = (Z^1, \dots, Z^m)$: $U \to C^m$ is a diffeomorphism of U onto $Z(U)$ and that U is the domain of local coordinates x_j $(1 \le j \le m)$ all vanishing at a "central point" which will be denoted by O. We will suppose $Z(o) = o$ and denote by Z_x the Jacobian matrix of the Z^j with respect to the x_k. Substitution of $Z_x(o)^{-1}Z(x)$ for $Z(x)$ will allow us to assume that $Z_x(o) =$ the identity matrix. Since the differential of $\operatorname{Im} Z_x$ is zero at the origin, after shrinking U if necessary, we can find a number K, $0 < K < 1$, such that for all x, y in U and for all ξ in R_m.

$$|\operatorname{Im} Z_x^*(x)\xi| \le K|\operatorname{Re} Z_x^*(x)\xi| \quad \text{and} \tag{3.1}$$

$$\begin{aligned} \operatorname{Re}(\sqrt{-1}Z_x^*(x)\xi \cdot (Z(x) - Z(y)) - \langle Z_x^*(x)\xi\rangle(Z(x) - Z(y))^2) \\ \le -K|\xi|\,|Z(x) - Z(y)|^2 \end{aligned} \tag{3.2}$$

where $\langle\zeta\rangle^2 = \zeta_1^2 + \cdots + \zeta_m^2$ for $\zeta \in C_m$, $|\operatorname{Re}\zeta| < |\operatorname{Im}\zeta|$.

Let u be a compactly supported distribution in U. We shall refer to

$$F(u,z,\zeta) = \int_U \exp(\sqrt{-1}\zeta(z - Z(y)) - \langle\zeta\rangle(z - Z(y))^2)u(y)\,dZ(y)$$

as the Fourier-Bros-Iagolnitzer (in short, FBI) transform of u (see **[1]**). Here $z \in C^m$, $\zeta \in C_m$ with $|\operatorname{Im}\zeta| < |\operatorname{Re}\zeta|$.

In **[1]** the authors established a connection between the concept of the FBI transform and the notion of hypo-analytic wave front set.

Their theorems that showed the equivalence between exponential decay in the FBI transform of u and microlocal hypo-analyticity may be consolidated into the following:

THEOREM 3.1. *The following two properties are equivalent:*

(i) *u is hypo-analytic at $(0, \xi^0) \in T^*U\backslash\{0\}$.*

(ii) *There is an open neighborhood V of 0 in C^m, a conic open neighborhood $\mathscr{C}_0$ of ξ^0 in C_m and constants $c, r > 0$ such that $|F(u, z, \zeta)| \leq ce^{-r|\zeta|}$ for all z in V and for all ζ in $\mathscr{C}_0$.*

We emphasize here that in **[1]** the assumption that $\operatorname{Im} Z_x(0) = 0$ was exploited in proving the sufficiency part, that is, the implication (ii)⇒(i) of the above theorem. We don't know whether (ii) implies (i) at a point $(x, \xi) \in T^*U\backslash\{0\}$ without the additional hypothesis that $\operatorname{Im} Z_x(x) = 0$.

In fact Theorem 3.1 does not give us any information regarding the points $(x, \xi) \in T^*U\backslash\{0\}$ unless $\operatorname{Im} Z_x(x) = 0$. Here we would like to show that such an assumption is not needed for the necessity part of Theorem 3.1. More precisely, we have:

THEOREM 3.2. *Suppose the distribution u of compact support is hypo-analytic at $(x_0, \xi^0) \in T^*U\backslash\{0\}$. Then there is an open neighborhood V of $Z(x_0)$ in C^m, a conic open neighborhood $\mathscr{C}_0$ of $Z_x^*(x_0)\xi^0$ in C_m and constants $c, r > 0$ such that $|F(u, z, \zeta)| \leq ce^{-r|\zeta|}$ for all z in V and for all ζ in $\mathscr{C}_0$.*

Proof. According to the definition, it suffices to prove the result when u is the boundary value of a holomorphic function f of tempered growth defined in a set of the form

$$\{Z(x) + \sqrt{-1}Z_x(x)v \colon x + \sqrt{-1}v \in (W + \sqrt{-1}\Gamma),\ |v| < \delta_0\}$$

where $W \subset U$ is an open neighborhood of x_0, δ_0 a positive number and Γ an acute open cone in $R_m\backslash\{0\}$ such that for very $v \in \Gamma$, $\xi^0 \cdot v < 0$. Thus for $\varphi \in C_c^\infty(W)$,

$$\langle u, \varphi \rangle = \lim_{t \to +0} \int f(Z(x) + t\sqrt{-1}Z_x(x)v)\varphi(x)\, dZ(x), \qquad v \in \Gamma.$$

After contracting Γ if necessary, we may assume that there is a number $c_0 > 0$ such that $\xi^0 \cdot v \leq -c_0|v|\,|\xi^0|$ whenever v is in Γ. We shall need the following lemma.

LEMMA 3.1. *Suppose $u \in E'(U)$ vanishes in an open neighborhood of $x_0 \in U$. Then there is an open neighborhood V of $Z(x_0)$ in C^m, a conic neighborhood $\mathscr{C}$ of $\{Z_x^*(x_0)\xi : \xi \in R_m \backslash \{0\}\}$ in C_m and constants c, r such that $|F(u,z,\zeta)| \le ce^{-r|\zeta|}$ for all z in V and for all ζ in $\mathscr{C}$.*

Proof of Lemma 3.1. For $z \in C^m$ and $\zeta \in C_m$, $|\operatorname{Im}\zeta| < |\operatorname{Re}\zeta|$ we consider the FBI

$$F(u,z,\zeta) = \int \exp(\sqrt{-1}\zeta \cdot (z - Z(y)) - \langle\zeta\rangle(z - Z(y))^2) u(y)\, dZ(y).$$

Let

$$Q(z,\zeta,y) = \operatorname{Re}\left\{\sqrt{-1}\frac{\zeta}{|\zeta|}\cdot(z - Z(y)) - \frac{\langle\zeta\rangle}{|\zeta|}(z - Z(y))^2\right\}.$$

We first freeze z to $Z(x_0)$ and ζ to $Z_x^*(x_0)\cdot\xi^0$ for some $\xi^0 \in R_m$, $|\xi^0| = 1$.

$$Q(Z(x_0), Z_x^*(x_0)\xi^0, y) = \operatorname{Re}\left\{\sqrt{-1}\frac{Z_x^*(x_0)\xi^0}{|Z_x^*(x_0)\xi^0|}(Z(x_0) - Z(y)) - \frac{\langle Z_x^*(x_0)\xi^0\rangle}{|Z_x^*(x_0)\xi^0|}(Z(x_0) - Z(y))^2\right\}.$$

Condition (3.2) tells us that

$$Q(Z(x_0), Z_x^*(x_0)\xi^0, y) \le -K|Z(x_0) - Z(y)|^2.$$

Suppose d is a positive number such that $|y - x_0| \ge d$ whenever $y \in \operatorname{Supp} u$.

Then in the support of u, $Q(Z(x_0), Z_x^*(x_0)\xi^0, y) \le -Kd^2$. By continuity, there are open neighborhoods $\tilde{V}$ of $Z(x_0)$ in C^m and $\tilde{\mathscr{C}}$ of $Z_x^*(x_0)\xi^0$ in C_m such that

$$Q(z,\zeta,y) \le -\frac{Kd^2}{2} \quad \text{for all } z \text{ in } \tilde{V},\ \zeta$$

in $\tilde{\mathscr{C}}$. By compactness of the unit sphere in R_m, we may assume that the open set $\tilde{\mathscr{C}}$ contains the set $\{Z_x^*(x_0)\xi : \xi \in R_m,\ |\xi| = 1\}$.

Moreover, the homogeneity of Q implies that there is a conic neighborhood $\mathscr{C}$ of $\{Z_x^*(x_0)\xi : \xi \in R_m \backslash \{0\}\}$ in C_m such that

$$\operatorname{Re}\{\sqrt{-1}\zeta \cdot (z - Z(y)) - \langle\zeta\rangle(z - Z(y))^2\} \le -\frac{Kd^2}{2}|\zeta|$$

whenever z is in $\tilde{V}$ and ζ is in $\mathscr{C}$. This gives us the required decay of $F(u,z,\zeta)$.

End of proof of Theorem 3.2. Let $g \in C_c^\infty(W)$, $g \equiv 1$ near x_0. Since $(1-g)u$ vanishes near x_0, by Lemma 3.1 we know that $F((1-g)u, z, \zeta)$ decays exponentially in the sets of interest. Therefore, it suffices to show a similar decay for $F(gu, z, \zeta)$. Let $\chi \in C_c^\infty(W)$, $\chi \equiv 1$ near x_0 and $\operatorname{supp}\chi \subset \{x\colon g(x) \equiv 1\}$. Fix $v \in \Gamma$, $|v| = 1$. When s is a suitably small positive number, we can deform the contour of integration in $F(gu, z, \zeta)$ under the mapping:

$$Z(y) \to \tilde{Z}(y) = Z(y) + \sqrt{-1}sZ_y(y)\chi(y)v.$$

Thus

$$\begin{aligned} F(gu, z, \zeta) &= \int_U \exp(\sqrt{-1}\zeta\cdot(z-\tilde{Z}(y)) - \langle\zeta\rangle(z-\tilde{Z}(y))^2)f(\tilde{Z}(y)) \\ &\qquad \cdot g(y)\,d\tilde{Z}(y). \end{aligned}$$

We focus on the quantity

$$Q(z,\zeta,y,s) = \operatorname{Re}\left\{\sqrt{-1}\frac{\zeta}{|\zeta|}\cdot(z-\tilde{Z}(y)) - \frac{\langle\zeta\rangle}{|\zeta|}(z-\tilde{Z}(y))^2\right\}$$

and write it as $Q = Q_1 + Q_2$ where

$$Q_1(z,\zeta,y) = \operatorname{Re}\left\{\sqrt{-1}\frac{\zeta}{|\zeta|}\cdot(z-Z(y)) - \frac{\langle\zeta\rangle}{|\zeta|}\cdot(z-Z(y))^2\right\}$$

and

$$\begin{aligned} &Q_2(z,\zeta,y,s) \\ &= \operatorname{Re}\left\{\frac{\zeta}{|\zeta|}\cdot(sZ_y(y)\chi(y)v)\right. \\ &\qquad \left. + \frac{\langle\zeta\rangle}{|\zeta|}[2\sqrt{-1}s(z-Z(y))\cdot(\chi(y)Z_y(y)v) + s^2|\chi(y)Z_y(y)v|^2]\right\}. \end{aligned}$$

We first consider these quantities when $z = Z(x_0)$, $\zeta = Z_x^*(x_0)\cdot\xi^0$ and y varies in the support of g. From (3.2) we have:

$$Q_1(Z(x_0), Z_x^*(x_0)\xi^0, y, s) \le -\kappa|Z(x_0) - Z(y)|^2.$$

To estimate $Q_2(Z(x_0), Z_x^*(x_0)\xi^0, y, s)$, we note that for s sufficiently small, say $0 < s \le s_0$:

$$\begin{aligned} &Q_2(Z(x_0), Z^*(x_0)\xi^0, x_0, s) \\ &= \operatorname{Re}\left\{\frac{s(\xi^0\cdot v)}{|Z_x^*(x_0)\xi^0|} + \frac{\langle Z_x^*(x_0)\xi^0\rangle}{|Z_x^*(x_0)\xi^0|}s^2|Z_x(x_0)v|^2\right\} \le -sc_0/4. \end{aligned}$$

Therefore, by continuity we can find a number $d > 0$ satisfying:

$$|y - x_0| \leq d \Rightarrow Q_2(Z(x_0), Z_x^*(x_0)\xi^0, y, s) \leq -sc_0/4.$$

We may assume that $\chi(y) = 1$ whenever $|y - x_0| \leq d$. On the other hand, whatever y,

$$Q_2(Z(x_0), Z_x^*(x_0)\xi^0, y, s) \leq 4s\chi(y)(|Z(x_0) - Z(y)| + s).$$

Hence when $|y - x_0| \leq d$,

$$Q(Z(x_0), Z_x^*(x_0)\xi^0, y, s) \leq -\kappa|Z(x_0) - Z(y)|^2 - sc_0/2,$$

while when $|y - x_0| \geq d$ then $|Z(y) - Z(x_0)| \geq d$ so that

$$\begin{aligned} &Q(Z(x_0), Z_x^*(x_0)\xi^0, y, s) \\ &\quad \leq -\kappa d|Z(x_0) - Z(y)| + 4s\chi(y)(|Z(x_0) - Z(y)| + s). \end{aligned}$$

Therefore, by choosing s small in comparison with d, we get a positive number δ such that

$$Q(Z(x_0), Z_x^*(x_0)\xi^0, y, s) \leq -\delta, \qquad y \in \operatorname{Supp} g.$$

By continuity, there are open neighborhoods $\tilde{V}$ of $Z(x_0)$ in C^m and $\tilde{\mathscr{C}}$ of $Z_x^*(x_0)\xi^0$ in C_m such that $Q(z, \zeta, y, s) \leq -\delta/2$ for all $z \in \tilde{V}$, $\zeta \in \tilde{\mathscr{C}}$ and for all $y \in \operatorname{Supp} g$. Now Q is positive homogeneous of degree 0 in ζ. Therefore, there is an open conic neighborhood $\mathscr{C}$ of $Z_x^*(x_0)\xi^0$ in C_m such that

$$\operatorname{Re}\left\{\sqrt{-1}\zeta \cdot (z - \tilde{Z}(y)) - \langle\zeta\rangle(z - \tilde{Z}(y))^2\right\} \leq -\frac{\delta}{2}|\zeta|$$

whenever z is in $\tilde{V}$ and ζ in $\mathscr{C}$. From this, we get the required decay of $F(u, z, \zeta)$.

Proof of Theorem 2.1. For $x, y, \zeta \in C^m$, let

$$\psi(x, y, \zeta) = (y - x)\frac{\zeta}{|\zeta|} + \frac{\sqrt{-1}}{2}\frac{\langle\zeta\rangle}{|\zeta|}(y - x)^2.$$

The Cauchy-Kovalevska Theorem implies that there exists a holomorphic function $\phi = \phi(t, x, y, \zeta)$ such that

$$\begin{aligned} &\frac{\partial\phi}{\partial t}(t, x, y, \zeta) = p(x, -\phi_x(t, x, y, \zeta)), \\ &\phi(0, x, y, \zeta) = \psi(x, y, \zeta). \end{aligned} \tag{3.3}$$

Since $\gamma(t)$ is a bicharacteristic for P, we have:

(i) $\phi(t, \tilde{x}(t), 0, \xi^0) = 0$,

(ii) $\phi_x(t, \tilde{x}(t), 0, \xi^0) = -\tilde{\xi}(t)$,

(iii) $\operatorname{Im}\phi_{xx}(t,\tilde{x}(t),0,\xi^0) > 0$.

Now (iii) clearly holds when $t = 0$. Therefore, this will persist for t small enough. To see (ii), we note that

$$\frac{d}{dt}(\phi_x(t,\tilde{x}(t),0,\xi^0)) = \phi_{xt} + \phi_{xx}\frac{d\tilde{x}}{dt}$$

and (3.3) implies that $\phi_{tx} = p_x(x,-\phi_x) - p_\zeta(x,-\phi_x)\phi_{xx}$. It follows that

$$\begin{aligned}\frac{d}{dt}&(-\phi_x(t,\tilde{x}(t),0,\xi^0))\\ &= -p_x(\tilde{x}(t),-\phi_x(t,\tilde{x}(t),0,\xi^0))\\ &\quad + p_\xi(\tilde{x}(t),-\phi_x(t,\tilde{x}(t),0,\xi^0))\phi_{xx}(t,\tilde{x}(t),0,\xi^0)\\ &\quad - \phi_{xx}(t,\tilde{x}(t),0,\xi^0)\frac{d\tilde{x}}{dt}.\end{aligned}$$

$\tilde{\xi}(t)$ also satisfies the same differential equation and $\phi_x(0,0,0,\xi^0) = -\tilde{\xi}(0)$.

Hence $\phi_x(t,\tilde{x}(t),0,\xi^0) = -\tilde{\xi}(t)$.

To see (i), we observe that

$$\begin{aligned}\frac{d}{dt}&\phi(t,\tilde{x}(t),0,\xi^0)\\ &= \phi_t + \phi_x\frac{d\tilde{x}}{dt} = \phi_t - \tilde{\xi}(t)\frac{\partial p}{\partial\zeta} \quad \text{and this in turn by (3.3)}\\ &= p(\tilde{x}(t),\tilde{\xi}(t)) - \tilde{\xi}(t)\frac{\partial p}{\partial\zeta}(\tilde{x}(t),\tilde{\xi}(t))\\ &= p(\tilde{x}(t),\tilde{\xi}(t)) - kp(\tilde{x}(t),\tilde{\xi}(t)) = 0.\end{aligned}$$

Also $\phi(0,\tilde{x}(0),0,\xi^0) = 0$. Hence $\phi(t,\tilde{x}(t),0,\xi^0) = 0$.

Since P is of principal type, we can construct an analytic amplitude $a = \sum_{l=0}^{\infty} a_l\lambda^{-l}$ satisfying

$$\left(D_t + \frac{{}^tp(z,D_z)}{\lambda^{k-1}}\right)e^{i\lambda\phi(t,z,y,\zeta)}a(t,z,y,\zeta,\lambda) = 0 \tag{3.4}$$

approximately and $a(0,z,y,\zeta,\lambda) = 1$. The error in the approximation has an exponential decay. Indeed, substitution of $a = \sum_{l=0}^{\infty} a_l\lambda^{-l}$ in (3.4) gives the transport equations:

$$\begin{aligned}&La_0 = 0, && a_0|_{t=0} = 1,\\ &La_1 + f_1(a_0) = 0, && a_1|_{t=0} = 0,\\ &\quad\vdots\\ &La_n + f_n(a_0,\dots,a_{n-1}) = 0, && a_n|_{t=0} = 0,\\ &\quad\vdots\end{aligned}$$

where

$$L = \frac{\partial}{\partial t} + \sum_{j=1}^{m} \frac{\partial p^t}{\partial \zeta_j}(x, \phi_x)\frac{\partial}{\partial x_j} + s(x, y, \zeta)$$

with s holomorphic and $f_n(a_0, \dots, a_{n-1})$ a linear expression with holomorphic coefficients of the derivatives of $a_0, \dots, a_{n-1}$. These equations can be solved successively in a complex domain independent of n. The fact that a becomes an analytic amplitude is shown for example in **[5]**.

Suppose now $\gamma(t_0) = (x(t_0), \xi(t_0)) \notin Wf_{\text{ha}}u$ for some t_0. Then we claim that

(3.5)

$$F(t_0, z, \zeta, \lambda) = \int e^{\sqrt{-1}\lambda\phi(t_0, Z(x), z, \zeta)} a(t_0, Z(x), z, \zeta, \lambda) u(x)\, dZ(x)$$

decays exponentially (i.e. $|F(t_0, z, \zeta, \lambda)| \le c_1 e^{-c_2\lambda}$, $\lambda > 0$) for z near 0 and ζ near ξ^0.

To prove this assertion, we look at the Taylor's expansion of ϕ around the point $(t_0, \tilde{x}(t_0), 0, \xi^0)$.

For $x, z \in C^m$ $\zeta \in C_m$ and t_0 fixed, we have:

$$\begin{aligned}\phi(t_0, x, z, \zeta) = {} & \phi(t_0, \tilde{x}(t_0), 0, \xi^0) + \phi_x(t_0, \tilde{x}(t_0), 0, \xi^0)(x - \tilde{x}(t_0)) \\ & + \phi_z(t_0, \tilde{x}(t_0), 0, \xi^0) z + \phi_\zeta(t_0, \tilde{x}(t_0), 0, \xi^0)(\zeta - \xi^0) \\ & + \phi_{xx}(t_0, \tilde{x}(t_0), 0, \xi^0)(x - \tilde{x}(t_0))^2 + \cdots .\end{aligned}$$

Since $\phi(t_0, \tilde{x}(t_0), 0, \xi^0) = 0$ and $\phi_x(t_0, \tilde{x}(t_0), 0, \xi^0) = -\tilde{\xi}(t_0)$, we can write

$$\begin{aligned}\phi(t_0, x, z, \zeta) = {} & \tilde{\xi}(t_0) \cdot (\tilde{x}(t_0) - x) \\ & + \phi_{xx}(t_0, \tilde{x}(t_0), 0, \xi^0)(\tilde{x}(t_0) - x)^2 + R(t_0, x, z, \zeta)\end{aligned}$$

where $R(t_0, x, z, \zeta)$ is a sum of terms each of which has at least one of $\zeta - \xi^0, z$ or $(\tilde{x}(t_0) - x)^3$ as a factor. Thus we have:

$$\begin{aligned}\phi(t_0, Z(x), z, \zeta) = {} & Z_x^*(x(t_0))\xi(t_0) \cdot (Z(x(t_0)) - Z(x)) \\ & + \phi_{xx}(t_0, Z(x(t_0)), 0, \xi^0)(Z(x(t_0)) - Z(x))^2 \\ & + R(t_0, Z(x), z, \zeta).\end{aligned}$$

We recall that $\operatorname{Im} \phi_{xx}(t_0, Z(x(t_0)), 0, \xi^0) > 0$. Since $(x(t_0), \xi(t_0)) \notin WF_{\text{ha}}u$, Theorem 3.2 implies that

$$\begin{aligned}\int \exp(\sqrt{-1}\lambda[& Z_x^*(x(t_0))\xi(t_0) \cdot (Z(x(t_0)) - Z(x)) \\ & + \phi_{xx}(t_0, Z(x(t_0)), 0, \xi^0)(Z(x(t_0)) - Z(x))^2] \\ & \cdot a(t_0, Z(x), z, \zeta, \lambda) u(x)\, dZ(x)\end{aligned}$$

decays exponentially. In fact, the decay persists if we replace $Z_x^*(x(t_0))\xi(t_0)$ by ζ and $Z(x(t_0))$ by ω as long as they are sufficiently close to $Z_x^*(x(t_0))\xi(t_0)$ and $Z(x(t_0))$ respectively. Moreover, $R(t_0, Z(x(t_0)), 0, \xi^0) = 0$. It follows that there are open neighborhoods U_1, U_2 of 0 and ξ^0 respectively such that if $z \in U_1$ and $\zeta \in U_2$, then

$$\int e^{\sqrt{-1}\lambda\phi(t_0, Z(x), z, \zeta)} a(t_0, Z(x), z, \zeta, \lambda) u(x)\, dZ(x)$$

decays exponentially uniformly in $U_1 \times U_2$.

In the above considerations, the open set U may have to be contracted. However, the contraction is independent of the distribution u.

Now let

$$F(t, z, \zeta, \lambda) = \int e^{\sqrt{-1}\lambda\phi(t, Z(x), z, \zeta)} a(t, Z(x), z, \zeta, \lambda) u(x)\, dZ(x).$$

We have:

$$\begin{aligned}
\left|\frac{\partial}{\partial t} F(t, z, \zeta, \lambda)\right| &= \left|\int \frac{\partial}{\partial t}(e^{\sqrt{-1}\lambda\phi} a) u(x)\, dZ(x)\right| \\
&= \frac{1}{\lambda^{k-1}} \left|\int {}^tP(e^{\sqrt{-1}\lambda\phi} a) u(x)\, dZ(x)\right| \\
&= \frac{1}{\lambda^{k-1}} \left|\int e^{\sqrt{-1}\lambda\phi} a Pu(x)\, dZ(x)\right|.
\end{aligned}$$

Since Pu is hypo-analytic at each point $(x(t), \xi(t))$, our arguments imply that for each t, the last integral decays exponentially for z near 0 and ζ near ξ^0. Moreover, this decay persists in a compact set of t's. It follows that there exist positive constants c_1, c_2 and open neighborhoods W_1, W_2 of 0 and ξ^0 respectively such that

$$\left|\frac{\partial}{\partial t} F(t, z, \zeta, \lambda)\right| \le c_1 e^{-c_2\lambda} \quad \text{for } \lambda > 0,$$

$z \in W_1$, $\zeta \in W_2$ and $0 \le t \le t_0$. Since we also had a similar decay for $F(t_0, z, \zeta, \lambda)$, it follows that $|F(0, z, \zeta, \lambda)| \le d_1 e^{-d_2\lambda}$ for some positive constants d_1, d_2 and (z, ζ) near $(0, \xi^0)$. But

$$\begin{aligned}
F(0, z, \zeta, \lambda) = \int \exp\Bigg(\sqrt{1}\lambda\Bigg[(z - Z(x))\frac{\zeta}{|\zeta|} \\
+\sqrt{-1}\frac{\langle\zeta\rangle}{|\zeta|}(z - Z(x))^2\Bigg]\Bigg) u(x)\, dZ(x).
\end{aligned}$$

Therefore, by Theorem 3.1 of the previous section $(0, \xi^0) \notin WF_{\text{ha}}u$. We have thus shown that there is an open neighborhood N in γ of

$(0,\xi^0)$, N independent of the distribution u such that $(x,\xi) \in N$ and $(x,\xi) \notin WF_{\text{ha}}u \Rightarrow (0,\xi^0) \notin WF_{\text{ha}}u$. This means that $WF_{\text{ha}}u \cap \gamma$ is open in γ. But this set is always closed. Since γ is connected, the theorem follows.

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