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HARDY INTERPOLATING SEQUENCES OF HYPERPLANES

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A sufficient condition is given on unions of complex hyperplanes in the unit ball of C^n so that they allow extension of functions in the Hardy H^1 space. The result is compared to Varopoulos' theorem about zeros of H^p functions.

1. Notations and definitions. For $z, w \in C^n$,

$$z \cdot \bar{w} = \sum_{i=1}^{n} z_i \bar{w}_i,$$
$$B^n = \{z \in C^n : |z|^2 = z \cdot \bar{z} < 1\}.$$

For $a_k \in B^n$, $a_k \neq 0$,

$$a_k^* = \frac{a_k}{|a_k|}.$$

λ_p = p real-dimensional Lebesgue measure. For instance, on C, $-\frac{i}{2} dz \wedge d\bar{z} = d\lambda_2$.

Automorphisms of the ball.

$$\phi_k(z) := \phi_{a_k}(z) := \frac{a_k - P_k(z) - s_k Q_k(z)}{1 - z \cdot \bar{a}_k}$$

where $P_k(z) := \frac{z \cdot \bar{a}_k}{|a_k|^2} a_k$ is the projection onto the complex line through a_k, $Q_k(z) := z - P_k(z)$ is the projection onto the complex hyperplane perpendicular to a_k, $s_k^2 := 1 - |a_k|^2$.

The map ϕ_k is an involution of the ball (see Rudin [4]). Note that

$$Q_k(B^n) = \{z : P_k(z) = 0\} = \{z : z \cdot \bar{a}_k = 0\}.$$

We write

$$d_G(z, w)^2 := |\phi_w(z)|^2 = 1 - \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - z \cdot \bar{w}|^2}.$$

This is an *invariant* distance: if ϕ is an automorphism of the ball (i.e. any composition of unitary transformations and the above involutions), $d_G(\phi(z), \phi(w)) = d_G(z, w)$.

We will study hyperplanes in the ball, denoted by:

$$V_j := \{z \in B^n : z \cdot \bar{a}_j = |a_j|^2\}.$$

The point a_j is the point in V_j closest to the origin. It is also the center of the $n-1$-complex-dimensional ball which V_j defines inside B^n. This definition makes no sense when $a_j = 0$, so we will not consider that case. However, the problem we will consider is automorphism-invariant and if there is a hyperplane going through the origin, applying to the whole sequence an automorphism ϕ_a, with $|a|$ small enough, will preserve the hypotheses (at the expense of a change in the value of δ, see below) and yield the conclusion. We define c^0_{jk} to be the "center" of the hyperplane $\phi_k(V_j)$, i.e.

$$\phi_k(V_j) = \phi_k^{-1}(V_j) = \{z \in B^n : z \cdot \bar{c}^0_{jk} = |c^0_{jk}|^2\}.$$

We further consider the angle between $\phi_k(V_j)$ and V_k:

$$\cos\theta_{jk} := \frac{|c^0_{jk} \cdot \bar{a}_k|}{|c^0_{jk}||a_k|}.$$

LEMMA 1.

(1)
$$c^0_{jk} = \frac{l_{jk} c_{jk}}{|c_{jk}|^2}$$

where

$$c_{jk} := \left((1 - s_k)\frac{a_j^* \cdot \bar{a}_k}{|a_k|^2} - |a_j|\right) a_k + s_k a_j^*,$$

$$l_{jk} := a_k \cdot \bar{a}_j^* - |a_j|^2 = (a_k - a_j) \cdot \bar{a}_j^*;$$

$$|c_{jk}|^2 = |l_{jk}|^2 + (1 - |a_j|^2)(1 - |a_k|^2).$$

(2)
$$\cos^2\theta_{jk} = \left(\frac{|c_{jk} \cdot \bar{a}_k|}{|c_{jk}||a_k|}\right)^2 = \frac{|a_k^* \cdot \bar{a}_j^* - |a_j||a_k||^2}{|l_{jk}|^2 + (1 - |a_j|^2)(1 - |a_k|^2)},$$

(3)
$$|c^0_{jk}|^2 = \frac{|(a_k - a_j) \cdot \bar{a}_j|^2}{|(a_k - a_j) \cdot \bar{a}_j|^2 + |a_j|^2(1 - |a_j|^2)(1 - |a_k|^2)},$$

$$1 - |c^0_{jk}|^2 = \frac{(1 - |a_j|^2)(1 - |a_k|^2)}{|l_{jk}|^2 + (1 - |a_j|^2)(1 - |a_k|^2)}.$$

The proofs of all lemmas are deferred until §4.

The interpolation problem. The *Hardy space* $H^p(B^n)$ is the space of functions f holomorphic on the ball and verifying

$$\|f\|_{H^p}^p := \sup_{r<1} \int_{\partial B^n} |f(r\zeta)|^p \, d\sigma(\zeta) < \infty,$$

where σ is $2n-1$-dimensional Lebesgue measure on ∂B^n.

The *Bergman space* $A^p(V_k)$ is the space of functions α holomorphic on the hyperplane V_k and verifying

$$\|\alpha\|_{A^p(V_k)}^p := \int_{V_k} |\alpha(z)|^p \, d\lambda_{2n-2}(z) < \infty.$$

DEFINITION. $l^p(A^p(V_k), 1-|a_k|^2)$ is the product of the Bergman spaces on each hyperplane, endowed with the following norm: if $\alpha = \{\alpha_k\}_{k\in Z_+}$, where α_k is a function defined and holomorphic on V_k,

$$\|\alpha\|_B^p := \|\alpha\|_{l^p(A^p(V_k),1-|a_k|^2)}^p = \sum_k (1-|a_k|^2)\|\alpha_k\|_{A^p(V_k)}^p.$$

Notice that $\phi_k|_{V_k}$ is just an affine map from V_k to $Q_k(B^n) \simeq B^{n-1}$, so that we can rewrite

$$\begin{aligned}\|\alpha\|_B^p &= \|\alpha\|_{l^p(A^p(B^{n-1}),(1-|a_k|^2)^n)}^p \\ &= \sum_k (1-|a_k|^2)^n \int_{Q_k(B^n)} |\alpha_k \circ \phi_k(w)|^p \, d\lambda_{2n-2}(w).\end{aligned}$$

Given a function $f \in H(B^n)$, the space of holomorphic functions, we consider the following map

$$\begin{aligned} T\colon H(B^n) &\to \prod_{i=1}^{\infty} H(V_i), \\ f &\mapsto \{f|_{V_i}\}_{i\geq 1}. \end{aligned}$$

DEFINITION. We say that $\{V_j\}_{j\in Z_+}$ is an H^p-interpolating sequence of hyperplanes if T maps $H^p(B^n)$ onto $l^p(A^p(V_k), 1-|a_k|^2)$.

Equivalently, given $\{\alpha_k\}$ a sequence of functions holomorphic on V_k, such that

$$\sum_k (1-|a_k|^2) \int_{V_k} |\alpha_k(z)|^p \, d\lambda_{2n-2}(z) < \infty,$$

there exists $f \in H^p(B^n)$ such that

$$f|_{V_k} = \alpha_k.$$

This definition is the one given by Amar [1] and reduces in the case $n = 1$ to that of Shapiro and Shields [5].

REMARK. With this definition, if a sequence of hyperplanes is H^p-interpolating and we take points $b_k \in V_k$, $\forall k$, then the sequence $\{b_k\}$ is H^p-interpolating (in the sense of [2]).

Proof. If we are given a sequence of complex numbers $\{\beta_k\}$ such that

$$\sum_k (1 - |b_k|^2)^n |\beta_k|^p < \infty,$$

then define

$$\alpha_k(z) = \left(\frac{1 - |b_k|^2}{1 - z \cdot \bar{b}_k}\right)^n \beta_k.$$

Then

$$\int_{V_k} |\alpha_k(z)|^p \, d\lambda_{2n-2}(z)$$
$$= \int_{Q_k(B^n)} |\beta_k|^p \left|\frac{1 - |b_k|^2}{1 - \psi(w) \cdot \bar{b}_k}\right|^{np} |J_\psi(w)| \, d\lambda_{2n-2}(w),$$

where $\psi(w) = a_k + s_k w$.

$$|J_\psi(w)| = s_k^{2n-2} = (1 - |a_k|^2)^{n-1}, \quad \text{and}$$
$$1 - \psi(w) \cdot \bar{\psi}(w') = (1 - |a_k|^2)(1 - w \cdot \bar{w}'),$$

so, setting $b_k' = \psi^{-1}(b_k)$, we get

$$\int_{V_k} |\alpha_k(z)|^p \, d\lambda_{2n-2}(z)$$
$$= |\beta_k|^p (1 - |a_k|^2)^{n-1} \int_{Q_k(B^n)} \left|\frac{1 - |b_k'|^2}{1 - w \cdot \bar{b}_k'}\right|^{np} d\lambda_{2n-2}(w)$$
$$\leq C |\beta_k|^p (1 - |a_k|^2)^{n-1} (1 - |b_k'|^2)^n \quad \text{because } np > n - 1,$$
$$= C |\beta_k|^p \frac{(1 - |b_k|^2)^n}{1 - |a_k|^2}.$$

It follows that

$$\sum_k (1 - |a_k|^2) \int_{V_k} |\alpha_k(z)|^p \, d\lambda_{2n-2}(z) \leq C \sum_k (1 - |b_k|^2)^n |\beta_k|^p,$$

and the function $f \in H^p$ which we get by interpolating the α_k on the hyperplanes verifies $f(b_k) = \alpha_k(b_k) = \beta_k$. □

Taking $b_k = a_k$, we get from **[8]** (for $p \geq 1$) the following necessary condition:

$$\sup_k \sum_j \left(\frac{(1-|a_k|^2)(1-|a_j|^2)}{|1-a_k \cdot \overline{a_j}|^2} \right)^n < \infty.$$

We also get that any sequence $\{b_k\}$ must be separated in the Gleason distance; thus there exists $\delta > 0$ such that if $j \neq k$, then

$$d_G(V_j, V_k) = \inf\{d_G(z, w), z \in V_j, w \in V_k\} \geq \delta > 0.$$

We say that the hyperplanes are *separated*.

2. The main result. We are looking for a sufficient geometric condition to ensure that a sequence of hyperplanes be H^1-interpolating. To do so, we define another family of neighborhoods for the hyperplanes.

DEFINITION. Given δ a positive number, we call *tube* around V_k the following open subset of B^n:

$$T_\delta(V_k) := \{z \in B^n \colon |(z - a_k) \cdot \bar{a}_k^*| < \delta(1 - |a_k|^2)\}.$$

Those neighborhoods of the hyperplanes will be larger than those given by separatedness in the Gleason distance. This will follow from:

LEMMA 2. (1) *Given any* $z \in B^n$,

$$d_G(z, V_k)^2 = \frac{|P_k \circ \phi_k(z)|^2}{|P_k \circ \phi_k(z)|^2 + (1 - |\phi_k(z)|^2)}.$$

(2) $\overline{V}_j \cap \overline{V}_k = \varnothing \Leftrightarrow \cos^2\theta_{jk} > (1 - |c_{jk}^0|^2)$.

(3) *If* (2) *is satisfied,*

$$1 - d_G^2(V_j, V_k) = \frac{(1-|a_j|^2)(1-|a_k|^2)}{|a_k^* \cdot \bar{a}_j^* - |a_j||a_k||^2} = \frac{(1-|c_{jk}^0|^2)}{\cos^2\theta_{jk}}.$$

(4)

$$d_G(V_j, V_k) \geq \delta_1 > 0 \Leftrightarrow (1 - \delta_1^2)\cos^2\theta_{jk} \geq (1 - |c_{jk}^0|^2).$$

From this we can prove that all points of the ball which are close enough to V_k in the invariant distance must be within the tube. Indeed, by applying Lemma 2(1) and the fact that

$$P_k \circ \phi_k(z) = -\frac{(z-a_k)\cdot \bar{a}_k}{1 - z\cdot \bar{a}_k}\frac{a_k}{|a_k|^2},$$

we see that

$$d_G(z, V_k)^2 = \frac{|(z-a_k)\cdot \bar{a}_k|^2}{|(z-a_k)\cdot \bar{a}_k|^2 + |a_k|^2(1-|a_k|^2)(1-|z|^2)}.$$

Clearly then, if $z \in \partial T_\delta(V_k)$,

$$d_G(z, V_k)^2 = \frac{\delta^2}{\delta^2 + |a_k|^2\frac{1-|z|^2}{1-|a_k|^2}} > \frac{\delta^2}{\delta^2 + 2(1+\delta)},$$

which shows the inequality holds for $z \notin T_\delta(V_k)$.

THEOREM. *There exists a number $c_0 = c_0(\delta) > 0$ such that if*

(i) $$\sup_k \sum_{j:\, j\neq k} \left(\frac{(1-|a_k|^2)(1-|a_j|^2)}{|1-a_k\cdot \overline{a_j}|^2}\right)^n < c_0$$

and

(ii) *for any $j \neq k$,* $T_\delta(V_j)\cap T_\delta(V_k) = \varnothing$,

then $\{V_k\}_{k\in Z_+}$ is an $H^1(B^n)$-interpolating sequence of hyperplanes.

REMARKS. (1) It was proved in **[6]** that (ii) together with

(B) $$\sup_k \sum_j \frac{(1-|a_k|^2)(1-|a_j|^2)}{|1-a_k\cdot \overline{a_j}|^2} < \infty$$

forms a sufficient condition for $\{V_k\}$ to be an H^∞ interpolating sequence of hyperplanes.

(2) A similar result holds for a sequence of points, but condition (i) is enough, with any constant $c_0 < 1$ **[8]**. Here c_0 will have to be even smaller; therefore condition (i) by itself is enough to ensure separatedness of the points, since in particular each term of the sum must be less then c_0.

Proof of the Theorem. We will construct an approximate extension, i.e. an operator

$$\tilde{E}\colon l^1(A^1(V_k), 1-|a_k|^2) \to H^1(B^n)$$

such that

$$\text{(E1)} \qquad \|\tilde{E}\|_{\text{op}} < \infty$$

and

$$\text{(E2)} \qquad \|T\tilde{E} - I\|_{\text{op}} < 1.$$

Then $T\tilde{E}$ is invertible, and one can write a true extension by letting $E = \tilde{E}(T\tilde{E})^{-1}$. The operator TE will be the identity map on l^1 and for $\alpha \in l^1$, $E(\alpha)$ will be a solution to the interpolation problem.

Let

$$\tilde{E}(\alpha)(z) = \sum_{k \in Z_+} \left(\frac{1 - |a_k|^2}{1 - z \cdot \bar{a}_k} \right)^{2n} \tilde{\alpha}_k(z),$$

where $\tilde{\alpha}_k = \alpha_k \circ \phi_k \circ Q_k \circ \phi_k$ is an extension of α_k to B^n. Note that for $z \in V_j$, the jth term in the sum is exactly $1^{2n}\tilde{\alpha}_j(z) = \alpha_j(z)$. (E1) is easily checked, for the coefficient of $\tilde{\alpha}_k(z)$ is bounded and it follows from the computations in **[6]** that

$$\int_{\partial B^n} \left| \left(\frac{1 - |a_k|^2}{1 - z \cdot \bar{a}_k} \right)^{n} \tilde{\alpha}_k(z) \right| d\sigma(z) \leq C(1 - |a_k|^2) \int_{V_k} |\alpha_k(z)| \, d\lambda_{2n-2}(z).$$

This step fails for $p > 1$, and prevents us from proving H^p results for hyperplanes similar to those for points in **[8]**.

The theorem reduces to:

MAIN LEMMA. *For c_0 small enough, there exists $c_1 < 1$ such that for any $\alpha \in l^1(A^1(V_k), 1 - |a_k|^2)$,*

$$\sum_j (1 - |a_j|^2) \int_{V_j} \left| \sum_{k:\, k \neq j} \left(\frac{1 - |a_k|^2}{1 - z \cdot \bar{a}_k} \right)^{2n} \tilde{\alpha}_k(z) \right| d\lambda_{2n-2}(z) \leq c_1 \sum_k (1 - |a_k|^2) \int_{V_k} |\alpha_k(z)| \, d\lambda_{2n-2}(z).$$

Comparison with zero-set results. Clearly, if $\{V_k\}_{k \in Z_+}$ satisfy the hypotheses of the theorem, then their union will be a subset of a zero set for H^1 functions. To see it, simply adjoin to the sequence a hyperplane V_0 such that (i) and (ii) still hold (this can be achieved by taking

a_0^* on $\partial B^n \setminus \bigcup_{k\leq 1} T_{2\delta}(V_k)$ and $|a_0|$ very close to 1); then interpolate 1 on V_0 and 0 everywhere else.

This needs to be compared to the results of N. Th. Varopoulos, at least in the special case of a divisor made up of a countable union of complex hyperplanes **[9**, §8**]**. In that case, he showed:

PROPOSITION 8.2. *There exist constants* $C_1, \dots, C_4$ *such that if*

$$\sum_{j:\ |1-a_j\cdot\bar{a}_k|\leq C_1(1-|a_k|^2)} (1-|a_j|^2)^n \leq C_2(1-|a_k|^2)^n \tag{8.18}$$

and

$$\operatorname{Card}\{j\colon V_j \cap K_h(\zeta) \neq \varnothing,\ V_j \not\subseteq K_{C_3 h}(\zeta)\} \leq C_4 \tag{8.19}$$

where $K_h(\zeta) := \{z \in B^n\colon |1 - z\cdot\bar{\zeta}| < h\}$, *then there exists* $p > 0$ *such that* $\bigcup_k V_k$ *is a zero set for* $H^p(B^n)$.

It can be shown (see e.g. **[3]**) that (8.18), which is a Carleson measure condition, is equivalent to

$$\sup_k \sum_j \left(\frac{(1-|a_k|^2)(1-|a_j|^2)}{|1-a_k\cdot\overline{a_j}|^2} \right)^n < \infty.$$

On the other hand, if we assume separatedness in the invariant distance, (8.19) is satisfied in the following stronger form:

$$\exists C_5 > 0 \text{ such that } \operatorname{Card}\{j\colon V_j \cap K_h(\zeta) \neq \varnothing,\ V_j \not\subseteq K_{C_5 h}(\zeta)\} \leq 1.$$

Note that the above set is non-empty only when $h \leq 2/C_5$.

The idea of the proof is first to use the triangle inequality for the Koranyi distance to reduce oneself to the case where $\zeta \in V_j \cap \partial B^n$; then to apply an automorphism to bring V_j to $\phi_j(V_j)$, which is a hyperplane through the center of B^n. The region $K_h(\zeta)$ is transformed into a similar region, because a_j, by the assumption that j is in the above set, is far enough away from ζ. If another index k was also in the set, the hyperplane $\phi_j(V_k)$ would pass through $\phi_j(K_h(\zeta))$, and thus its projection onto $\phi_j(V_j)$ would come too close to the boundary, violating the conclusion of Lemma 5, given below.

Varopoulos' theorem, as he pointed out, provides no control over the value of p (which could indeed be very small, if one works out the constants involved). This is essentially because the norm of the Carleson measure supported by the divisor *cannot* be made arbitrarily small. For this very special structure of the divisor $\bigcup_j V_j$, our result provides additional control on the exponent, although the actual zero set involved could be much larger than $\bigcup_j V_j$. Namely:

PROPOSITION. *If $\{V_k\}_{k\in Z_+}$ satisfies*

$$(\mathrm{i}_M) \qquad \sup_k \sum_{j:\, j\neq k} \left(\frac{(1-|a_k|^2)(1-|a_j|^2)}{|1-a_k\cdot\overline{a_j}|^2}\right)^n < 2^M c_0$$

and

$$(\mathrm{ii}_N) \qquad \text{for any } k, \quad \mathrm{Card}\{j\colon T_\delta(V_j)\cap T_\delta(V_k)\neq\varnothing\}\leq N,$$

where $M\geq 0$, $N\geq 0$, are integers, then there exists $f\not\equiv 0$, $f\in H^{1/2^M(N+1)}(B^n)$, such that $f|_{V_k}\equiv 0$ for all k.

Proof. An elementary combinatorial argument shows that under (ii_N), the sequence can be split into $N+1$ subsequences, each of which satisfies (ii), and of course (i_M). Then Mills' Lemma **[8]** allows us to split each such subsequence into 2^M further subsequences verifying (i). Thus we are reduced to the case $M=0$, $N=0$, i.e. the assumptions of the theorem; by the argument given at the beginning of this section, each subsequence has a nonzero H^1 function vanishing on it. Taking the product of the annihilating functions, we find $f\in H^{1/2^M(N+1)}(B^n)$.

3. Proof of the main lemma. For convenience, we shall introduce the notation $A_k=\alpha_k\circ\phi_k$. Thus A_k is a function defined on $A_k(B^n)\simeq B^{n-1}$, and

$$\begin{aligned}(1-|a_k|^2)^n &\int_{Q_k(B^n)} |A_k(z)|\,d\lambda_{2n-2}(z)\\ &= (1-|a_k|^2)\int_{V_k}|\alpha_k(z)|\,d\lambda_{2n-2}(z).\end{aligned}$$

Furthermore, $\tilde{\alpha}_k=A_k\circ Q_k\circ\phi_k$. With this new notation, it is enough to bound

$$\sum_k\sum_{j:\, j\neq k}(1-|a_j|^2)(1-|a_k|^2)^{2n}\int_{V_j}\frac{|A_k\circ Q_k\circ\phi_k(z)|}{|1-z\cdot\bar{a}_k|^{2n}}\,d\lambda_{2n-2}(z).$$

The integral in question is equal to

$$\int_{\phi_k(V_j)}\frac{|A_k\circ Q_k(w)|}{|1-z\cdot\bar{a}_k|^{2n}}|J_{\phi_k|V_j}(z)|^{-1}\,d\lambda_{2n-2}(w),$$

where $J_{\phi_k|V_j}(z)$ is the Jacobian of the map ϕ_k restricted to V_j.

LEMMA 3.
$$|J_{\phi_k|V_j}(z)| = \frac{(1-|a_k|^2)^{n-1}}{|1-z\cdot\bar{a}_k|^{2n}}[|(a_k-a_j)\cdot\bar{a}_j^*|^2+(1-|a_j|^2)(1-|a_k|^2)]$$
$$= \frac{(1-|a_k|^2)^{n-1}}{|1-z\cdot\bar{a}_k|^{2n}}|c_{jk}|^2,$$
with the notations from Lemma 1.

Thus the terms in the sum reduce to:
$$\frac{(1-|a_j|^2)(1-|a_k|^2)^{n+1}}{|l_{jk}|^2+(1-|a_j|^2)(1-|a_k|^2)}\int_{\phi_k(V_j)}|A_k\circ Q_k(w)|\,d\lambda_{2n-2}(w)$$
$$=\frac{(1-|a_j|^2)(1-|a_k|^2)^{n+1}}{|c_{jk}|^2}\int_{Q_k\circ\phi_k(V_j)}|A_k(u)||J_{Q_k|_{\phi_k(V_j)}}(w)|^{-1}\,d\lambda_{2n-2}(u).$$

LEMMA 4. *Given $a\in B^n$, let $V=\{z\in B^n\colon z\cdot\bar{a}=|a|^2\}$. Then*
(1)
$$|J_{Q_k|V}|=\left(\frac{|a\cdot\bar{a}_k|}{|a||a_k|}\right)^2 =: \cos^2\theta.$$
(2) *In the case where $a\cdot\bar{a}_k\neq 0$, $Q_k(V)$ is the subset of $Q_k(B^n)$ given by the equation*
$$\left(\frac{|a\cdot\bar{a}_k|}{|a||a_k|}\right)^{-2}|w_1-Q_k(a)|^2+|w_2|^2<1-|a|^2,$$
where w_1 is the coordinate in the $Q_k(a)$ complex direction, and w_2 represents the $n-2$ complex coordinates in the orthogonal directions within $Q_k(B^n)$. $Q_k(V)$ is thus an ellipsoid of radii $(\cos\theta)(1-|a|^2)^{1/2}$ in the w_1 direction, and $(1-|a|^2)^{1/2}$ in each of the w_2 directions. In the case where $a\cdot\bar{a}_k=0$, we get simply $Q_k(B^n)\cap V$ as the projection.
(3)
$$\max_{Q_k(V)}|z| = |a|\sin\theta+(1-|a|^2)^{1/2}\cos\theta.$$

We apply this lemma with $a=c_{jk}^0$ and $\theta=\theta_{jk}$. Since, under the separatedness condition, $V_j\cap V_k=\emptyset$, we always have $|c_{jk}^0\cdot\bar{a}_k| = |a_k||c_{jk}^0|\cos\theta_{jk}>|a_k||c_{jk}^0|(1-|c_{jk}^0|^2)^{1/2}>0$, i.e. $c_{jk}^0\cdot\bar{a}_k\neq 0$. Replacing the Jacobian by its value (see Lemma 1(2)), we get for each term of the sum:
$$=\frac{(1-|a_j|^2)(1-|a_k|^2)^{n+1}|a_k|^2}{|a_k\cdot\bar{c}_{jk}|^2}\int_{Q_k\circ\phi_k(V_j)}|A_k(u)|\,d\lambda_{2n-2}(u).$$

We now make use of (ii):

LEMMA 5. *If* $T_\delta(V_j) \cap T_\delta(V_k) = \varnothing$, *then there exists* $\delta_1 = \delta_1(\delta) > 0$ *such that*

$$\max\{|z|: z \in Q_k \circ \phi_k(V_j)\} \le \sqrt{1-\delta_1^2} < 1.$$

Thus the distance to ∂B^n from $Q_k \circ \phi_k(V_j)$ is at least $\delta_2 = 1 - \sqrt{1-\delta_1^2}$. By the classical theory of Bergman spaces, this implies that A_k satisfies a uniform estimate on $Q_k \circ \phi_k(V_j)$:

$$|A_k(u)| \le \frac{C}{\delta_2^{2n-2}} \int_{Q_k(B^n)} |A_k(u)|\, d\lambda_{2n-2}(u).$$

It follows from Lemma 4(2), applied with $a = c_{jk}^0$, that

$$\lambda_{2n-2}(Q_k \circ \phi_k(V_j)) = \cos^2\theta_{jk}(1-|c_{jk}^0|^2)^{n-1}.$$

Thus each term in our sum is bounded by

$$C(\delta)\frac{(1-|a_j|^2)(1-|a_k|^2)^{n+1}(1-|c_{jk}^0|^2)^{n-1}}{|c_{jk}|^2}\int_{Q_k(B^n)} |A_k(u)|\, d\lambda_{2n-2}(u)$$

which Lemma 1(3) and some arithmetic reduces to:

$$= C(\delta)\frac{(1-|a_j|^2)^n(1-|a_k|^2)^{2n}}{|c_{jk}|^{2n}}\int_{Q_k(B^n)} |A_k(u)|\, d\lambda_{2n-2}(u).$$

We must estimate $|c_{jk}|^2 = |l_{jk}|^2 + (1-|a_j|^2)(1-|a_k|^2)$ from below. Simply writing that $a_k \notin T_\delta(V_j)$, condition (ii) implies $|l_{jk}| > \delta(1-|a_j|^2)$.

Case 1. $(1-\delta)|1-a_j\cdot\bar a_k| \le 2(1-|a_j|)$. Then

$$|l_{jk}| > \delta(1-|a_j|^2) \ge \frac{\delta(1-\delta)}{2}|1-a_j\cdot\bar a_k|.$$

Case 2. $(1-\delta)|1-a_j\cdot\bar a_k| > 2(1-|a_j|)$. Then

$$\begin{aligned}|l_{jk}| &= |a_k\cdot\bar a_j^* - |a_j||\\ &= |1-a_k\cdot\bar a_j - (1-|a_j|)(1+a_k\cdot\bar a_j^*)| \ge \delta|1-a_k\cdot\bar a_j|.\end{aligned}$$

In either case, $|c_{jk}|^{2n} > |l_{jk}|^{2n} \geq C(\delta)|1 - a_j \cdot \bar{a}_k|^{2n}$, and our whole sum is majorized by

$$C(\delta)\sum_k (1 - |a_k|^2)^n$$
$$\times \left(\sum_{j:\, j\neq k} \left[\frac{(1 - |a_j|^2)(1 - |a_k|^2)}{|1 - a_j \cdot \overline{a_k}|^2} \right]^n \right) \int_{Q_k(B^n)} |A_k(u)|\, d\lambda_{2n-2}(u)$$
$$\leq c_0 C(\delta) \sum_k (1 - |a_k|^2) \int_{V_k} |\alpha_k(z)|\, d\lambda_{2n-2}(z).$$

It will now be enough to pick

$$c_0 < \frac{1}{C(\delta)} (\approx \delta_2^{2(n-1)} (\delta(1-\delta))^{2n} \approx \delta^{6n-4}),$$

which concludes the proof of the Main Lemma.

4. Proof of the Lemmas.

Proof of Lemma 1. Since $\phi_k = \phi_k^{-1}$,

$$\phi_k(V_j) = \phi_k^{-1}(V_j) = \{z \in B^n : \phi_k(z) \cdot \bar{a}_j = |a_j|^2\}.$$

This equation becomes:

$$a_k \cdot \bar{a}_j - \frac{z \cdot \bar{a}_k}{|a_k|^2} a_k \cdot \bar{a}_j (1 - s_k) - s_k z \cdot \bar{a}_j = |a_j|^2 (1 - z \cdot \bar{a}_k),$$
$$z \cdot \left(\left(|a_j|^2 - \frac{a_k \cdot \bar{a}_j}{|a_k|^2}(1 - s_k) \right) \bar{a}_k - s_k \bar{a}_j \right) = |a_j|^2 - a_k \cdot \bar{a}_j.$$

Let $|a_j| c_{jk} := ((1 - s_k)(a_j \cdot \bar{a}_k / |a_k|^2) - |a_j|^2) a_k + s_k a_j$, $l_{jk} := a_k \cdot \bar{a}_j^* - |a_j|$.

The equation now reads $z \cdot \bar{c}_{jk} = l_{jk}$, or equivalently

$$z \cdot \frac{\bar{l}_{jk} \bar{c}_{jk}}{|c_{jk}|^2} = \frac{|l_{jk}|^2}{|c_{jk}|^2} = \left| \frac{l_{jk} c_{jk}}{|c_{jk}|^2} \right|^2.$$

We need to compute $|c_{jk}|^2$. Note first that

$$\begin{aligned} |a_j| c_{jk} \cdot \bar{a}_k &= (1 - s_k) a_j \cdot \bar{a}_k - |a_j|^2 |a_k|^2 + s_k a_j \cdot \bar{a}_k \\ &= a_j \cdot \bar{a}_k - |a_j|^2 |a_k|^2; \end{aligned}$$

and

$$|a_j| c_{jk} \cdot \bar{a}_j = (1 - s_k) \frac{|a_j \cdot \bar{a}_k|^2}{|a_k|^2} - |a_j|^2 a_k \cdot \bar{a}_j + s_k |a_j|^2.$$

Thus

$$
\begin{aligned}
|a_j|^2|c_{jk}|^2 &= |a_j|c_{jk}\cdot|a_j|\bar{c}_{jk}\\
&= |a_j|c_{jk}\cdot\bar{a}_k\left((1-s_k)\frac{a_j\cdot\bar{a}_k}{|a_k|^2}-|a_j|^2\right)+(|a_j|c_{jk}\cdot\bar{a}_j)s_k\\
&= (1-s_k)\frac{|a_j\cdot\bar{a}_k|^2}{|a_k|^2}-(1-s_k)a_k\cdot\bar{a}_j|a_j|^2-|a_j|^2a_j\cdot\bar{a}_k+|a_j|^4|a_k|^2\\
&\quad+s_k(1-s_k)\frac{|a_j\cdot\bar{a}_k|^2}{|a_k|^2}-s_k|a_j|^2a_k\cdot\bar{a}_j+s_k^2|a_j|^2\\
&= |a_j\cdot\bar{a}_k|^2-|a_j|^2(a_k\cdot\bar{a}_j+a_j\cdot\bar{a}_k)+|a_j|^2(1-|a_k|^2)+|a_j|^4|a_k|^2\\
&= |a_j\cdot\bar{a}_k-|a_j|^2|^2+(|a_j|^2-|a_j|^4)(1-|a_k|^2)\\
&= |a_j|^2(|a_k\cdot\bar{a}_j^*-|a_j||^2+(1-|a_j|^2)(1-|a_k|^2)).
\end{aligned}
$$

This proves (1).

We get from the above

$$\cos^2\theta_{jk}=\frac{|a_j\cdot\bar{a}_k-|a_j|^2|a_k|^2|^2}{|a_k|^2|a_j|^2(|l_{jk}|^2+|a_j|^2(1-|a_j|^2)(1-|a_k|^2))},$$

which proves (2) after cancelling $|a_k|^2|a_j|^2$ from top and bottom. Finally,

$$|c_{jk}^0|^2=\left|\frac{l_{jk}}{c_{jk}}\right|^2=\frac{|l_{jk}|^2}{|l_{jk}|^2+(1-|a_j|^2)(1-|a_k|^2)},$$

from which (3) follows.

Proof of Lemma 2. Since d_G is automorphism-invariant, we can compute $d_G(\phi_k(V_k),z)$ first. But $P_k(z)=a_k$ for $z\in V_k$, so $\phi_k(V_k)=Q_k(B^n)$. Now fix $z\in B^n$. We need to find

$$
\begin{aligned}
&\inf_{w\in Q_k(B^n)}\left(1-\frac{(1-|z|^2)(1-|w|^2)}{|1-z\cdot\bar{w}|^2}\right)\\
&\qquad=1-(1-|z|^2)\sup_{w\in Q_k(B^n)}\frac{1-|w|^2}{|1-z\cdot\bar{w}|^2}.
\end{aligned}
$$

If $z\cdot\bar{w}=Q_k(z)\cdot\bar{w}$ remains fixed, the largest value is obtained for $|w|$ minimal, i.e. w parallel to $Q_k(z)$. Set $w=\alpha Q_k(z)^*$, with $\alpha\in\Delta=B^1\subset C$. We have to study

$$\max_{\alpha\in\Delta}\frac{1-|\alpha|^2}{|A+B\alpha|^2},$$

with $A=1$, $B=Q_k(z)^*\cdot\bar{z}=|Q_k(z)|<1$. This function is always differentiable and the gradient vanishes for $\alpha=-\overline{B}/\overline{A}$. The maximum equals $(|A|^2 - |B|^2)^{-1} = 1/(1 - |Q_k(z)|^2)$.

$$1 - \frac{1 - |z|^2}{1 - |Q_k(z)|^2} = \frac{|z|^2 - |Q_k(z)|^2}{1 - |Q_k(z)|^2} = \frac{|P_k(z)|^2}{1 - |z|^2 + |P_k(z)|^2}.$$

That gives the distance from z to $\phi_k(V_k)$. By invariance under automorphisms, $d_G(V_k, z) = d_G(\phi_k(V_k), \phi_k(z))$, and we get (1) by substituting $\phi_k(z)$ into the above formula.

Now we want to minimize $d_G(\phi_k(V_k), z)$ over $z \in \phi_k(V_j)$, i.e. for $z \cdot \overline{c_{jk}^0} = |c_{jk}^0|^2$. Recall that $P_k(z) = z \cdot \bar{a}_k^*$. Let

$$\Psi(z) := \frac{|z \cdot \bar{a}_k^*|^2}{|z \cdot \bar{a}_k^*|^2 + 1 - |z|^2} = \frac{1}{1 + (1 - |z|^2)/|z \cdot \bar{a}_k^*|^2},$$

so to minimize Ψ we have to maximize $1 - |z|^2/|z \cdot \bar{a}_k^*|^2$. We can reduce ourselves to the case where $z \in \operatorname{Span}(a_k, c_{jk}^0)$; otherwise, projecting z onto it will not change $z \cdot \bar{a}_k^*$ and will increase $1 - |z|^2$. If $z \in \phi_k(V_j) \cap \operatorname{Span}(a_k, c_{jk}^0)$, we can write

$$z = c_{jk}^0 + (1 - |c_{jk}^0|^2)^{1/2} \widetilde{c_{jk}^0} \alpha,$$

where α is a complex number, $\alpha \in \Delta$, and $|\widetilde{c_{jk}^0}| = 1$, $\widetilde{c_{jk}^0} \in \operatorname{Span}(a_k, c_{jk}^0)$, and $\widetilde{c_{jk}^0} \cdot \overline{c_{jk}^0} = 0$. With this notation,

$$1 - |z|^2 = (1 - |c_{jk}^0|^2)(1 - |\alpha|^2),$$

$$z \cdot \bar{a}_k^* = c_{jk}^0 \cdot \bar{a}_k^* + (1 - |c_{jk}^0|^2)^{1/2} \alpha \widetilde{c_{jk}^0} \cdot \bar{a}_k^* =: A + B\alpha.$$

Note that

$$\frac{|c_{jk}^0 \cdot \bar{a}_k^*|^2}{|c_{jk}^0|^2} + |\widetilde{c_{jk}^0} \cdot \bar{a}_k^*|^2 = 1,$$

so that

$$\begin{aligned} |A|^2 &= |c_{jk}^0 \cdot \bar{a}_k^*|^2 = |c_{jk}^0|^2 \cos^2 \theta_{jk}, \\ |B|^2 &= (1 - |c_{jk}^0|^2) \left(1 - \frac{|c_{jk}^0 \cdot \bar{a}_k^*|^2}{|c_{jk}^0|^2}\right) \\ &= (1 - |c_{jk}^0|^2)(1 - \cos^2 \theta_{jk}). \end{aligned}$$

As above, the maximum of $(1 - |\alpha|^2)/|A + B\alpha|^2$ is $(|A|^2 - |B|^2)^{-1}$, provided that $|A| > |B|$. This last condition simply means that $\phi_k(\overline{V}_k) \cap \phi_k(\overline{V}_j) = \varnothing$, i.e. $\overline{V}_k \cap \overline{V}_j = \varnothing$. This is equivalent to $|A|^2 > |B|^2$, which is easily rewritten into (2).

Getting back to $1 - \inf\{d_G^2(z, w),\ z \in V_j,\ w \in V_k\}$, we find

$$\begin{aligned}\frac{1}{1 + (1 - |c_{jk}^0|^2)/(|A|^2 - |B|^2)} &= \frac{1 - |c_{jk}^0|^2}{|A|^2 - |B|^2 + (1 - |c_{jk}^0|^2)} \\ &= \frac{1 - |c_{jk}^0|^2}{\cos^2 \theta_{jk}}.\end{aligned}$$

Writing $d_G^2(V_j, V_k) \geq \delta_1^2$ gives (4) immediately. (3) follows from substituting the values given by Lemma 1 (2) and (3).

Proof of Lemma 3. Recall from **[4]** that the global Jacobian of ϕ_k is

$$J_{\phi_k} = \left(\frac{1 - |a_k|^2}{|1 - z \cdot \bar{a}_k|^2} \right)^{n+1}.$$

To restrict to V_j, we must divide out the dilation corresponding to the directions orthogonal to the source set, $a_j^* \perp V_j$, and to the target set, $c_{jk} \perp \phi_k(V_j)$. This will be $|D_{a_j^*}(\phi_k(z) \cdot \bar{c}_{jk}/|\bar{c}_{jk}|)|^2$, where $D_{a_j^*}$ denotes the derivative in the complex direction of a_j^*.

$$\begin{aligned}&\phi_k(z) \cdot \bar{c}_{jk} \\ &\quad = \frac{a_k(1 - (1 - s_k) z \cdot \bar{a}_k / |a_k|^2) - s_k z}{1 - z \cdot \bar{a}_k} \\ &\qquad \cdot \left[\left((1 - s_k) \frac{a_j^* \cdot \bar{a}_k}{|a_k|^2} - |a_j| \right) a_k + s_k a_j^* \right] \\ &\quad = \frac{1}{1 - z \cdot \bar{a}_k} \Bigg[(1 - s_k) a_k \cdot \bar{a}_j^* - |a_j||a_k|^2 + s_k a_k \cdot \bar{a}_j^* \\ &\qquad + \Bigg[(1 - s_k)^2 \frac{a_j^* \cdot \bar{a}_k}{|a_k|^2} - (1 - s_k)|a_j| - s_k(1 - s_k) \frac{a_j^* \cdot \bar{a}_k}{|a_k|^2} \\ &\qquad\qquad + s_k |a_j| - s_k(1 - s_k) \frac{a_j^* \cdot \bar{a}_k}{|a_k|^2} \Bigg] z \cdot \bar{a}_k - s_k^2 z \cdot \bar{a}_j^* \Bigg] \\ &\quad = \frac{1}{1 - z \cdot \bar{a}_k} [a_k \cdot \bar{a}_j^* - |a_j||a_k|^2 \\ &\qquad + (a_j - a_k) \cdot \bar{a}_j^* (z \cdot \bar{a}_k) - (1 - |a_k|^2) z \cdot \bar{a}_j^*].\end{aligned}$$

Since $z \cdot \bar{a}_k$ and $z \cdot \bar{a}_j^*$ are linear forms,

$$D_{a_j^*}(z \cdot \bar{a}_j^*) = a_j^* \cdot \bar{a}_j^* = 1 \quad \text{and} \quad D_{a_j^*}(z \cdot \bar{a}_k) = a_j^* \cdot \bar{a}_k.$$

Thus

$$D_{a_j^*}(\phi_k(z)\cdot\bar{c}_{jk}) = \frac{\phi_k(z)\cdot\bar{c}_{jk}}{1-z\cdot\bar{a}_k}a_j^*\cdot\bar{a}_k + \frac{1}{1-z\cdot\bar{a}_k}[-l_{jk}a_j^*\cdot\bar{a}_k-(1-|a_k|^2)].$$

For $z\in V_j$, $z\cdot\bar{a}_j=|a_j|^2$ and $\phi_k(z)\cdot\bar{c}_{jk}=l_{jk}$, so that all that remains is the second term inside the square brackets:

$$\left|D_{a_j^*}\left(\phi_k(z)\cdot\frac{\bar{c}_{jk}}{|\bar{c}_{jk}|}\right)\right|^2=\frac{(1-|a_k|^2)^2}{|c_{jk}|^2|1-z\cdot\bar{a}_k|^2}.$$

Dividing the global Jacobian by this quantity yields the result.

Proof of Lemma 4. (1) At any point of V, split the tangent space $\mathscr{V}$ into an orthogonal direct sum:

$$\mathscr{V}=\mathscr{V}\cap\operatorname{Span}(a,a_k)\oplus\mathscr{V}'.$$

The projection Q_k induces the identity on $\mathscr{V}'$, so it is enough to consider the situation on the complex line $\mathscr{V}\cap\operatorname{Span}(a,a_k)=\operatorname{Span}(\vec{u})$, where $\vec{u}:=a_k-(a_k\cdot\bar{a}/|a|^2)a$. Thus

$$|J_{Q_k|_V}|=\frac{|Q_k(\vec{u})|^2}{|\vec{u}|^2},$$

and an easy computation gives (1).

(2) If $a\cdot\bar{a}_k\neq 0$, then $Q_k|_V$ is one-to-one. Let $(Q_k|_V)^{-1}(w)=w+\lambda a_k$, where $\lambda\in C$.

$$(w+\lambda a_k)\cdot\bar{a}=|a|^2\Rightarrow\lambda=\frac{|a|^2-w\cdot\bar{a}}{a_k\cdot\bar{a}}.$$

Since we want the image under the projection of those points inside the ball,

$$Q_k(V)=\left\{w\in Q_k(B^n)\colon |w|^2+\frac{||a|^2-w\cdot\bar{a}|^2}{|a_k\cdot\bar{a}|^2}|a_k|^2<1\right\}.$$

Using the w_1, w_2 notation, the above equation is written

$$|w_1|^2+|w_2|^2+\frac{||a|^2-w_1\cdot\bar{a}|^2}{|a_k\cdot\bar{a}|^2}|a_k|^2<1.$$

Notice that $w_1\cdot\bar{a}=w_1\cdot\overline{Q_k(a)}$, $|w_1\cdot\overline{Q_k(a)}|^2=|w_1|^2|Q_k(a)|^2$, and

$$|a|^2=|Q_k(a)|^2+\frac{|a\cdot\bar{a}_k|^2}{|a_k|^2}.$$

The equation becomes:

$$|w_1|^2\left(1+\frac{|a_k|^2}{|a_k\cdot\bar a|^2}\right)-\frac{|a_k|^2|a|^2}{|a_k\cdot\bar a|^2}(w_1\cdot\overline{Q_k(a)}+\bar w_1\cdot Q_k(a))$$
$$+\frac{|a_k|^2|a|^4}{|a_k\cdot\bar a|^2}+|w_2|^2<1$$

which simplifies to

$$\frac{|a_k|^2|a|^2}{|a_k\cdot\bar a|^2}|w_1-Q_k(a)|^2+|w_2|^2<1-|a|^2.$$

(3) In the above ellipsoid, the minimum distance to the boundary is attained when $w_2=0$, and equals

$$1-|Q_k(a)|-(1-|a|^2)^{1/2}\cos\theta=1-|a|\sin\theta-(1-|a|^2)^{1/2}\cos\theta.$$

Proof of Lemma 5. First, since $V_j\cap T_\delta(V_k)=\varnothing$, $\phi_k(V_j)\cap\phi_k(T_\delta(V_k))=\varnothing$. Although tubes have no reason to be invariant under automorphisms, $\phi_k(T_\delta(V_k))$ is not far from being a tube around $Q_k(B^n)=\phi_k(V_k)$. More precisely, if $|P_k(z)|<\delta/(1+\delta)$, then $\phi_k^{-1}(z)=\phi_k(z)\in T_\delta(V_k)$. Indeed,

$$(\phi_k(z)-a_k)\cdot\bar a_k^*=\frac{-(1-|a_k|^2)P_k(z)}{1-z\cdot\bar a_k},$$

$$|(\phi_k(z)-a_k)\cdot\bar a_k^*|\le(1-|a_k|^2)\frac{|P_k(z)|}{1-|a_k|^2|P_k(z)|}$$
$$\le(1-|a_k|^2)\frac{|P_k(z)|}{1-|P_k(z)|}<\delta(1-|a_k|^2)$$

under the above hypothesis. It follows that for $z\in\phi_k(V_j)$, since $z\notin\phi_k(T_\delta(V_k))$, $|P_k(z)|\ge\delta/(1+\delta)=:\delta_1$, and consequently $|Q_k(z)|=(1-|P_k(z)|^2)^{1/2}\le\sqrt{1-\delta_1^2}$. □

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References

[1] E. Amar, *Extension de fonctions analytiques avec estimation*, Ark. Mat., **17** No. 1 (1979), 123–138.

[2] ———, *Suites d'interpolation pour les classes de Bergman de la boule et du polydisque de C^n*, Canad. J. Math., **XXX** No. 4 (1978), 711–737.

[3] A. M. Mantero, *Sur la condition de Carleson dans la boule unité de C^m*, Boll. Un. Mat. Ital., A. **6** 2 (1983), 163–169.

[4] W. Rudin, *Function Theory in the Unit Ball of C^n*, Springer Verlag, New York, 1980.

[5] H. Shapiro and A. L. Shields, *On some interpolation problems for analytic functions*, Amer. J. Math., **83** (1961), 513–532.

[6] P. J. Thomas, *Interpolating Sequences of Complex Hyperplanes in the Unit Ball of C^n*, Ann. Inst. Fourier Grenoble, **36** No. 3 (1986), 167–181.

[7] ———, *Properties of Interpolating Sequences in the Unit Ball*, Ph.D. dissertation, University of California, Los Angeles, 1986.

[8] ———, *Hardy space interpolation in the unit ball*, Nederl. Akad. Wetensch. Indag. Math., **90** No. 3 (1987), 325–351.

[9] N. Th. Varopoulos, *Zeros of H^p functions in several complex variables*, Pacific J. Math., **88** No. 8 (1980), 189–246.

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